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WEAK APPROACHABILITY IN A TWO-PERSON GAME[1]

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1. Introduction. Let $M = \|m_{ij}\|$ be a 2×2 matrix whose elements m_{ij} are probability distributions on the Borel sets of a closed bounded convex subset X of Euclidean 2-space. We associate with M a game between two players, I and II, with the following infinite sequence of engagements: At the nth engagement, $n = 1, 2, \cdots$, player I selects a number p_n and player II selects a number q_n from the unit interval (each selection is made without either player knowing the choice of the other player), a point $Y_n \varepsilon X$ is selected according to the distribution

$$(1.1)\quad p_nq_nm_{11} + p_n(1 - q_n)m_{12} + (1 - p_n)q_nm_{21} + (1 - p_n)(1 - q_n)m_{22},$$

and then Y_n, p_n and q_n are announced to both players.

A strategy for player I is a function P defined on the set of all n tuples $(Y_1, p_1, q_1; \cdots; Y_n, p_n, q_n)$, $n = 1, 2, \cdots$, with values $P(Y_1, p_1, q_1; \cdots; Y_n, p_n, q_n) = p_{n+1}$ in the unit interval, and $p_1 = P(\varnothing)$, where $\varnothing$ is the empty sequence, is simply a point in the unit interval. A strategy Q for player II is similar: $Q(Y_1, p_1, q_1; \cdots; Y_n, p_n, q_n) = q_{n+1}$, $0 \leqq q_{n+1} \leqq 1$, and $q_1 = Q(\varnothing)$ is a point in the unit interval. For a given M, each pair P, Q of strategies determines a sequence of random variables $Y_1, Y_2, \cdots$ in X.

In this paper we investigate the controllability of the behavior of the random variable $\bar{Y}_N = \sum_1^N Y_n/N$ for each N, especially N large. For a given M and a set S in 2-space, can one of the players guarantee that $\bar{Y}_N$ is in or arbitrarily near S, with probability approaching 1 as N tends to infinite?

We paraphrase here the following definitions given by Blackwell [1]: For a given M, a set S in 2-space is said to be weakly approachable in M by $I(II)$ if for every $\nu > 0$ there is an N_0 such that, for every $N \geqq N_0$ there is a strategy P^* for I (Q^* for II) such that

$$(1.2)\qquad \text{Prob}\,\{\delta_N > \nu\} < \nu \qquad \text{for all} \quad Q(P),$$

where $\delta_N = \delta(\bar{Y}_N, S)$ denotes the distance of the point $\bar{Y}_N$ from S, and $Y_1, \cdots, Y_N$ are the variables determined by P^*, $Q(Q^*, P)$. The set S is weakly excludable in M by $I(II)$ if there exists a $\Delta > 0$ such that for every $\nu > 0$ there is an N_0 such that for every $N \geqq N_0$ there is a strategy P^* for I (Q^* for II)

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with

$$\text{(1.3)} \qquad \text{Prob}\,\{\delta_N < \Delta\} < \nu \qquad \text{for all} \quad Q(P).$$

In terms of the above definitions, we investigate the collection of sets in 2-space which are weakly approachable by I(II) and the collection of sets which are weakly excludable by $I(II)$. The solution of a special case ($m_{11} = m_{12} = (0, 0)$, $m_{21} = (1, 0)$, $m_{22} = (1, 1)$) has been mentioned by Blackwell [1]. In this paper we consider the general case.

It is clear that the weak approachability and the weak excludability for a set S are the same for its closure; hence, we may assume that S is closed. Clearly each superset of a weakly approachable set is weakly approachable, each subset of a weakly excludable set is weakly excludable, and no set is weakly approachable by one player and also weakly excludable by the other player. For any pair of disjoint closed sets S, S', if S is weakly approachable by a player, then S' is weakly excludable by the same player. Thus any condition for weak approachability implies a condition for weak excludability, and we may focus our attention only on weak approachability.

For each matrix M, let $\bar{M}$ be the matrix whose elements $\bar{m}_{ij}$ are the expectations of the distributions m_{ij}. We write $\bar{m}_{11} = a$, $\bar{m}_{12} = b$, $\bar{m}_{21} = c$, and $\bar{m}_{22} = d$ as a matter of convenience. For each p, $0 \leqq p \leqq 1$, let $R(p)$ be the convex hull of the points $pa + (1 - p)c$ and $pb + (1 - p)d$. For each q, $0 \leqq q \leqq 1$, let $T(q)$ be the convex hull of the points $qa + (1 - q)b$ and $qc + (1 - q)d$. Let Ω^* be the convex hull of the points a, b, c, d, and let Ω be the union of $R(p)$ for all p; in fact, Ω is also the union of $T(q)$ for all q and is a subset of Ω^*.

For a given M with a convex Ω, which means that the quadrilateral $abdc$ is convex and $\Omega = \Omega^*$, it turns out that every set in 2-space is either weakly approachable by one player or weakly excludable by the other player (Theorems 2 and 4). There exists a collection of sets $\mathcal{E}^*$ such that a set in 2-space is weakly approachable by one player if and only if it contains a set of $\mathcal{E}^*$ (Theorems 3 and 5). Sufficient conditions for weak approachability are given in Section 2 and Section 3, and necessary conditions are given in Section 4.

Sufficient conditions for weak approachability for matrices with nonconvex Ω are introduced in Section 5, Section 6, and Section 7.

REMARK. For a matrix of degenerate probability distributions, that is $M = \bar{M}$ almost everywhere, we neglect the null set and write $M = \bar{M}$.

We introduce the following example to illustrate some ideas of this paper.

EXAMPLE 1. Let $M = \bar{M}$, $a = (\frac{1}{2}, 1)$, $b = (0, 1)$, $c = (1, 0)$ and $d = (0, 0)$.

Every continuous graph from the (line) segment $\overline{b\,d}$ to the segment $\overline{ac}$ in the trapezoid $abdc$ is weakly approachable by player I, provided the slope is bounded between -1 and $+2$ (slopes of the two diagonals of the trapezoid) (Theorem 1). However, many nonconnected graphs are also weakly approachable by I. For example, the set $B_1 \cup B_2$, where B_1 is the segment $\overline{(0, \frac{1}{3})(\frac{7}{12}, \frac{1}{3})}$ and B_2 is the segment $\overline{(\frac{1}{4}, \frac{2}{3})(\frac{2}{3}, \frac{2}{3})}$, is weakly approachable by I (Theorem 2). For every N, player I has a strategy which guarantees that $\bar{Y}_{3N} \in B_1 \cup B_2$, as follows: $p_n = 0$

for $1 \leqq n \leqq N$ and $p_n = 1$ for $N < n \leqq 2N$, so that $\bar{Y}_{2N} = (u, \frac{1}{2})$; if $u \leqq \frac{3}{8}$, $p_n = 0$ for $2N < n \leqq 3N$, then $\bar{Y}_{3N} \varepsilon B_1$; and if $u > \frac{3}{8}$, $p_n = 1$ for $2N < n \leqq 3N$, then $\bar{Y}_{3N} \varepsilon B_2$.

2. Preliminary sufficient condition for the case $\Omega = \Omega^*$. Throughout Sections 2, 3, and 4, it is assumed that the quadrilateral *abdc* is convex ($\Omega = \Omega^*$).

Except for the special case in Lemma 10, in which Ω has empty interior, we may assume $d = (0, 0)$, $b = (0, b^y)$, $b^y \geqq 0$, $a^x > 0$, $c^x > 0$, where ω^x, ω^y are the x- and y-coordinates of the point $\omega = (\omega^x, \omega^y)$ in 2-space.

Let F be the graph of some continuous function f which satisfies that

(i) there is a point $u_f = (u_f^x, f(u_f^x)) \varepsilon T(1)$,

(ii) f is defined and $(x, f(x)) \varepsilon \Omega$ for all $0 \leqq x \leqq u_f^x$,

(iii) $\rho_* = (c^y - b^y)/c^x \leqq (f(x_2) - f(x_1))/(x_2 - x_1) \leqq a^y/a^x = \rho^*$ for all $0 \leqq x_1 < x_2 \leqq u_f^x$, where ρ_* and ρ^* are the slopes of the two diagonals $\overline{bc}$ and $\overline{da}$,

(iv) $F = \{(x, f(x)):0 \leqq x \leqq u_f^x\}$;

and let $\mathfrak{F}$ denote the collection of all such subsets F of Ω. Then we have the following theorem.

THEOREM 1. *If a set S in 2-space contains an F, $F \varepsilon \mathfrak{F}$, then S is weakly approachable by player* I *and he has a pure strategy* (i.e., $p_n = 0$ *or* 1 *for all* n).

We begin by showing that this theorem holds for those M with degenerate distributions, that is $M = \bar{M}$. Let

$$(2.1)\quad \omega_n = z(p_n, q_n) = p_n q_n a + p_n(1 - q_n)b + (1 - p_n)q_n c + (1 - p_n)(1 - q_n) d$$

be the point in Ω chosen by the two players at the nth engagement with p_n, q_n as their strategies, and let $\bar{\omega}_n = \sum_1^n \omega_i/n$. For a fixed positive integer N, let $M_0 = M_{0,N} = \bar{M}$ and

$$(2.2)\quad M_n = M_{n,N} = \| n\bar{\omega}_n/N + (N - n)\bar{m}_{ij}/N \| = \begin{Vmatrix} a_n & b_n \\ c_n & d_n \end{Vmatrix} \quad \text{for each} \quad 1 \leqq n \leqq N.$$

We define $\Omega_n = \Omega_{n,N}$, $R_n(p)$, $T_n(q)$, and $z_n(p, q)$, $0 \leqq n \leqq N$, and $0 \leqq p, q \leqq 1$, with respect to M_n in the same way that we define Ω, $R(p)$, $T(q)$ with respect to $\bar{M}$ as follows:

$$(2.3)\quad R_n(p) = \{\alpha[pa_n + (1 - p)c_n] + (1 - \alpha)[pb_n + (1 - p) d_n]:0 \leqq \alpha \leqq 1\},$$

$$(2.4)\quad T_n(q) = \{\alpha[qa_n + (1 - q)b_n] + (1 - \alpha)[qc_n + (1 - q) d_n]:0 \leqq \alpha \leqq 1\},$$

$$(2.5)\quad \Omega_n = \bigcup_{0 \leqq p \leqq 1} R_n(p) = \bigcup_{0 \leqq q \leqq 1} T_n(q),$$

and

$$z_n(p, q) = pqa_n + p(1 - q)b_n + (1 - p)qc_n + (1 - p)(1 - q)d_n, \tag{2.6}$$

where $a_0 = a$, $b_0 = b$, $c_0 = c$, $d_0 = d$, $R_0(p) = R(p)$, $T_0(q) = T(q)$, $\Omega_0 = \Omega$ and $z_0(p, q) = z(p, q)$. For a given $\bar{\omega}_n$, $1 \leqq n \leqq N$, Ω_n is the conditional range of $\bar{\omega}_N$ and $R_n(p)(T_n(q))$ is the conditional range of $\bar{\omega}_N$ if $p_m = p(q_m = q)$ for $n < m \leqq N$. $z_n(p, q)$ is the point in Ω_n chosen by the two players if they play one engagement with M_n and adopt p, q as their strategies.

LEMMA 1. *If we denote $M_{m,N}$ by $M_{0,N-m}$, $0 < m < N$, then we have*

$$M_n = M_{n,N} = M_{n-m,N-m} \tag{2.7}$$

for all $m < n \leqq N$; that is, M_n can be determined by M_m, p_i and q_i for $m < i \leqq n$.

PROOF. Since

$$M_{n-m,N-m} = \|(n - m)\bar{\omega}_{m,n}/(N - m) + (N - n)(m\bar{\omega}_m/N + (N - m)\bar{m}_{ij}/N)/(N - m)\|,$$

where $\bar{\omega}_{m,n} = \sum_{m+1}^{n} z_m(p_i, q_i)/(n - m) \,\varepsilon\, \Omega_m$; we have

$$M_{n-m,N-m} = \|n\bar{\omega}_n/N + (N - n)\bar{m}_{ij}/N\| = M_{n,N}. \qquad \text{Q.E.D.}$$

COROLLARY 1. *$\Omega_n = \Omega_{n,N}$ is monotonically decreasing to the point $\bar{\omega}_N$ as n tends to N.*

PROOF. Since $(n\bar{\omega}_n/N + (N - n)\bar{m}_{ij}/N) \,\varepsilon\, \Omega$ and $\Omega_n \subset \Omega$ for all $1 \leqq n \leqq N$ and $1 \leqq i, j \leqq 2$, we have

$$\Omega_n = \Omega_{n-m,N-m} \subset \Omega_{0,N-m} = \Omega_m \tag{2.8}$$

for all $0 \leqq m \leqq n \leqq N$ by Lemma 1 and $\Omega_N = \bar{\omega}_N$ by definition. Q.E.D.

COROLLARY 2. *For each $1 \leqq n \leqq N$, we have*

$$\max_{\omega,\omega' \varepsilon \Omega_n} \delta(\omega, \omega') \leqq k(N - n)/N \tag{2.9}$$

and

$$\max_{0 \leqq p,q \leqq 1} \delta(z_{n-1}(p, q), z_n(p, q)) \leqq k/N, \tag{2.10}$$

where $k = \max_{\omega,\omega' \varepsilon \Omega} \delta(\omega, \omega')$ is the diameter of Ω.

PROOF. Since $\delta(z_n(p, q), z_n(p', q')) = (N - n)\delta(z(p, q), z(p', q'))/N$, $z_n(p, q) = z_{n-1}(p, q) + (z(p_n, q_n) - z(p, q))/N$, and $\delta(z(p_n, q_n), z(p, q)) \leqq k$ for all $0 \leqq p, p', p_n, q, q', q_n \leqq 1$, the corollary follows immediately. Q.E.D.

LEMMA 2. *If $FT_n(\mu) = \varnothing$ ($FS = F \cap S$ and $\varnothing$ is an empty set) for some $F \,\varepsilon\, \mathfrak{F}$, $1 \leqq n \leqq N$, and $0 \leqq \mu \leqq 1$, then*

$$\delta(T_n(\mu), F) = \min_{\lambda = 0 \text{ or } 1} \delta(z_n(\lambda, \mu), F), \tag{2.11}$$

where $z_n(0, \mu)$ and $z_n(1, \mu)$ are the vertexes of $T_n(\mu)$.

PROOF. Suppose $n < N$ and this lemma does not hold. Then there exists a λ^* with $0 < \lambda^* < 1$ such that

$$\delta(T_n(\mu), F) = \delta(z_n(\lambda^*, \mu), F) < \min_{\lambda = 0 \text{ or } 1} \delta(z_n(\lambda, \mu), F).$$

Let $s = (s^x, f(s^x))$ be the closest point in F to $v = z_n(\lambda^*, \mu)$. Since there exists an open neighborhood of the point v in $T_n(\mu)$, which is a segment, the segment $\overline{sv}$ must be perpendicular to $T_n(\mu)$.

Transform the given space linearly to a new 2-space such that

$$(2.12) \qquad v' = (0, 0) \quad \text{and} \quad z_n'(1, \mu) = (0, \delta(v, z_n(1, \mu))),$$

where ω'^x, ω'^y are the x- and y-coordinates of the image point $\omega' = (\omega'^x, \omega'^y)$ of ω in the new space. Thus $s' = (-\delta(v, s), 0)$ or $(\delta(v, s), 0)$. The convexity of quadrilateral Ω and the definition of F imply that $s_1'^x \leqq s_2'^x$ for each pair of points $s_1, s_2 \varepsilon F$ with $s_1^x < s_2^x$.

Suppose $s' = (-\delta(v, s), 0)$ and let $T_n'(\mu)$ be the image of $T_n(\mu)$ in the new space. Then there exists a $\Delta > 0$ such that $s_\Delta = (s^x + \Delta, f(s^x + \Delta)) \varepsilon F$ and either $\delta(T_n(\mu), s_\Delta) \doteq \delta(T_n'(\mu), s_\Delta') < \delta(v, s)$, which is a contradiction, or

$$s_\Delta' = (-\delta(v, s), -\delta(v, z_n(0, \mu))) \quad \text{or} \quad (-\delta(v, s), \delta(v, z_n(1, \mu))),$$

which implies $\min_{\lambda=0 \text{ or } 1} \delta(z_n(\lambda, \mu), s_\Delta) = \delta(v, s)$ and contradicts the assumption.

For $s' = (\delta(v, s), 0)$, the argument is similar. Q.E.D.

LEMMA 3. *For each triple $M = \bar{M}$, $F \varepsilon \mathfrak{F}$, and N, there exists a pure strategy for* I *such that*

$$(2.13) \qquad \delta_n = \max_{0 \leqq q \leqq 1} \delta(T_n(q), F) \leqq k/N$$

for all $1 \leqq n \leqq N$ and for all Q.

PROOF. For each $1 \leqq n \leqq N$, let n^0, n^1 be the last pair of nonnegative integers less than n if any such that $p_{n^0+1} = 1$, $p_{n^1+1} = 0$. For each $z_n(\lambda, q)$, $\lambda = 0$ or 1, let $v_n(\lambda, q)$ be the point in $R_{n^\lambda}(\lambda)$ such that the segment $\overline{v_n(\lambda, q) z_n(\lambda, q)}$ is parallel to the segment $\overline{d_{n^0} d_{n^0+1}}$ or $\overline{b_{n^1} b_{n^1+1}}$ according as $\lambda = 0$ or 1. For each $0 \leqq q \leqq 1$ and $\lambda = 0$ or 1, let

$$(2.14) \qquad T_n^\lambda(q) = \{\alpha z_n(\lambda, q) + (1 - \alpha) v_n(\lambda, q) : 0 \leqq \alpha < 1\}$$

and $T_n^\lambda(q) = \varnothing$ if $p_m = \lambda$ for all $1 \leqq m \leqq n$. Since $d_{n^0}^x \leqq d_{n^0+1}^x$, $b_{n^1}^x \leqq b_{n^1+1}^x$, $\rho^* \leqq (d_{n^0+1}^y - d_{n^0}^y)/(d_{n^0+1}^x - d_{n^0}^x) \leqq \infty$, and $-\infty \leqq (b_{n^1+1}^y - b_{n^1}^y)/(b_{n^1+1}^x - b_{n^1}^x) \leqq \rho_*$, we have

$$(2.15) \qquad FT_n^{1-\lambda}(q) = \varnothing \qquad \text{for all} \quad 0 \leqq q \leqq 1$$

if

$$FT_n^\lambda(q) \neq \varnothing \qquad \text{for some} \quad 0 \leqq q \leqq 1, \quad \lambda = 0 \text{ or } 1.$$

In Ω, each continuous graph from $T(0)$ to $T(1)$ intersects every continuous graph from $R(0)$ to $R(1)$. If we let $p_1 = 0$, then $F[T_1(q) \cup T_1^1(q)] \neq \varnothing$ for all $0 \leqq q \leqq 1$. Assume that

$$F[T_n(q) \cup T_n^0(q) \cup T_n^1(q)] \neq \varnothing \qquad \text{for all} \quad 0 \leqq q \leqq 1.$$

If $FT_n^1(q) \neq \varnothing$ for some $0 \leqq q \leqq 1$, then

$$FT_n^0(q) = \varnothing \quad \text{and} \quad F[T_n(q) \cup T_n^1(q)] \neq \varnothing \quad \text{for all} \quad 0 \leqq q \leqq 1;$$

and if we let $p_{n+1} = 1$, then $\bigcup_{0 \leqq q \leqq 1} T_{n+1}^1(q) \subset \bigcup_{0 \leqq q \leqq 1} T_n^1(q)$ and

$$F[T_{n+1}(q) \cup T_{n+1}^0(q) \cup T_{n+1}^1(q)] \neq \varnothing \quad \text{for all} \quad 0 \leqq q \leqq 1.$$

If $FT_n^1(q) = \varnothing$ for all $0 \leqq q \leqq 1$ and we let $p_{n+1} = 0$, then

$$\bigcup_{0 \leqq q \leqq 1} T_{n+1}^0(q) \subset \bigcup_{0 \leqq q \leqq 1} T_n^0(q)$$

and

$$F[T_{n+1}(q) \cup T_{n+1}^0(q) \cup T_{n+1}^1(q)] \neq \varnothing \quad \text{for all} \quad 0 \leqq q \leqq 1.$$

By induction, for each $1 \leqq n \leqq N$ we have

(2.16) $$F[T_n(q) \cup T_n^0(q) \cup T_n^1(q)] \neq \varnothing \quad \text{for all} \quad 0 \leqq q \leqq 1;$$

hence, there exists a $\lambda_n = 0$ or 1 such that

(2.17) $$F[T_n(q) \cup T_n^{\lambda_n}(q)] \neq \varnothing \quad \text{for all} \quad 0 \leqq q \leqq 1.$$

By Lemma 2 and Corollary 2, we have

$$\begin{aligned} \delta_n &= \max_{0 \leqq q \leqq 1, FT_n(q) = \varnothing} \{\min_{\lambda = 0 \text{ or } 1} \delta(z_n(\lambda, q), F)\} \\ &\leqq \max_{0 \leqq q \leqq 1, FT_n^{\lambda_n}(q) \neq \varnothing} \delta(z_n(\lambda_n, q), FT_n^{\lambda_n}(q)) \\ &\leqq \max(\delta(d_{n^0}, d_{n^0+1}), \delta(b_{n^1}, b_{n^1+1})) \leqq k/N, \end{aligned}$$

where $\delta_n = 0$ if $FT_n(q) \neq \varnothing$ for all $0 \leqq q \leqq 1$. Q.E.D.

Therefore, $\delta(\bar{\omega}_N, F) = \delta_N \leqq k/N \to 0$ as $N \to \infty$ by Corollary 1 and Theorem 1 follows for the case $M = \bar{M}$.

EXAMPLE 2. Let M be defined as in Example 1 and $F = \{(x, f(x)): 0 \leqq x \leqq \frac{3}{4}\}$, where $f(x) = 2x + \frac{1}{2}$ for $0 \leqq x \leqq \frac{1}{4}$ and $f(x) = \frac{5}{4} - x$ for $\frac{1}{4} \leqq x \leqq \frac{3}{4}$.

A weak approachable strategy for player I (according to Lemma 3) is described as follows: Let $p_1 = 0$; and for each $1 \leqq n < N$, $N \geqq 2$, let $p_{n+1} = 0$ if $c_n^x \leqq \frac{3}{4}$ and $f(c_n^x) \leqq c_n^y$ or $f(d_n^x) \leqq d_n^y$, and let $p_{n+1} = 1$ otherwise; then $\delta_n \leqq 2^{\frac{1}{2}}/N$ for all $1 \leqq n \leqq N$.

LEMMA 4. *For each $\nu > 0$, there exists a positive integer N_0' such that*

(2.18) $$\text{Prob}\{\delta(\bar{Y}_N, \bar{\omega}_N) > \nu/2 \quad \text{for some} \quad N \geqq N_0'\} < \nu,$$

where

$$\begin{aligned} \omega_n &= E(Y_n \mid p_1, q_1, Y_1; \cdots; p_{n-1}, q_{n-1}, Y_{n-1}; p_n, q_n) \\ &= E(Y_n \mid p_n, q_n) \qquad (2.19) \\ &= p_n q_n a + p_n(1 - q_n)b + (1 - p_n)q_n c + (1 - p_n)(1 - q_n)d \end{aligned}$$

for each $n \geqq 1$ and $E(Y \mid Z)$ denotes the conditional expectation of Y given Z.

PROOF. Let K be the diameter of X and $\langle u, v\rangle$ represent the inner product of the vectors u and v. For each $n > 1$, since

$$\delta(Y_{n-1}, \omega) \leqq K \qquad \text{for all} \quad \omega \varepsilon X,$$

$$\begin{aligned}\delta(\bar{Y}_n, \bar{\omega}_n) &\leqq \delta((n-1)\bar{Y}_{n-1}/n + \omega_n/n, \bar{\omega}_n) + \delta(\bar{Y}_n, (n-1)\bar{Y}_{n-1}/n + \omega_n/n)\\ &\leqq \delta(\bar{Y}_{n-1}, \bar{\omega}_{n-1}) + K/n,\end{aligned}$$

$$\delta(\bar{Y}_n, \bar{\omega}_n) \geqq \delta(\bar{Y}_{n-1}, \bar{\omega}_{n-1}) - 2K/n,$$

$$\begin{aligned}\delta^2(\bar{Y}_n, \bar{\omega}_n) = {} & (n-1)^2\delta^2(\bar{Y}_{n-1}, \bar{\omega}_{n-1})/n^2 + \delta^2(Y_n, \omega_n)/n^2\\ & + 2(n-1)\langle \bar{Y}_{n-1} - \bar{\omega}_{n-1}, Y_n - \omega_n\rangle/n^2,\end{aligned}$$

we have

$$(2.20) \quad |\delta(\bar{Y}_n, \bar{\omega}_n) - \delta(\bar{Y}_{n-1}, \bar{\omega}_{n-1})| \leqq 2K/n,$$

$$\begin{aligned}E\{\delta^2(\bar{Y}_n, \bar{\omega}_n) \mid p_1, q_1, Y_1; \cdots; p_n, q_n\}&\\ \leqq (n-1)^2\delta^2(\bar{Y}_{n-1}, \bar{\omega}_{n-1})/n^2 + K^2/n^2&,\\ E[\delta^2(\bar{Y}_n, \bar{\omega}_n)] \leqq (n-1)^2E[\delta^2(\bar{Y}_{n-1}, \bar{\omega}_{n-1})]/n^2 + K^2/n^2&.\end{aligned}$$

By induction, we have

$$(2.21) \quad E[\delta^2(\bar{Y}_n, \bar{\omega}_n)] \leqq K^2/n \quad \text{and} \quad \text{Prob}\{\delta(\bar{Y}_n, \bar{\omega}_n) \geqq \Delta\} \leqq K^2/(n\Delta^2)$$

for each $n \geqq 1$ and $\Delta > 0$. Hence

$$\textstyle\sum_{m=1}^{\infty} \text{Prob}\{\delta(\bar{Y}_{m^2}, \bar{\omega}_{m^2}) \geqq \Delta\} \leqq (K^2/\Delta^2)\sum_{m=1}^{\infty} 1/m^2 < \infty$$

and

$$(2.22) \qquad \text{Prob}\{\limsup_m \delta(\bar{Y}_{m^2}, \bar{\omega}_{m^2}) \geqq \Delta\} = 0$$

for each $\Delta > 0$ by Borel-Cantelli lemma.

For each integer n with $m^2 \leqq n < (m+1)^2$, we have

$$\begin{aligned}(2.23) \quad |\delta(\bar{Y}_n, \bar{\omega}_n) - \delta(\bar{Y}_{m^2}, \bar{\omega}_{m^2})|&\\ \leqq \textstyle\sum_{i=m^2}^{n-1} |\delta(\bar{Y}_{i+1}, \bar{\omega}_{i+1}) - \delta(\bar{Y}_i, \bar{\omega}_i)| \leqq 4K/m&\end{aligned}$$

by (2.20). Thus $\delta(\bar{Y}_n, \bar{\omega}_n)$ converges to 0 almost everywhere. Q.E.D.

PROOF OF THEOREM 1. For each $\nu > 0$, let $N_0(\nu) \geqq \max(N_0', 2k/\nu)$. For each $N \geqq N_0(\nu)$, let $p_n = 0$ or 1 for $1 \leqq n \leqq N$ depending on $(\omega_1, \cdots, \omega_{n-1})$ as in Lemma 3, then $\delta(\bar{\omega}_N, S) \leqq \delta(\bar{\omega}_N, F) \leqq \nu/2$, where ω_n is defined in (2.19). Since

$$(2.24) \qquad \delta_N = \delta(\bar{Y}_N, S) \leqq \delta(\bar{Y}_N, \bar{\omega}_N) + \delta(\bar{\omega}_N, S),$$

we have Prob $\{\delta_N > \nu\} < \nu$ for all Q by Lemma 4. Q.E.D.

3. Sufficient condition for the case $\Omega = \Omega^*$. For each pair $\omega \varepsilon \Omega$ and $0 \leqq \lambda \leqq 1$, define $V(\omega, \lambda)$, $R_{V(\omega,\lambda)}(p)$, and $T_{V(\omega,\lambda)}(q)$ from the matrix $M(\omega, \lambda) =$

$\|\lambda\omega + (1-\lambda)\bar{m}_{ij}\|$ as we defined Ω, $R(p)$, and $T(q)$ from $\bar{M}$ respectively. We omit the arguments from $V(\omega, \lambda)$ and simply write it as V. Denote

$$(3.1) \qquad \vartheta = \{V:\omega \varepsilon \Omega, 0 \leqq \lambda \leqq 1\},$$

$$(3.2) \qquad \mathfrak{F}(V) = \{F(V):F \varepsilon \mathfrak{F}\},$$

where $F(V) = \{\lambda\omega + (1-\lambda)\omega'':\omega'' \varepsilon F\}$ is an $\mathfrak{F}$-type set with respect to V (instead of Ω). For each closed set $S \subset \Omega$, let

$$(3.3) \qquad \delta_S^{\,l} = \delta(S, T(0)), \qquad \delta_S^{\,r} = \delta(S, T(1)),$$

and l_S, r_S be the closest points in S to $T(0)$, $T(1)$ respectively. Define

$$(3.4) \qquad \mathcal{E}^1 = \{B:B = \{(x, f(x)):0 \leqq \alpha \leqq x \leqq \beta \leqq u_f^{\,x} \quad \text{for some } F \text{ in } \mathfrak{F} \text{ and some } 0 \leqq \alpha \leqq \beta \leqq u_f^{\,x}\}\};$$

$$(3.5) \qquad \mathcal{E}_*^{\,1} = \{B:B \varepsilon \mathcal{E}^1, l_B \varepsilon T(0), r_B \varepsilon T(1)\} = \mathfrak{F};$$

$$(3.6) \qquad \mathcal{E}^2 = \{D:D = B_1 \cup B_2\},$$

where the following conditions are satisfied:

(i) $B_1, B_2 \varepsilon \mathcal{E}^1$, $\delta_{B_1}^l \leqq \delta_{B_2}^l$, $\delta_{B_1}^r \geqq \delta_{B_2}^r$,

(ii) there exists a $V_1 \varepsilon \vartheta$ such that $r_{B_1} \varepsilon T_{V_1}(1)$, $l_{B_2} \varepsilon T_{V_1}(0)$, and B_1V_1, $B_2V_1 \varepsilon \mathcal{E}_*^{\,1}(V_1)$ [Remark: $\mathcal{E}_*^{\,1}(V_1) = \{B(V_1):B \varepsilon \mathcal{E}_*^{\,1}\}$, where $B(V_1) = B(V_1(\omega, \lambda)) = \{\lambda\omega + (1-\lambda)\omega'':\omega'' \varepsilon B\}$];

$$(3.7) \qquad \mathcal{E}_*^{\,2} = \{D:D \varepsilon \mathcal{E}^2, l_D \varepsilon T(0), r_D \varepsilon T(1)\}.$$

Suppose we have completed the constructions of $\mathcal{E}^2$, $\mathcal{E}_*^{\,2}$, $\cdots$, $\mathcal{E}^{m-1}$, $\mathcal{E}_*^{\,m-1}$ with $m \geqq 3$; let

$$(3.8) \qquad \mathcal{E}^m = \{D:D = \textstyle\bigcup_1^m B_u = [\bigcup_1^t B_u] \cup [\bigcup_{t+1}^m B_u], \quad \text{some } 1 \leqq t \leqq m-1\},$$

where the following conditions are satisfied:

(i) $B_u \varepsilon \mathcal{E}^1$ for all $1 \leqq u \leqq m$, $[\bigcup_1^t B_u] \varepsilon \mathcal{E}^t$ (there exist $V_1, \cdots, V_{t-1}$ associated with $B_1, \cdots, B_t$ by the definition of an $\mathcal{E}^t$ set), $[\bigcup_{t+1}^m B_u] \varepsilon \mathcal{E}^{m-t}$ (there exist $V_{t+1}, \cdots, V_{m-1}$ associated with $B_{t+1}, \cdots, B_m$), $\delta_{B_1}^l \leqq \delta_{B_{t+1}}^l$, $\delta_{B_t}^r \geqq \delta_{B_m}^r$,

(ii) there exists a $V_t \varepsilon \vartheta$ such that $r_{B_t} \varepsilon T_{V_t}(1)$, $l_{B_{t+1}} \varepsilon T_{V_t}(0)$, $[\bigcup_1^t B_u]V_t \varepsilon \mathcal{E}_*^{t'}(V_t)$ for some $1 \leqq t' \leqq t$, and $[\bigcup_{t+1}^m B_u]V_t \varepsilon \mathcal{E}_*^{t''}(V_t)$ for some $1 \leqq t'' \leqq m-t$,

(iii) for each pair i, j with $1 \leqq i \leqq t \leqq j \leqq m-1$, there exists a $V_{t,i,j} \varepsilon \vartheta$ such that

$$V_{t,i,j} \supset [\textstyle\bigcup_i^j V_u] \cup [\bigcup_{i+1}^j B_u], \qquad T_{V_{t,i,j}}(0) \supset T_{V_i}(0) \text{ or } T_{V_t}(0)$$

according as $V_i \not\subset V_t$ or $V_i \subset V_t$, $T_{V_{t,i,j}}(1) \supset T_{V_j}(1)$ or $T_{V_t}(1)$ according as $V_j \not\subset V_t$ or $V_j \subset V_t$, and $V_{t,i,j} \supset V_{t,i',j'}$ for all $1 \leqq i \leqq i' \leqq t \leqq j' \leqq$

$j \leqq m - 1$; and let

(3.9) $$\mathcal{E}_*^m = \{D: D \varepsilon \mathcal{E}^m, l_D \varepsilon T(0), r_D \varepsilon T(1)\}.$$

An example of $\mathcal{E}_*^3$-set is shown in Figure 1.

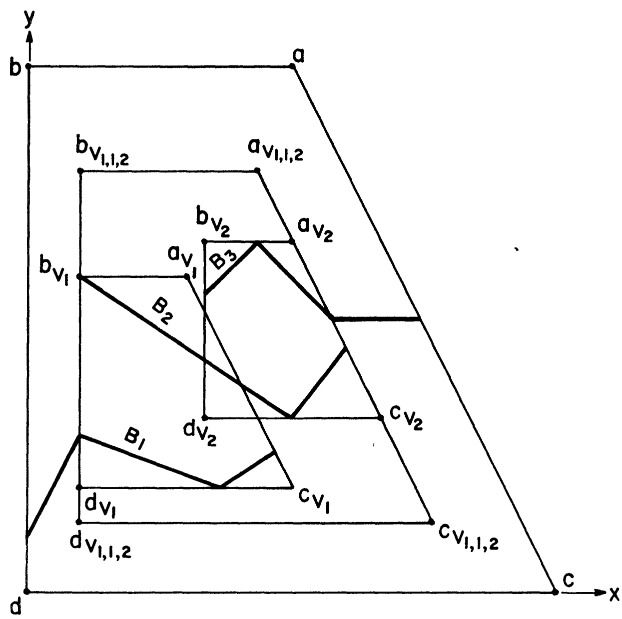

FIG. 1

THEOREM 2. *If a set S in 2-space contains a set of $\mathcal{E} = \bigcup_{m=1}^{\infty} \mathcal{E}_*^m$, then player* I *has a pure strategy such that S is weakly approachable by him.*

LEMMA 5. *For a given $M = \bar{M}$ and a $D \varepsilon \mathcal{E}_*^2$,* I *has a pure strategy such that*

(3.10) $$\delta_n = \max_{0 \leqq q \leqq 1} \delta(T_n(q), D) \leqq k/N \quad \text{for all} \quad 1 \leqq n \leqq N.$$

PROOF. From the definition of $\mathcal{E}_*^2$ in (3.6) and (3.7), we have $D = B_1 \cup B_2$ with B_1, $B_2 \varepsilon \mathcal{E}^1$, $l_{B_1} \varepsilon T(0)$, $r_{B_2} \varepsilon T(1)$, $r_{B_1} \varepsilon T_{V_1}(1)$, $l_{B_2} \varepsilon T_{V_1}(0)$, and $B_1 V_1$, $B_2 V_1 \varepsilon \mathcal{E}_*^1(V_1)$ for some $V_1 \varepsilon \vartheta$. If we let

$$F = [D \sim V_1] \cup [(x, v_1^y + (x - v_1^x)(v_2^y - v_1^y)/(v_2^x - v_1^x)) : v_1^x \leqq x \leqq v_2^x],$$

where $v_1 = B_1 T_{V_1}(0)$ and $v_2 = B_2 T_{V_1}(1)$, then F is an element of $\mathfrak{F}$. As indicated in Lemma 3, I has a pure strategy associated with this F for the first n_1 engagements, n_1 is the first n such that $V_1 \not\subset \Omega_n$, such that

$$F[T_n(q) \cup T_n^{\lambda_n}(q)] \neq \varnothing \quad \text{for all} \quad 0 \leqq q \leqq 1,$$

$$FT_n^{1-\lambda_n}(q) = \varnothing \quad \text{for all} \quad 0 \leqq q \leqq 1, \quad \text{and} \quad \max_{0 \leqq q \leqq 1} \delta(T_n(q), F) \leqq k/N$$

for all $n < n_1$, where $\lambda_n = 0$ or 1. Since $V_1 \subset \Omega_{n_1 - 1}$, we have

$$V_1[T_n^0(q) \cup T_n^1(q)] = \varnothing$$

for all $0 \leqq q \leqq 1$ and $n < n_1$; that is, p_n, $1 \leqq n \leqq n_1$, is independent of DV_1. Therefore, in relation to the first n_1 choices of player I as above, the set $D \sim V_1$ is equivalent to the set F and to the set $D \cup V_1$, and

$$\delta_n \leqq \max_{0 \leqq q \leqq 1} \delta(T_n(q), F) \leqq k/N \qquad \text{for all} \quad n < n_1 .$$

At $n = n_1$, we have one of the following two cases.

(1) $\Omega_{n_1} T_{V_1}(\mu) = \varnothing$ for $\mu = 0$ or 1.

In this case we have $\delta^l_{\Omega_{n_1}} \geqq \delta^l_{B_2}$ and

$$B_2[T_{n_1}(q) \cup T^0_{n_1}(q) \cup T^1_{n_1}(q)] \neq \varnothing \qquad \text{for all} \quad 0 \leqq q \leqq 1 \quad \text{for} \quad \mu = 0,$$

or $\delta^r_{\Omega_{n_1}} \geqq \delta^r_{B_1}$ and

$$B_1[T_{n_1}(q) \cup T^0_{n_1}(q) \cup T^1_{n_1}(q)] \neq \varnothing \qquad \text{for all} \quad 0 \leqq q \leqq 1 \quad \text{for} \quad \mu = 1;$$

hence, the set B_2 or B_1 is equivalent to an $\mathfrak{F}$ set (in the sense of Lemma 3) for all $n \geqq n_1$. As indicated in Lemma 3, player I has a pure strategy associated with B_i such that $\delta_n \leqq \max_{0 \leqq q \leqq 1} \delta(T_n(q), B_i) \leqq k/N$ for all $n_1 \leqq n \leqq N$, where $i = 2$ or 1 according as $\mu = 0$ or 1.

(2) $\Omega_{n_1} T_{V_1}(\mu) \neq \varnothing$ for $\mu = 0$ and 1, $\Omega_{n_1} R_{V_1}(\lambda) = \varnothing$ for $\lambda = 0$ or 1.

Let $p_n = \lambda$ for all n with $n_1 < n \leqq n_2$, where n_2 is the first $n > n_1$ such that $\Omega_n T_{V_1}(\mu) = \varnothing$ for $\mu = 0$ or 1. In this case we have $\Omega_n R_{V_1}(1 - \lambda) \neq \varnothing$ for all $n_1 \leqq n < n_2$ and $R_n(\lambda) \subset R_{n_1}(\lambda)$ for all $n_1 \leqq n \leqq n_2$. Hence,

$$[D \cup V_1] T_n^{1-\lambda}(q) = \varnothing \qquad \text{for all} \quad 0 \leqq q \leqq 1,$$

$$\max_{\omega \varepsilon V_1} \delta(\omega, \Omega_n) = \max_{\omega \varepsilon R_{V_1}(\lambda)} \delta(\omega, R_n(\lambda)) \leqq k/N,$$

$$\max_{0 \leqq q \leqq 1} \delta(T_n(q), F) \leqq \max_{0 \leqq q \leqq 1, FT^\lambda_{n_1}(q) \neq \varnothing} \delta(z_{n_1}(\lambda, q), FT^\lambda_{n_1}(q))$$
$$\leqq k/N,$$

$$\delta_n \leqq \max\{\max_{\omega \varepsilon V_1} \delta(\omega, \Omega_n), \max_{0 \leqq q \leqq 1} \delta(T_n(q), F)\} \leqq k/N$$
$$\text{for all} \quad n_1 \leqq n < n_2; \qquad \text{and}$$

$$\bigcup_{0 \leqq q \leqq 1} T_n^\lambda(q) \subset \bigcup_{0 \leqq q \leqq 1} T^\lambda_{n_1}(q) \qquad \text{for all} \quad n_1 \leqq n \leqq n_2 .$$

For $n \geqq n_2$, we use the same argument as in (1) to complete the proof. Q.E.D.

We may rewrite the above strategy for player I for the first n^* engagements, n^* is the first n such that $\Omega_n T_{V_1}(\mu) = \varnothing$ for $\mu = 0$ or 1, as follows. For each $n < n^*$, if $\Omega_n R_{V_1}(1) = \varnothing$ or $[D \sim V_1] T_n^1(q) \neq \varnothing$ for some $0 \leqq q \leqq 1$, we let $p_{n+1} = 1$; otherwise, we let $p_{n+1} = 0$. Clearly, this part of the strategy depends only on the triple $B_1 \sim V_1$, $B_2 \sim V_1$, and V_1.

For a given $n < n^*$, the conditions required for player I to play the next $n^* - n$ engagements as in Lemma 5 are as follows:

(i) $\Omega_n T_{V_1}(\mu) \neq \varnothing$ for $\mu = 0$ and 1;

(ii) (a) if $\Omega_n R_{V_1}(\lambda) = \varnothing$ for $\lambda = 0$ or 1, then $\Omega_n R_{V_1}(1 - \lambda) \neq \varnothing$ and $R_n(\lambda) \subset R_{n_1}(\lambda)$, where n_1 is the first n such that $\Omega_n R_{V_1}(\lambda) = \varnothing$;

or

(b) if $\Omega_n \supset V_1$, then $[D \cup V_1][T_n(q) \cup T_n^0(q) \cup T_n^1(q)] \neq \emptyset$ for all $0 \leqq q \leqq 1$.

For each $D = \bigcup_1^m B_i \varepsilon \mathcal{E}_*^m$, $m \geqq 2$, there exists a $D' = \bigcup_1^2 B_i' \varepsilon \mathcal{E}_*^2$ such that the triple $B_1 \sim V_{t,1,m-1}$, $B_m \sim V_{t,1,m-1}$, $V_{t,1,m-1}$ is exactly equal to the triple $B_1' \sim V_1'$, $B_2' \sim V_1'$, V_1' for D'. Therefore, player I can achieve the same result as in Lemma 5 and (i) and (ii) hold for all $n < n^*$ with respect to the triple $B_1 \sim V_{t,1,m-1}$, $B_m \sim V_{t,1,m-1}$, $V_{t,1,m-1}$, where n^* is the first n such that $\Omega_n T_{V_{t,1,m-1}}(\mu) = \emptyset$ for $\mu = 0$ or 1.

LEMMA 6. *If we assume that $M = \bar{M}$, and suppose that for each*

$$D' = \bigcup_1^{m'} B_i' \varepsilon \mathcal{E}_*^{m'},$$

some $2 \leqq m' < m$, there exists a pure strategy for I, *with the first $n^{*\prime}$ choices made depending on the triple $B_1' \sim V'_{t',1,m'-1}$, $B'_{m'} \sim V'_{t',1,m'-1}$, $V'_{t',1,m'-1}$ as in Lemma* 5 *such that*

$$(3.11) \qquad \max_{0 \leqq q \leqq 1} \delta(T_n'(q), D') \leqq k/N \qquad \text{for all} \quad 1 \leqq n \leqq N,$$

where $n^{*\prime}$ is the first n such that $\Omega_n' T_{V'_{t',1,m'-1}}(\mu) = \emptyset$ for $\mu = 0$ or 1. Then for each $D = \bigcup_1^m B_i \varepsilon \mathcal{E}_*^m$ there exists a pure strategy for I, *with the first n^* choices made depending on the triple $B_1 \sim V_{t,1,m-1}$, $B_m \sim V_{t,1,m-1}$, $V_{t,1,m-1}$ as in Lemma* 5, *such that*

$$(3.12) \qquad \delta_n = \max_{0 \leqq q \leqq 1} \delta(T_n(q), D) \leqq k/N \qquad \text{for all} \quad 1 \leqq n \leqq N,$$

where n^* is the first n such that $\Omega_n T_{V_{t,1,m-1}}(\mu) = \emptyset$ for $\mu = 0$ or 1.

PROOF. As indicated in the last paragraph, player I has a pure strategy as in Lemma 5 for the first n^* engagements such that (i) and (ii) hold for all $n < n^*$ with respect to the triple $B_1 \sim V_{t,1,m-1}$, $B_m \sim V_{t,1,m-1}$, $V_{t,1,m-1}$; therefore,

$$\delta_n \leqq k/N \qquad \text{for all} \quad n < n^*.$$

We have one of the following two cases.

(1) $\Omega_{n^*} T_{V_t}(\mu) \neq \emptyset$ for $\mu = 0$ and 1. Since $V_{t,t,t} = V_t$ by definition in (3.8), we can find a smallest integer i with $1 \leqq i \leqq t$ and a largest integer j with $t \leqq j \leqq m - 1$ such that $(j + 1) - i + 1 = m' < m$,

$$\Omega_{n^*} T_{V_{t,i,j}}(\mu) \neq \emptyset \qquad \text{for} \quad \mu = 0 \quad \text{and} \quad 1,$$

and then $$\delta^l_{B_i} \leqq \delta^l_{\Omega_{n^*}}, \qquad \delta^r_{B_{j+1}} \leqq \delta^r_{\Omega_{n^*}}.$$

Player I may pay his attention only to $\bigcup_i^{j+1} B_u$, an $\mathcal{E}^{m'}$-set, for the future engagements because of the following argument. If $\Omega_{n^*} R_{V_{t,i,j}}(\lambda) = \emptyset$ for $\lambda = 0$ or 1, then $\Omega_{n^*} R_{V_{t,i,j}}(1 - \lambda) \neq \emptyset$ and $R_{n^*}(\lambda) \subset R_{n_1}(\lambda)$ by Lemma 5, where n_1 is the first n such that $\Omega_n R_{V_{t,1,m-1}}(\lambda) = \emptyset$ and is also the first n such that $\Omega_n R_{V_{t,i,j}}(\lambda) = \emptyset$.

If $\Omega_{n^*} \supset V_{t,i,j}$ we have either $i = 1$ or $B_i \subset V_{t,1,m-1}$ and either $j = m - 1$ or $B_{j+1} \subset V_{t,1,m-1}$. Since (i) and (ii) hold for $n = n^* - 1$ for the triple

$B_1 \sim V_{t,1,m-1}$, $B_m \sim V_{t,1,m-1}$, $V_{t,1,m-1}$, and p_{n^*} is chosen according to Lemma 5, we have

$$[B_i \cup B_{j+1} \cup V_{t,i,j}][T_{n^*}(q) \cup T^0_{n^*}(q) \cup T^1_{n^*}(q)] \neq \emptyset \qquad \text{for all} \quad 0 \leqq q \leqq 1.$$

The above results are exactly the conditions (i) and (ii) with respect to the triple $B_1' \sim V'_{t',1,m'-1}$, $B'_{m'} \sim V_{t',1,m'-1}$, $V'_{t',1,m'-1}$ for $n = n^* < n^{*\prime}$, where $V'_{t',1,m'-1} = V_{t,i,j}$, $m' = j + 2 - i < m$, and B_1', $B'_{m'} \varepsilon \mathcal{E}^1$ with $l_{B_1'} \varepsilon T(0)$, $B_1'\{\omega{:}\omega^x \geqq l^x_{B_i}\} = B_i$, $r_{B'_{m'}} \varepsilon T(1)$, and $B'_{m'}\{\omega{:}\omega^x \leqq r^x_{B_{j+1}}\} = B_{j+1}$. We have $B_1' \cup [\bigcup_{i+1}^{j} B_u] \cup B'_{m'} = D' \varepsilon \mathcal{E}^{m'}_*$ and $\delta_n \leqq k/N$ for all $n^* \leqq n \leqq N$ by hypothesis.

(2) (a) $\Omega_{n^*}T_{V_t}(1) = \emptyset$: We have $\delta^r_{\Omega_{n^*}} \geqq \delta^r_{B_t}$ and $\bigcup_1^t B_i = [\bigcup_1^{t'} B_u] \cup [\bigcup_{t'+1}^{t} B_u] \varepsilon \mathcal{E}^t$ for some $1 \leqq t' < t$ associated with this $\mathcal{E}^t$-set by definition. Then either there exists a smallest integer i with $1 \leqq i \leqq t'$ and a largest integer j with $t' \leqq j \leqq t - 1$ such that $m' = j + 2 - i \leqq t < m$ and

$$\Omega_{n^*}T_{V'_{t',i,j}}(\mu) \neq \emptyset \qquad \text{for} \quad \mu = 0 \quad \text{and} \quad 1,$$

and the remaining proof follows from (1); or $\Omega_{n^*}T_{V'_{t'}}(\mu) = \emptyset$ for $\mu = 0$ or 1, and we may repeat the argument of (2) for a lower order set. In the case of $t = 1$, there exists a pure strategy for I such that

$$\delta_n \leqq \max_{0 \leqq q \leqq 1} \delta(T_n(q), B_1) \leqq k/N \qquad \text{for all} \quad n^* \leqq n \leqq N$$

as in Lemma 5.

(b) $\Omega_{n^*}T_{V_t}(0) = \emptyset$: the argument is similar to (a). Q.E.D.

PROOF OF THEOREM 2. By Lemma 5, Lemma 6, and induction, if we assume that $M = \bar{M}$, then for each $D \varepsilon \mathcal{E} = \bigcup_{m=1}^{\infty} \mathcal{E}^m_*$ there exists a pure strategy for player I such that

$$\delta_n = \max_{0 \leqq q \leqq 1} \delta(T_n(q), D) \leqq k/N \qquad \text{for all} \quad n \leqq N.$$

By Lemma 4, the proof of the theorem follows as in Theorem 1. Q.E.D.

THEOREM 3. *Let $\mathcal{E}^* = \{D^*\}$ be the collection of sets in 2-space generated by $\mathcal{E}$ such that, for each $D^* \varepsilon \mathcal{E}^*$ there exists a sequence of sets $\{D_n\}$ belonging to $\mathcal{E}$ with*

$$\delta^*(D_n) \to 0 \qquad \text{as} \quad n \to \infty, \tag{3.13}$$

where $\delta^*(D_n) = \max_{\omega \varepsilon D_n - D^*} \delta(\omega, D^*)$. If a set S in 2-space contains a D^*, then S is weakly approachable by player I, *and he has a pure strategy.*

PROOF. For a given $D^* \varepsilon \mathcal{E}^*$ and a $\nu > 0$, let D_{n^*} be the set of $\mathcal{E}$ such that $\delta^*(D_{n^*}) \leqq \nu/2$.

According to Theorem 2, D_{n^*} is weakly approachable by player I and he has a pure strategy; therefore, there is an $N_0(\nu)$ such that, for every $N \geqq N_0(\nu)$ there is a pure strategy for I such that $P\{\delta(\bar{Y}_N, D_{n^*}) > \nu/2\} < \nu/2 < \nu$ for all Q. Since

$$\delta(\bar{Y}_N, D^*) \leqq \delta(\bar{Y}_N, D_{n^*}) + \delta^*(D_{n^*}) \leqq \delta(\bar{Y}_N, D_{n^*}) + \nu/2,$$

we have $$P\{\delta_N = \delta(\bar{Y}_N, D^*) > \nu\} < \nu \qquad \text{for all} \quad Q. \qquad \text{Q.E.D.}$$

4. Necessary condition for the case $\Omega = \Omega^*$.

THEOREM 4. *If S is a set in 2-space such that $SD \neq \emptyset$ for all $D \varepsilon \mathcal{E}$, then player* II *has a pure strategy such that S is weakly approachable by him; that is, S is not weakly excludable by player* I.

PROOF. We begin by showing that this theorem holds if $M = \bar{M}$. For each $1 \leqq n \leqq N, N \geqq 1$, let $q_n = 0$ or 1; and let n^0, n^1 be the last pair of nonnegative integers less than n if any such that $q_{n^0+1} = 1$, $q_{n^1+1} = 0$. For each n^0, n^1, and $D_n \varepsilon \mathcal{E}(\Omega_n) = \alpha_n$, let $D_n(0)$ be any left-extension set of D_n such that $D_n(0)$ is a continuous graph and a subset of some $F(\Omega_{n^0}) \varepsilon \mathfrak{F}(\Omega_{n^0})$ with $l_{D_n(0)} \varepsilon T_{n^0}(0)$ and $r_{D_n(0)} \varepsilon D_n T_n(0)$, and let $D_n(1)$ be any right-extension set of D_n such that $D_n(1)$ is a continuous graph and a subset of some $F(\Omega_{n^1}) \varepsilon \mathfrak{F}(\Omega_{n^1})$ with $l_{D_n(1)} \varepsilon D_n T_n(1)$ and $r_{D_n(1)} \varepsilon T_{n^1}(1)$.

LEMMA 7. *Given D_n', $D_n'' \varepsilon \alpha_n$, if player* II *uses a pure strategy, then*

$$D_n' \cup D_n'(q_n) \cup D_n'' \cup D_n''(1 - q_n) = D_{n-1} \cup D_{n-1}(q_n) \tag{4.1}$$

for some $D_{n-1} \varepsilon \alpha_{n-1}$.

PROOF OF LEMMA 7. $D_n', D_n'' \varepsilon \alpha_n$ imply that $D_n' \cup D_n'' \varepsilon \alpha_n$ and $n^{1-q_n} = n - 1$; therefore,

$$D_n'' \cup D_n''(1 - q_n) \varepsilon \alpha_{n-1}, \qquad D_n' \cup D_n'' \cup D_n''(1 - q_n) = D_{n-1} \varepsilon \alpha_{n-1}, \tag{4.2}$$

and $D_n'(q_n)$ is a left- (for $q_n = 0$) or a right- (for $q_n = 1$) extension set of this D_{n-1} by definition. Q.E.D.

LEMMA 8. *For each $M = \bar{M}$ and a set S in 2-space with $SD \neq \emptyset$ for all $D \varepsilon \mathcal{E}$, player* II *has a pure strategy with the following property for all $1 \leqq n \leqq N$: If there exists a $D_n^*(\mu)$ for some $D_n^* \varepsilon \alpha_n$ such that $S[D_n^* \cup D_n^*(\mu)] = \emptyset$, then $S[D_n \cup D_n(1 - \mu)] \neq \emptyset$ for all $D_n \varepsilon \alpha_n$ and $D_n(1 - \mu)$, where $\mu = 0$ or 1.*

PROOF OF LEMMA 8. If $SD_m \neq \emptyset$ for all $D_m \varepsilon \alpha_m$, $1 \leqq m \leqq N$, we may consider an $(N - m)$-engagement game associated with M_m and carry on the proof for $m \leqq n \leqq N$.

Thus we may assume that there exists at least one $D_n \varepsilon \alpha_n$ such that $SD_n = \emptyset$ for all $1 \leqq n < N$. Let

$$q_1 = 0 \qquad \text{and} \qquad q_2 = 1.$$

If $S[D_2 \cup D_2(1)] \neq \emptyset$ for all $D_2 \varepsilon \alpha_2$ and $D_2(1)$, we let $q_3 = 1$; otherwise, we let $q_3 = 0$. In the latter case, we have $S[D_2^* \cup D_2^*(1)] = \emptyset$ for some $D_2^* \varepsilon \alpha_2$ and some $D_2^*(1)$ for D_2^*. Since for any pair of $D_2 \varepsilon \alpha_2$ and $D_2(0)$ for D_2, $D_2^* \cup D_2^*(1) \cup D_2 \cup D_2(0) = D_1 \cup D_1(1) = D$ for some $D_1 \varepsilon \alpha_1$ and $D \varepsilon \mathcal{E}$ by Lemma 7, and $SD \neq \emptyset$ by assumption, we must have $S[D_2 \cup D_2(0)] \neq \emptyset$ for all $D_2 \varepsilon \alpha_2$ and $D_2(0)$.

The following properties are used in reference to player II's strategy for $n < N$;

(i) there exists a $D_n^*(0)$ for some $D_n^* \varepsilon \alpha_n$ such that $S[D_n^* \cup D_n^*(0)] = \emptyset$;

(ii) there exists a $D_n^*(1)$ for some $D_n^* \varepsilon \alpha_n$ such that $S[D_n^* \cup D_n^*(1)] = \emptyset$;

(iii) player II chooses $q_{n+1} = 0$ (or 1) only if (i) (or (ii)) does not hold;

(iv) property (iii) occurs and (i) (or (ii)) implies that (ii) (or (i)) does not hold.

If we assume that property (iv) holds for $n = i - 1$, the $(i + 1)$th choice of player II is one of the following:

(1) If $q_i = 0$ and (i) does not hold for $n = i$, then player II chooses $q_{i+1} = 0$.

(2) If $q_i = 0$ and (i) holds for $n = i$, then player II chooses $q_{i+1} = 1$. (ii) does not hold for $n = i$ by Lemma 7 and the assumption of $q_i = 0$.

(3) If $q_i = 1$ and (ii) does not hold for $n = i$, then player II chooses $q_{i+1} = 1$.

(4) If $q_i = 1$ and (ii) holds for $n = i$, then player II chooses $q_{i+1} = 0$. (i) does not hold for $n = i$ by Lemma 7 and the assumption of $q_i = 1$.

The above results show that (iv) holds for $n = i$.

Since (iv) holds for $n = 2$, the induction is complete and the lemma follows. Q.E.D.

LEMMA 9. *Theorem* 4 *holds if* $M = \bar{M}$.

PROOF OF LEMMA 9. Since $D_N = \bar{\omega}_N = \Omega_N$ by Corollary 1 and $SD_N(0) \neq \varnothing$ or $SD_N(1) \neq \varnothing$ by Lemma 8, we have

$$\delta_N \leqq \delta(D_N, S[D_N(0) \cup D_N(1)]) \leqq k/N.$$

The lemma follows with $N_0(\nu) \geqq k/\nu$ for each $\nu > 0$. Q.E.D.

Theorem 4 follows from Lemmas 4 and 9 as in Theorem 1. Q.E.D.

LEMMA 10. *If* $\Omega = \Omega^*$ *with empty interior, each set* S *in* 2*-space is either weakly approachable by one player or weakly excludable by the other player.*

PROOF. The condition implies that a, b, c, and d are colinear. By Blackwell's Theorem 4 [1], if $M = \bar{M}$, then S is either approachable by one player or excludable by the other player.

Approachability implies weak approachability and excludability implies weak excludability by their definitions [1]. Q.E.D.

REMARK. Lemma 10 can be proved by a similar argument as in Theorem 1.

THEOREM 5. *A set* S *in* 2*-space is weakly approachable by player* I *if and only if* S *contains a set of* $\mathcal{E}^*$.

PROOF. Suppose S does not contain any $D^* \varepsilon \mathcal{E}^*$, that is, $\delta^*(D) = \max_{\omega \varepsilon D \sim S} \delta(\omega, S) \geqq \Delta$ for all $D \varepsilon \mathcal{E}$ for some $\Delta > 0$. If we let

$$S' = \{\omega_D : \omega_D \varepsilon D \sim S \quad \text{and} \quad \delta(\omega_D, S) = \delta^*(D) \quad \text{for each} \quad D \varepsilon \mathcal{E}\}$$

then $\delta(S, S') \geqq \Delta$ and S' is weakly approachable by player II according to Theorem 4. Therefore, S is not weakly approachable by player I. The proof is completed by using Theorem 3. Q.E.D.

5. $\Omega \neq \Omega^*$ **and** $a = d$ **or** $b = c$. For the case $\Omega \neq \Omega^*$, there exist $0 \leqq p, p', q, q' \leqq 1$ such that $z(p, q) = z(p', q')$ and $p \neq p'$ or $q \neq q'$; and there exist $0 \leqq p, p', p'', q, q' \leqq 1$ such that $0 \leqq p < p'' < p' \leqq 1$ and $z(p, q)$ and $z(p', q')$, are in the same half plane of $R^*(p'')$, a line of X and a superset of $R(p'')$; where $z(p, q)$ was defined in (2.1). Therefore, the technique in the previous sections could not be applied directly.

In this section we may assume that $a = d = (0, 0)$, $b^x < 0$, $c^x > 0$, $c^y/c^x = -b^y/b^x = \rho$, and $0 < \rho < \infty$; hence

$$(5.1) \qquad z(p, q) = z(1 - q, 1 - p) \qquad \text{for all} \quad 0 \leqq p, \quad q \leqq 1.$$

Let Λ be a continuous graph and a subset of Ω such that

$$\Lambda = \{z(\lambda(\mu), \mu): 0 \leqq \mu \leqq 1\}, \tag{5.2}$$

where

$$\lambda(\mu) = 1 - \mu. \tag{5.3}$$

We introduce $\lambda(\mu)$ here that most definitions and results in this section can be carried over to the next two sections. Let $\mathfrak{F} = \{F\}$ denote the collection of all F, subsets of Ω, such that each F is the graph of some continuous function f which satisfies that

(i) there are points $v_f = (v_f^x, f(v_f^x)) \varepsilon T(0)$ and $u_f = (u_f^x, f(u_f^x)) \varepsilon T(1)$,

(ii) f is defined and $(x, f(x)) \varepsilon \Omega$ for all $v_f^x \leqq x \leqq u_f^x$,

(iii) there is a point $t_f = (t_f^x, f(t_f^x)) \varepsilon \Lambda$ with $v_f^x \leqq t_f^x \leqq u_f^x$,

(iv) $-\rho \leqq (f(x_2) - f(x_1))/(x_2 - x_1) \leqq \rho$ for all $v_f^x \leqq x_1 < x_2 \leqq u_f^x$,

(v) $F = \{(x, f(x)): v_f^x \leqq x \leqq u_f^x\}$.

Then we have the following theorem for weak approachability.

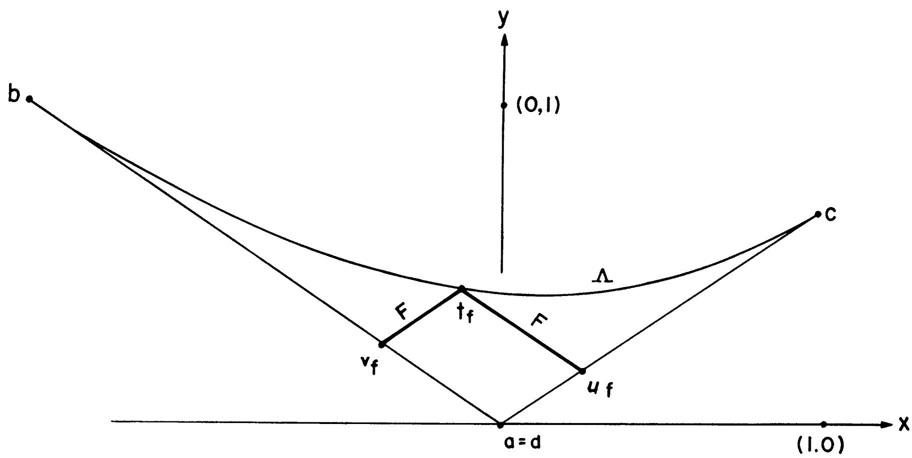

FIG. 2

THEOREM 6. *If a set S in 2-space contains an F of $\mathfrak{F}$, then S is weakly approachable by player* I.

PROOF. We may assume that $M = \bar{M}$ by Lemma 4. For each $0 \leqq n \leqq N$, $N \geqq 1$, let

$$\Lambda_n = \{z_n(\lambda(\mu), \mu): 0 \leqq \mu \leqq 1\}, \tag{5.4}$$

where $z_n(p, q)$ was defined in (2.6). For each $0 \leqq n < N$, let $A_n(\mu)$, $0 \leqq \mu \leqq 1$, be the ray of X with vertex $z_n(\lambda(\mu), \mu)$ such that $A_n(\mu)$ is perpendicular to the segment $R_n(\lambda(\mu))$ at $z_n(\lambda(\mu), \mu)$ from below (wrt the y-coordinate), hence

$$(5.5)\quad A_n(\mu) = \{\omega : \omega \,\varepsilon\, X, \omega^x = z_n^x(\lambda(\mu), \mu) - \rho^\mu(\omega^y - z_n^y(\lambda(\mu), \mu)), \text{ and } \omega^y \leqq z_n^y(\lambda(\mu), \mu)\},$$

where

$$(5.6)\quad \rho^\mu = [z^y(\lambda(\mu), 1) - z^y(\lambda(\mu), 0)]/[z^x(\lambda(\mu), 1) - z^x(\lambda(\mu), 0)]$$

is the slope of $R(\lambda(\mu))$ and $z_n(p, q) = (z_n^x(p, q), z_n^y(p, q))$; and let A_n be the subset of X such that it contains every point of and below the graph $A_n(0) \cup \Lambda_n \cup A_n(1)$, hence

$$(5.7)\quad \Omega_n \subset A_n .$$

LEMMA 11. *For a point $\omega_0 \,\varepsilon\, A_n$, $0 \leqq n < N$, let $z_n(\lambda(\mu_0), \mu_0)$ be the closest point in Λ_n to ω_0, then the segment $\overline{\omega_0 z_n(\lambda(\mu_0),\mu_0)}$ is perpendicular to the segment $R_n(\lambda(\mu_0))$ at $z_n(\lambda(\mu_0), \mu_0)$, that is $\omega_0 \,\varepsilon\, A_n(\mu_0)$, and then*

$$(5.8)\quad A_n = \textstyle\bigcup_{0 \leqq \mu \leqq 1} A_n(\mu).$$

PROOF OF LEMMA 11. $\Lambda = \Lambda_0$ is the graph of a convex function f_Λ defined for $z^x(\lambda(0), 0) = b^x \leqq x \leqq c^x = z^x(\lambda(1), 1)$ which satisfies

$$(5.9)\quad f_\Lambda(z^x(\lambda(\mu), \mu)) = z^y(\lambda(\mu), \mu) = -\rho(1 - \mu)^2 b^x + \rho\mu^2 c^x$$

and has derivative

$$(5.10)\quad \rho^\mu = [z^y(\lambda(\mu), 1) - z^y(\lambda(\mu), 0)]/[z^x(\lambda(\mu), 1) - z^x(\lambda(\mu), 0)]$$
$$= \rho[\mu c^x + (1 - \mu)b^x]/[\mu c^x - (1 - \mu)b^x]$$

at $x = z^x(\lambda(\mu), \mu) = (1 - \mu)^2 b^x + \mu^2 c^x$ for $0 \leqq \mu \leqq 1$.

(5.10) is a continuous increasing function of μ for $0 \leqq \mu \leqq 1$ (and of x for $b^x \leqq x \leqq c^x$) with $-\rho \leqq \rho^\mu \leqq \rho$ and is equal to the slope of $R(\lambda(\mu)) = R(1 - \mu)$. Hence, $R(\lambda(\mu))$ is a segment of the tangent to Λ at $z(\lambda(\mu), \mu)$ and $z(\lambda(\mu), \mu)$ is the closest point in Λ to every point of $A_0(\mu)$ by the convexity of Λ. The continuity of the derivative of f_Λ in (5.10) implies that $\bigcup_{0 \leqq \mu \leqq 1} A_0(\mu) = A_0$.

For $1 \leqq n < N$, the argument is similar (rescaling). Q.E.D.

LEMMA 12. *For a given $\bar{\omega}_n = \sum_1^n z(p_i, q_i)/n \,\varepsilon\, A_0$, $1 \leqq n < N$, let $z(\lambda(\mu_n), \mu_n)$ be the closest point in Λ to $\bar{\omega}_n$ and $p_{n+1} = \lambda(\mu_n)$, then*

$$(5.11)\quad \delta(\bar{\omega}_{n+1}, \Lambda) \leqq (n^2\delta^2(\bar{\omega}_n, \Lambda) + k^2)^{\frac{1}{2}}/(n + 1).$$

For a given $\omega \,\varepsilon\, A_n$, $0 \leqq n < N$, let $z_n(\lambda(\mu_n), \mu_n)$ be the closest point in Λ_n to ω and $p_{n+1} = \lambda(\mu_n)$, then

$$(5.12)\quad \delta(\omega, \Lambda_{n+1}) \leqq (\delta^2(\omega, \Lambda_n) + k^2/N^2)^{\frac{1}{2}}.$$

PROOF OF LEMMA 12. Since

$$\begin{aligned}\delta(\bar{\omega}_{n+1}, z(\lambda(\mu_n), \mu_n)) &= [n^2\delta^2(\bar{\omega}_n, z(\lambda(\mu_n), \mu_n))/(n+1)^2 + \delta^2(\omega_{n+1}, z(\lambda(\mu_n), \mu_n))/(n+1)^2]^{\frac{1}{2}} \\ &\leqq (n^2\delta^2(\bar{\omega}_n, \Lambda) + k^2)^{\frac{1}{2}}/(n+1)\end{aligned}$$

and

$$\begin{aligned}\delta(\omega, z_{n+1}(\lambda(\mu_n), \mu_n)) &= [\delta^2(\omega, z_n(\lambda(\mu_n), \mu_n)) + \delta^2(\omega_{n+1}, z(\lambda(\mu_n), \mu_n))/N^2]^{\frac{1}{2}} \\ &\leqq (\delta^2(\omega, \Lambda_n) + k^2/N^2)^{\frac{1}{2}}\end{aligned}$$

by Lemma 11 and Corollary 2, we have (5.11) and (5.12). Q.E.D.

LEMMA 13. *The set Λ is weakly approachable by player I.*

PROOF OF LEMMA 13. Since $\omega_1 \varepsilon A_0$, $\delta(\omega_1, \Lambda) \leqq k$, and $R(\lambda(\mu)) \subset A_0$ for all $0 \leqq \mu \leqq 1$; we have $\bar{\omega}_{n+1} \varepsilon A_0$, $1 \leqq n < N$, if $\bar{\omega}_n \varepsilon A_0$ and $p_{n+1} = \lambda(\mu_n)$ as in Lemma 12. Hence

$$(5.13) \qquad \delta(\bar{\omega}_n, \Lambda) \leqq k/n^{\frac{1}{2}} \qquad \text{for all} \quad 1 \leqq n \leqq N$$

by Lemma 12 and induction. Let $N_0(\nu) \geqq k^2/\nu^2$ for every $\nu > 0$, then the lemma follows.

LEMMA 14. *For a given $\bar{\omega}_{m,n} = \sum_{m+1}^{n} z_m(p_i, q_i)/(n-m) \varepsilon A_m$ and a point $z_n(\lambda(\mu), \mu) \varepsilon A_m\Lambda_n$, some $0 \leqq m < n \leqq N$, let $z_m(\lambda(\mu_n), \mu_n)$ be the closest point in Λ_m to $\bar{\omega}_{m,n}$ and let $e^{\mu}_{m,n}$ be the point in Λ_m such that the segment $\overline{z_n(\lambda(\mu), \mu)e^{\mu}_{m,n}}$ is parallel to the segment $\overline{z_n(\lambda(\mu_n), \mu_n)z_m(\lambda(\mu_n), \mu_n)}$ and $e^{\mu}_{m,n}$ is closer to $z_n(\lambda(\mu), \mu)$ than other point in Λ_m with same property, then*

$$(5.14) \qquad \delta(z_n(\lambda(\mu), \mu), e^{\mu}_{m,n}) \leqq \delta(z_n(\lambda(\mu_n), \mu_n), z_m(\lambda(\mu_n), \mu_n)).$$

If $p_{i+1} = \lambda(\mu_i)$ for $m < i < n$, then

$$(5.15) \qquad \delta(z_n(\lambda(\mu), \mu), e^{\mu}_{m,n}) \leqq k(n-m)^{\frac{1}{2}}/N.$$

PROOF OF LEMMA 14. For each $0 \leqq \mu \leqq 1$, $z_n(\lambda(\mu), \mu)$ is a point of the segment $\overline{\bar{\omega}_{m,n}z_m(\lambda(\mu), \mu)}$,

$$(5.16) \quad \delta(z_n(\lambda(\mu), \mu), z_m(\lambda(\mu), \mu)) = (n-m)\delta(\bar{\omega}_{m,n}, z_m(\lambda(\mu), \mu))/(N-m),$$

and the segment $\overline{z_n(\lambda(\mu_n), \mu_n)z_n(\lambda(\mu), \mu)}$ is parallel to the segment $\overline{z_m(\lambda(\mu_n), \mu_n)z_m(\lambda(\mu), \mu)}$. If we let s^{μ} be the point of $\overline{z_m(\lambda(\mu_n), \mu_n)z_m(\lambda(\mu), \mu)}$ such that the segment $\overline{z_n(\lambda(\mu), \mu)s^{\mu}}$ is parallel to $\overline{z_n(\lambda(\mu_n), \mu_n)z_n(\lambda(\mu_n), \mu_n)}$, then

$$\delta(z_n(\lambda(\mu), \mu), s^{\mu}) = \delta(z_n(\lambda(\mu_n), \mu_n), z_n(\lambda(\mu_n), \mu_n))$$

by the property of a parallelogram.

$\bar{\omega}_{m,n} \varepsilon A_m$, $z_n(\lambda(\mu), \mu) \varepsilon A_m\Lambda_n$, and the convexity of Λ_m imply that $e^{\mu}_{m,n}$ exists and is a point of $\overline{z_n(\lambda(\mu), \mu)s^{\mu}}$. Hence, (5.14) follows.

Since the diameter of Ω_m is $k(N-m)/N$ by Corollary 2 and $p_{i+1} = \lambda(\mu_i)$ for

$m < i < n$, we have

(5.17) $\delta(\bar{\omega}_{m,i}, \Lambda_m) \leqq k(N - m)/[N(i - m)^{\frac{1}{2}}]$ for $m < i \leqq n$

by a similar argument as in Lemma 13 and we have (5.15) by (5.14) and (5.16).
Q.E.D.

COROLLARY 3. *For each* $0 \leqq m < n \leqq N$ *and* $0 \leqq \mu \leqq 1$, *let*

$$C_{m,n}(\mu) = \{\alpha z_n(\lambda(\mu), \mu) + (1 - \alpha)e^{\mu}_{m,n}: 0 \leqq \alpha < 1\} \quad \text{if}$$

(5.18) $$\bar{\omega}_{m,n},\ z_n(\lambda(\mu), \mu) \,\varepsilon\, A_m \quad \text{and} \quad e^{\mu}_{m,n} \neq z_n(\lambda(\mu), \mu)$$

$$= \varnothing \qquad \text{otherwise}$$

and

(5.19) $$C_{m,n} = \bigcup_{0 \leqq \mu \leqq 1} C_{m,n}(\mu)$$

then

(5.20) $$\max_{\omega \varepsilon C_{m,n}} \delta(\omega, \Lambda_n) \leqq \delta(z_n(\lambda(\mu_n), \mu_n), z_m(\lambda(\mu_n), \mu_n))$$

by Lemma 14 *if* $C_{m,n} \neq \varnothing$ *and* $z_m(\lambda(\mu_n), \mu_n)$ *is the closest point in* Λ_m *to* $\bar{\omega}_{m,n}$.

For a given $F \,\varepsilon\, \mathfrak{F}$, we describe a weak approachable strategy for player I for each N as follows:

(1) Let $p_1 = \lambda(\mu_0)$, where $z(\lambda(\mu_0), \mu_0) \,\varepsilon\, F\Lambda$, then $\delta(F, \Lambda_1) \leqq k/N$.

(2) $FC_{0,m} = \varnothing$, $FA_m \neq \varnothing$, and $\delta(FA_m, \Lambda_m) \leqq km^{\frac{1}{2}}/N$, $0 \leqq m < N$. (We let $C_{0,0} = \varnothing$; hence, the hypothesis is true for $m = 0$.)

Let $z_m(\lambda(\mu_m), \mu_m)$ be the closest point in Λ_m to FA_m and $p_{m+1} = \lambda(\mu_m)$, then $\delta(F, \Lambda_{m+1}) \leqq k(m + 1)^{\frac{1}{2}}/N$ by Lemma 12. For the decision of p_n, $m + 1 < n \leqq m' + 1$ and some $m' < N$, we adopt one of the following three procedures in (3) associated with three possible conditions.

(3a) $FC_{m,n} \neq \varnothing$ for $m < n \leqq m'$. We have $\bar{\omega}_{m,n} \,\varepsilon\, A_m$ by definition. Let $z_m(\lambda(\mu_n), \mu_n)$ be the closest point in Λ_m to $\bar{\omega}_{m,n}$ and $p_{n+1} = \lambda(\mu_n)$ for $m < n \leqq m'$, then

(5.21) $$\delta(F, \Lambda_n) \leqq \max_{\omega \varepsilon FC_{m,n}} \delta(\omega, \Lambda_n) \leqq kn^{\frac{1}{2}}/N$$

by Corollary 3. If $FC_{m,m'+1} \neq \varnothing$, then (5.21) holds for $n = m' + 1$; and if $FC_{m,m'+1} = \varnothing$, then $FC_{0,m'+1} = \varnothing$,

$$\delta(F, \Lambda_{m'+1}) \leqq \max_{\omega \varepsilon \Lambda_{m'}} \delta(\omega, \Lambda_{m'+1}) \leqq k/N, \qquad \text{and}$$

$$\delta(F, \Lambda_{m'+1}) = \delta(FA_{m'+1}, \Lambda_{m'+1}) \qquad \text{if} \quad FA_{m'+1} \neq \varnothing$$

by the hypothesis of $FC_{0,m} = \varnothing$, the definition of F, and Corollary 2.

(3b) $FC_{n-1,n} = \varnothing$ and $FA_n \neq \varnothing$ for $m < n \leqq m'$. $FC_{0,n-1} = \varnothing$ and $FC_{n-1,n} = \varnothing$ imply $FC_{0,n} = \varnothing$; hence,

(5.22) $$FC_{0,n} = \varnothing \qquad \text{for all} \quad m < n \leqq m'.$$

Let $z_n(\lambda(\mu_n), \mu_n)$ be the closest point in Λ_n to FA_n and $p_{n+1} = \lambda(\mu_n)$ for

$m < n \leqq m'$, then $\delta(FA_n, \Lambda_n) \leqq \delta(s_{n-1}, \Lambda_n)$ by the definition of F and $FA_n \neq \varnothing$, and $\delta(F, \Lambda_n) \leqq \delta(FA_n, \Lambda_n) \leqq kn^{\frac{1}{2}}/N$ by Lemma 12, where s_{n-1} is the closest point in FA_{n-1} to Λ_{n-1}.

(3c) $FC_{m,m+1} = \varnothing$ and $FA_{m+1} = \varnothing$. We have $FC_{0,m+1} = \varnothing$ as in (2b). (The condition $FC_{m,m+1} = \varnothing$ can be replaced by $FC_{0,m+1} = \varnothing$.) This condition may hold only if $0 < \theta < \pi/2$, $(\pi - \theta)$ is the counterclockwise angle from $T(1)$ to $R(1)$, (i.e., $0 < \rho < 1$ in this section) and one of the following two conditions holds.

(3c-1) $u_f \varepsilon FT(1)$, the right vertex of F, is a point above the ray $A_{m+1}(0)$ and below the line

$$(5.23) \quad R^*_{m+1}(\lambda(0)) = \{\omega : \omega \varepsilon X, \omega^y = \rho^0[\omega^x - z^x_{m+1}(\lambda(0), 0)] + z^y_{m+1}(\lambda(0), 0)\}$$

which is a superset of $R_{m+1}(\lambda(0))$; that is

$$(5.24) \quad (u_f^x - z^x_{m+1}(1, 0))/\rho + z^y_{m+1}(1, 0) < u_f^y < \rho(z^x_{m+1}(1, 0) - u_f^x) + z^y_{m+1}(1, 0).$$

(3c-2) v_f, the left vertex of F, is a point above the ray $A_{m+1}(1)$ and below the line

$$(5.25) \quad R^*_{m+1}(\lambda(1)) = \{\omega : \omega \varepsilon X, \omega^y = \rho^1[\omega^x - z^x_{m+1}(\lambda(1), 1)] + z^y_{m+1}(\lambda(1), 1)\}.$$

Suppose that (3c-1) holds, then

$$(5.26) \quad \delta(u_f, R^*_{m+1}(\lambda(0))) \leqq \delta(F, \Lambda_{m+1}) \leqq k(m+1)^{\frac{1}{2}}/N.$$

Let $p_n = \lambda(0) = 1$ for $m + 1 < n \leqq m' + 1 = N$, then

$$(5.27) \quad \delta(F, \Lambda_n) \leqq \delta(u_f, R^*_{m+1}(\lambda(0)))/(\sin\theta) \leqq kn^{\frac{1}{2}}/(N \sin\theta) \qquad \text{for all} \quad m + 1 < n \leqq N.$$

If (3c-2) holds and we let $p_n = \lambda(1)$ for $m + 1 < n \leqq N$, then the result of (5.27) holds by the same argument.

(4a) (3a) holds and $m' + 1 < N$. We choose $p_{m'+2}$ according to (3a) if $FC_{m,m'+1} \neq \varnothing$ (replacing m' by $m' + 1$); according to (2) if $FC_{m,m'+1} = \varnothing$ and $FA_{m'+1} \neq \varnothing$, and according to (3c) if $FC_{m,m'+1} = \varnothing$ and $FA_{m'+1} = \varnothing$.

(4b) (3b) holds. Then $p_{m'+1}$ was selected according to (2).

(5) By induction, we have

$$(5.28) \quad \delta(F, \Lambda_n) \leqq kn^{\frac{1}{2}}/N \qquad \text{for all} \quad 0 \leqq n \leqq N$$

if (3c) never holds and $\delta(F, \Lambda_n) \leqq kn^{\frac{1}{2}}/(N \sin\theta)$ for all $0 \leqq n \leqq N$ otherwise.

Let $N_0(\nu) \geqq k^2/(\nu \sin\theta)^2$ if $0 < \rho < 1$ and $N_0(\nu) \geqq k^2/\nu^2$ otherwise, for every $\nu > 0$, then the proof of Theorem 6 is completed. Q.E.D.

EXAMPLE 3. Let

$$M = \begin{Vmatrix} (0, 0)(-\frac{3}{2}, 1) \\ (1, \frac{2}{3})(0, 0) \end{Vmatrix} \quad \text{and} \quad f(x) = \begin{cases} 2x/3 + \frac{1}{2} & \text{for} \quad -\frac{3}{8} \leqq x \leqq -\frac{1}{8} \\ (1 - 2x)/3 & \text{for} \quad -\frac{1}{8} \leqq x \leqq \frac{1}{4} \end{cases}$$

be exhibited in Figure 2, then $\Lambda = \{(x, 20 - 10x/3 - 8(6-2x)^{\frac{1}{2}}): -\frac{3}{2} \leqq x \leqq 1\}$ and $F = \{(x, f(x)): -\frac{3}{8} \leqq x \leqq \frac{1}{4}\}$ is weakly approachable by player I as follows: For each $N \geqq 1$, let $z_n(1-\mu_n, \mu_n)$, $0 \leqq n \leqq m < N$, be the closest point in Λ_n to $(-\frac{1}{8}, \frac{5}{12})$ (the closest point in F to Λ_n) and $p_{n+1} = 1 - \mu_n (p_1 = \frac{1}{2})$, where $(m+1)$ is the first $i \geqq 1$ such that

$$\tfrac{5}{12} > \min\{z_i^y(1, 0) - 3(z_i^x(1, 0) + \tfrac{1}{8})/2, z_i^y(0, 1) + 3(z_i^x(0, 1) + \tfrac{1}{8})/2\};$$

and let $p_n = 1$ if $\frac{5}{12} > z_{m+1}^y(1, 0) - 3(z_{m+1}^x(1, 0) + \frac{1}{8})/2$ and $p_n = 0$ if $\frac{5}{12} > z_{m+1}^y(0, 1) + 3(z_{m+1}^x(0, 1) + \frac{1}{8})/2$ for $m + 1 < n \leqq N$.

COROLLARY 4. *The set S in Theorem* 6 *is also weakly approachable by player* II.

PROOF. Since

$$z_m(p, q) = z_m(1-q, 1-p); \tag{5.29}$$

we have this corollary if we let $q_{n+1} = 1 - \lambda(\mu_n) = \mu_n$ for $0 \leqq n < N$. Q.E.D.

6. $\Omega \neq \Omega^*$ and $T(0)T(1) \neq \varnothing$ or $R(0)R(1) \neq \varnothing$. We may assume that $R(0)R(1) = (0, 0)$, $a^x, c^x > 0$, $b^x < 0$, $a^x/(a^x - b^x) \leqq c^x/(c^x - d^x)$, $(b^y - a^y)/(a^x - b^x) = (c^y - d^y)/(c^x - d^x) = \rho$, and $0 < \rho < \infty$; hence

$$(0, 0) = T(\mu')T(\mu'') \quad \text{for some} \quad 0 \leqq \mu' \leqq \mu'' < 1, \tag{6.1}$$

$$T(\mu') \subset R(1), \quad \text{and} \quad T(\mu'') \subset R(0). \tag{6.2}$$

For $\mu' < \mu''$, let

$$\begin{aligned} \lambda(\mu) &= 1 && \text{for } 0 \leqq \mu \leqq \mu' \\ &= (\mu - \mu'')/(\mu' - \mu'') && \text{for } \mu' \leqq \mu \leqq \mu'' \\ &= 0 && \text{for } \mu'' \leqq \mu \leqq 1 \end{aligned} \tag{6.3}$$

and

$$\Lambda^* = \{z(\lambda(\mu), \mu): \mu' \leqq \mu \leqq \mu''\}, \tag{6.4}$$

and define $\mathfrak{F} = \{F\}$ as in Section 5 except the replacement of Λ by Λ^* in (iii), then we have a similar theorem.

THEOREM 7. *If a set S in 2-space contains an F* of $\mathfrak{F}$, *then S is weakly approachable by player* I.

PROOF. We may assume that $M = \bar{M}$ by Lemma 4. For each $N \geqq 1$, let Λ_n, $A_n(\mu)$ for $0 \leqq \mu \leqq 1$, A_n, and $C_{m,n}$ be defined by substituting (6.3) to (5.4), (5.5), (5.8), and (5.19). An example of M and F is shown in Figure 3.

LEMMA 15. *In case of $\Omega \not\subset A$, let A_n^0, $0 \leqq n < N$, be the subset of X such that it contains every point of and below the graph $A_n'(0) \cup \Lambda_n \cup A_n(1)$, where $A_n'(0)$ is a superset of $T_n(0)$ and a ray of X with vertex $b_n = z_n(1, 0)$. If A_m, $0 \leqq m < N$, is replaced by A_m^0 in Lemma* 14 *and Corollary* 3, *then the results still hold.*

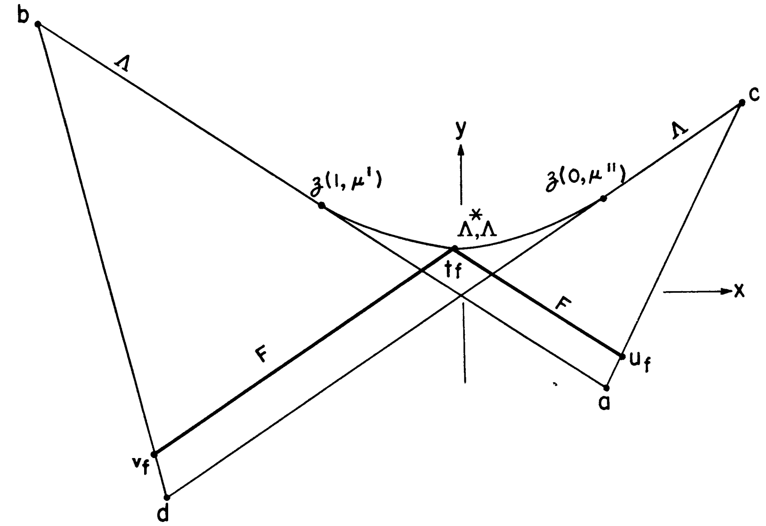

FIG. 3

PROOF OF LEMMA 15. Since our assumptions of this section imply that $\Omega_n \subset A_n$, $0 \leqq n < N$, if and only if $T(0) \subset A$, we have

$$A_n^0 \supset A_n \cup \Omega_n . \tag{6.5}$$

The proof of (5.14) is still valid. For (5.15) and (5.20), we need to prove (5.17) for the additional case that $\bar{\omega}_{m,i-1} \,\varepsilon\, [A_m^0 \sim A_m]$ and $\delta(\bar{\omega}_{m,i-1}, \Lambda_m) \leqq k(N-m)/[N(i-m-1)^{\frac{1}{2}}]$, $m + 1 < i \leqq N$. In this additional case, $z_m(\lambda(0), 0) = z_m(1, 0) = b_m$ is the closest point in Λ_m to $\bar{\omega}_{m,i-1}$ and the counterclockwise angle from the segment $\bar{\omega}_{m,i-1}b_m$ to $R_m(1)$ is greater than $\pi/2$. If we let $p_i = \lambda(\mu_{i-1}) = \lambda(0) = 1$, then

$$\begin{aligned}\delta(\bar{\omega}_{m,i}, \Lambda_m) &< [(i-m-1)^2\delta^2(\bar{\omega}_{m,i-1}, \Lambda_m) + (N-m)^2k^2/N^2]^{\frac{1}{2}}/(i-m) \\ &\leqq k(N-m)/[N(i-m)^{\frac{1}{2}}].\end{aligned} \tag{6.6}$$

Q.E.D.

From the definition of F, we have the following facts for $0 \leqq n < N$: If there exists a point $s \,\varepsilon\, F\{z_n(\lambda(\mu), \mu): 0 \leqq \mu < \mu''\}$, then

$$R_n(0)\{(x, f(x)): v_f^x \leqq x \leqq s^x\} = \varnothing;$$

if there exists points $s_1 \,\varepsilon\, F\Lambda_n$, $s_2 \,\varepsilon\, FR_n(0)$, and $s_1^x \leqq s_2^x$, then there exists a point $s_3 \,\varepsilon\, FT_n(1)$ such that $s_1^x \leqq s_2^x \leqq s_3^x$; if there exists a point $s \,\varepsilon\, F\{z_n(\lambda(\mu), \mu): \mu' < \mu \leqq 1\}$, then $R_n(1)\{(x, f(x)): s^x \leqq x \leqq u_f^x\} = \varnothing$; and if there exist points $s_1 \,\varepsilon\, F\Lambda_n$, $s_2 \,\varepsilon\, FR_n(1)$, and $s_2^x \leqq s_1^x$, then there exists a point $s_3 \,\varepsilon\, FT_n(0)$ such that $s_1^x \geqq s_2^x \geqq s_3^x$.

Based on the above analysis, Lemma 12, and Lemma 15 ($\Omega \not\subset A$) or Corollary

3 ($\Omega \subset A$), we have the proof of this theorem as in Theorem 6 (θ is replaced by θ_1 in (3c − 1) and by θ_2 in (3c − 2), where $\pi - \theta_1$ and $\pi - \theta_2$ are the counterclockwise angles from $T(1)$ to $R(1)$ and from $R(0)$ to $T(0)$ respectively). Q.E.D.

REMARK. The sets $\bigcup_{0 \leqq q \leqq \mu'} T(q)$ and $\bigcup_{\mu'' \leqq q \leqq 1} T(q)$ are convex sets; therefore, we may construct a collection of weakly approachable sets in $\bigcup_{0 \leqq q \leqq \mu'} T(q)$ for player II ($q_n = 0$ or μ') and a collection of weakly approachable sets in $\bigcup_{\mu'' \leqq q \leqq 1} T(q)$ for player II ($q_n = \mu''$ or 1) in a similar manner as in Theorems 1 and 2. The set $\bigcup_{\mu' \leqq q \leqq \mu''} T(q)$ has the same shape as Ω in Section 5; hence, we may construct a collection of weakly approachable sets in $\bigcup_{\mu' \leqq q \leqq \mu''} T(q)$ for player II ($\mu' \leqq q_n \leqq \mu''$) in a similar manner as in Theorem 6.

THEOREM 8. *If in addition* $c^x/(c^x - d^x) = a^x/(a^x - b^x)$, *then every set in* 2-*space is either weakly approachable by one player or weakly excludable by the* other player, and we may construct a collection of sets ε^* *in* 2-*space such that a set* S *is weakly approachable by player* I *if and only if* S *contains an element of* $\mathcal{E}^*$.

PROOF. In this case, we have

$$0 < \mu' = \mu'' = b^x/(b^x - a^x) = d^x/(d^x - c^x) < 1, \tag{6.7}$$

$$t_f = \Lambda^* = T(\mu') = R(0)R(1) = (0, 0), \tag{6.8}$$

and Ω is the union of two convex sets $\bigcup_{0 \leqq q \leqq \mu'} T(q)$ and $\bigcup_{\mu' \leqq q \leqq 1} T(q)$.

$\lambda(\mu')$ cannot be uniquely determined by (6.3) and we need the following adjustment. For each $0 \leqq n < N$, let

$$A_n(\mu') = \bigcup_{0 \leqq p \leqq 1} A_n(\mu_p'), \tag{6.9}$$

where $A_n(\mu_p') = \{\omega : \omega \,\varepsilon\, X, \omega^x = t_n^x - \rho^p(\omega^y - t_n^y)$, and $\omega^y \leqq t_n^y\}$, $\rho^p = [z^y(p, 1) - z^y(p, 0)]/[z^x(p, 1) - z^x(p, 0)]$, and $t_n = T_n(\mu')$; for each $\omega \,\varepsilon\, A_n(\mu_p')$ (or $\bar{\omega}_{n,m} \,\varepsilon\, A_n(\mu_p')$, $n < m \leqq N$), we say that $z_n(\lambda(\mu_p'), \mu')$ is the closest point in Λ_n to ω (or $\bar{\omega}_{n,m}$), where $\lambda(\mu_p') = p$; then every F of $\mathfrak{F}$ (defined in Theorem 7) is weakly approachable by player I as in Theorem 7. The necessary condition for weakly approachability is given in the following lemma.

LEMMA 16. *If* $c^x/(c^x - d^x) = a^x/(a^x - b^x)$ *and* S *is a set in* 2-*space such that* $SF \neq \emptyset$ *for all* $F \,\varepsilon\, \mathfrak{F}$ (defined in Theorem 7), *then* S *is weakly approachable by player* II.

PROOF OF LEMMA 16. Since $t_f = (0, 0)$ for all $F \,\varepsilon\, \mathfrak{F}$, we have either

$$S\{(x, f(x)) : v_f^x \leqq x \leqq 0\} \neq \emptyset \qquad \text{for all} \quad F \,\varepsilon\, \mathfrak{F} \tag{6.10}$$

or

$$S\{(x, f(x)) : 0 \leqq x \leqq u_f^x\} \neq \emptyset \qquad \text{for all} \quad F \,\varepsilon\, \mathfrak{F}.$$

Assuming the first case, we let $q_{n+1} = 0$ if $\delta(t_n, S) > k/N$ or $q_{n+1} = \mu'$ if $\delta(t_n, S) \leqq k/N$ for $0 \leqq n < N$, where $t_n = T_n(\mu')$ and $t_0 = t_f$. The graph of the line segments $\overline{t_0t_1}, \overline{t_1t_2}, \cdots, \overline{t_{N-1}t_N}$ is a subset of $\{(x, f(x)) : v_f^x \leqq x \leqq 0\}$ for some $F \,\varepsilon\, \mathfrak{F}$; hence, there exists an $1 \leqq n^* < N$ which is the first n such that $\delta(t_n, S)$

$\leqq k/N$. If we let $q_n = \mu'$ for all $n^* < n \leqq N$, then we have $t_n = t_{n^*}$ for all $n^* < n \leqq N$ and

$$\delta(\bar{\omega}_N, S) = \delta(t_N, S) \leqq k/N. \tag{6.11}$$

The proof of the second case is similar. Q.E.D.

Generate $\mathcal{E}^*$ from $\mathfrak{F}$ similar as $\mathcal{E}^*$ from $\mathcal{E}$ in Theorem 3, then Theorem 8 follows from Theorem 7 and Lemma 16 as in Theorem 5. Q.E.D.

7. $\Omega \neq \Omega^*$**,** $T(0)T(1) = \varnothing$ **and** $R(0)R(1) = \varnothing$**.** In this case, one element of $\bar{M}$ is an interior point of the convex hull of the other elements of $\bar{M}$. We may assume that a is this particular element, $d = (0, 0)$, $b^x = 0$, $b^y > 0$, and $c^x > 0$; hence,

$$a = z(1, 1) = z(\lambda', \mu') \quad \text{for some} \quad 0 < \lambda', \mu' < 1. \tag{7.1}$$

Let

$$\begin{aligned} \mu(\lambda) &= 1 && \text{for} \quad 0 \leqq \lambda \leqq \lambda' \\ &= 1 - (\lambda - \lambda')(1 - \mu')/(1 - \lambda') && \text{for} \quad \lambda' \leqq \lambda \leqq 1 \end{aligned} \tag{7.2}$$

and

$$\Lambda^* = \{z(\lambda, \mu(\lambda)): 0 \leqq \lambda \leqq 1\}; \tag{7.3}$$

and define $\mathfrak{F} = \{F\}$ as in Section 5 except the replacement of Λ by Λ^* in (iii) and the replacement of condition (iv) by

$$\begin{aligned} \rho_* = (a^y - b^y)/a^x &\leqq (f(x_2) - f(x_1))/(x_2 - x_1) \\ &\leqq z^y(\lambda', 1)/z^x(\lambda', 1) = \rho^* \end{aligned} \tag{7.4}$$

for all $v_f^x \leqq x_1 < x_2 \leqq u_f^x$, where $v_f^x = 0$ in this section.

THEOREM 9. *If a set S in 2-space contains an F of $\mathfrak{F}$, then S is weakly approachable by player* I.

PROOF. We may assume that $M = \bar{M}$ by Lemma 4. For each $0 \leqq n \leqq N$, $N \geqq 1$, let

$$\begin{aligned} \lambda(\mu) &= 1 && \text{for} \quad 0 \leqq \mu \leqq \mu' \\ &= \lambda' + (1 - \mu)(1 - \lambda')/(1 - \mu') && \text{for} \quad \mu' \leqq \mu \leqq 1, \end{aligned} \tag{7.5}$$

$$\Lambda_n = \{z_n(\lambda(\mu), \mu): 0 \leqq \mu \leqq 1\}, \tag{7.6}$$

$$\Lambda_n^* = \{z_n(\lambda, \mu(\lambda)): 0 \leqq \lambda \leqq 1\}; \tag{7.7}$$

and for each $0 \leqq n < N$, let $A_n(\mu)$ for $0 \leqq \mu \leqq 1$, A_n, $C_{n,m}$ for $n < m \leqq N$, and A_n^0 be defined as in Section 6. An example of M and F is shown in Figure 4.

From the definition of F, we have the following facts for each $0 \leqq n < N$: If there exists a point $s \in F\{z_n(\lambda(\mu), \mu): 0 \leqq \mu < 1\}$, then $R_n(0)\{(x, f(x)): 0 \leqq x \leqq s^x\} = \varnothing$ and $F\{z_n(0, q): 0 \leqq q < \mu'\} = \varnothing$; and if there exists

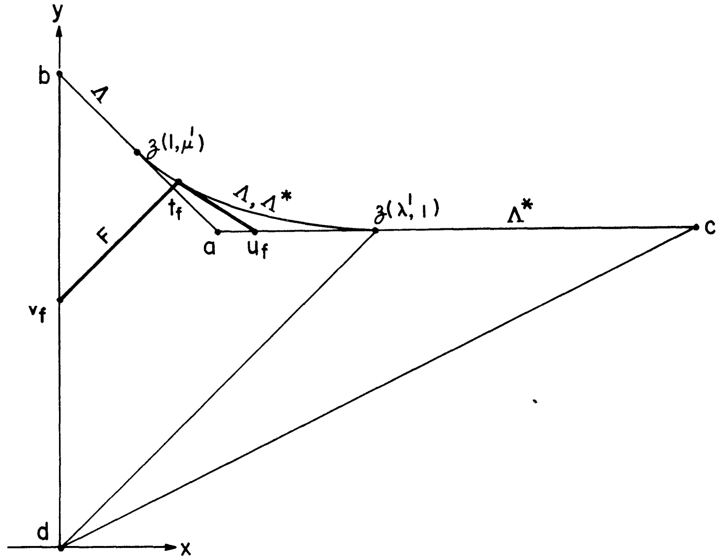

Fig. 4

$s_1 \, \varepsilon \, F\{z_n(\lambda(\mu), \mu) : 0 \leqq \mu < 1\}$ and, $s_2 \, \varepsilon \, FR_n(0)$, then there exists a point $s_3 \, \varepsilon \, FT_n(1)$ such that $s_1^x \leqq s_3^x \leqq s_2^x$.

Based on the above analysis, Theorems 1, 6, and 7, player I may play a strategy as follows: For each $0 \leqq n < N$, if $FC_{m,n} \neq \emptyset$ for some $0 \leqq m < n$ (as indicated in Section 5 (3a)) or $\delta(F, \Lambda_n) \leqq \delta(F, \Lambda_n^*)$, then I chooses p_{n+1} with respect to Λ_m or Λ_n as in Theorems 6 and 7; otherwise, I chooses $p_{n+1} = 0$ as in Theorem 1. Then results similar to Section 5 (5) (with respect to $\Lambda_n \cup \Lambda_n^*$ instead of Λ_n) can be obtained as in Theorem 6. Q.E.D.

Remarks. (i) If we fix the points b, c, d and move point a toward the line segment $\overline{bc}$, then $z(\lambda', 1)$ and $z(1, \mu')$ tend to point a, and $\mathfrak{F}$ tends to the collection of weakly approachable sets $\mathfrak{F}$ in Section 2; therefore, we conjecture that we may construct a collection of weakly approachable sets $\mathcal{E}$ for this case as in Section 3.

(ii) Let a, b, c be fixed and move point d toward a. Then $z(\lambda', 1)$ tends to c and $z(1, \mu')$ tends to b (i.e., λ' and μ' tend to 0), and we have the case of Section 5.

(iii) Weakly approachable sets for player II are similar.

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