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GENERALIZED LEAST-SQUARES ESTIMATION OF A SUBVECTOR OF PARAMETERS IN RANDOMIZED FRACTIONAL FACTORIAL EXPERIMENTS[1]

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1. Introduction and summary. In fractional replication, the problem is to estimate a subvector, α, of $S = p^s$ pre-assigned parameters in the presence of an additional subvector of nuisance parameters. Estimation procedures for randomized fractional replicates in 2^m factorial systems have been treated by Ehrenfeld and Zacks [1], [2] and Zacks [6], [7] under an orthogonalized form of the statistical model. Extension to systems with general prime p has recently been studied under a "fully orthogonalized" model by Lentner [4]; this is discussed in Section 3.

The generalized inverse operator is then applied to the study of estimation procedures in general $N = p^m$ fractional replication. The present work differs from that of Zacks [7] where the problem of estimating the *entire* vector of N parameters was considered.

The class of all type-g (generalized inverse) solutions for α is given and the class is investigated with respect to unbiasedness and optimality. Specifically, it is shown that there is a unique unbiased type-g estimator which coincides with the classical estimator under the assumption of zero nuisance parameters. Using the trace of the mean square error matrix as the risk function, it is shown that there is a coincidence of Bayes, minimax, admissible, and classical procedures under certain conditions.

2. Notation. Except for minor deviations, the symbolism and terminology to be used is that in [2] and [6]. For prime integer $p > 1$, a general factorial system of m factors each at p levels consists of $N = p^m$ treatments. The main effects and interactions of the m factors are measured by the set of N parameters $B = \{\beta_u : u = 0, 1, \cdots, N - 1\}$; α is the subvector of B of $S = p^s$ pre-assigned parameters of interest and β is the subvector of the remaining $N - S$ parameters of B, the nuisance parameters. There is no loss in generality in assuming that the parameters in α are $\beta_0, \beta_1, \cdots, \beta_{S-1}$ since one can merely relabel the original parameters.

The treatment designated by the m-tuple $(i_0, i_1, \cdots, i_{m-1})$, where $i_j = 0, 1, \cdots, p - 1$ for each $j = 0, 1, \cdots, m - 1$, when placed in correspondence with the point x_t, where $t = \sum_{j=0}^{m-1} i_j p^j$, invokes a *standard order* among the x_t. The set $X = \{x_t : t = 0, 1, \cdots, N - 1\}$ is partitioned into $M = p^{m-s}$ blocks of S

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treatments each, in such a way that the subvector α is unconfounded with blocks; specifically, for $v = 0, 1, \cdots, M - 1$, the blocks are given by

$$X_v = \{x_t : t = i + vS; i = 0, 1, \cdots, S - 1\}.$$

A randomized fractional factorial experiment is one in which block X_v is selected with probability ξ_v; the probability vector $\xi' = (\xi_0, \xi_1, \cdots, \xi_{M-1})$ completely determines such an experiment. The special randomized procedure, RPI, is specified by the probability vector $\xi^* = (1/M)1^{(M)}$, where $1^{(M)}$ is a vector of M ones. Another special class of fractional procedures are the non-randomized ones, those having a single ξ_v equal to unity.

Let the random variable y_t denote the response observed under treatment x_t. The expected value of y_t can be expressed as a linear function of the N parameters; viz., for each $t = 0, 1, \cdots, N - 1$,

$$E(y_t) = \textstyle\sum_{u=0}^{N-1} c_N(t, u)\beta_u. \tag{2.1}$$

The $N \times N$ matrix $C_N = [c_N(i, j)]$ can be obtained recursively by Kronecker direct multiplication from C_p, the matrix whose column vectors are orthogonal polynomial coefficients of order p; i.e., for $r = 2, 3, \cdots, m$

$$C_{p^r} = C_p \otimes C_{p^{r-1}}. \tag{2.2}$$

The random error associated with y_t is denoted by ϵ_t, independent of x_t.

Other terminology will be defined when first introduced.

3. Extension of estimation procedures to general p^m fractional factorial systems under RPI. As mentioned in [1], structural problems (singular coefficient matrices, non-orthogonal columns) may arise in fractional factorials when $p \geqq 3$ even if the orthogonalized form of the model is used. As proposed in [1] to overcome these difficulties, the S preassigned parameters are taken to be the subgroup generated by the first s main effects while the defining subgroup will be taken as the subgroup generated by the last $(m - s)$ main effects. It should be emphasized that these restrictions are unnecessary when $p = 2$; any set of s independent parameters can be taken as the generator of α and any set of $(m - s)$ independent parameters (which are also independent of the pre-assigned) can be taken as the generator of the defining subgroup.

The complete factorial model in orthogonalized form is

$$Y = C_N \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \epsilon, \tag{3.1}$$

where Y denotes the $N \times 1$ vector of observations and ϵ denotes the $N \times 1$ vector of random errors. Since the columns of C_N are orthogonal, it follows for $i, j = 0, 1, \cdots, N - 1$, that

$$C_N' C_N = \Delta_N^2 = [\delta_{ij} d_{i,N}^2],$$

where

$$d_{i,N}^2 = \textstyle\sum_{j=0}^{N-1} c_N^2(i, j)$$

and δ_{ij} is the Kronecker's delta. Being a positive-definite (diagonal) matrix, Δ_N^2 has a positive-definite (diagonal) square root; viz.,

$$\Delta_N = [\delta_{ij}\, d_{i,N}] \quad \text{for} \quad i, j = 0, 1, \cdots, N-1.$$

Then, letting $C_N^* = C_N \Delta_N^{-1}$, the complete factorial model (3.1) becomes

$$Y = C_N^* \Delta_N \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \epsilon. \tag{3.2}$$

Using the fact that $C_N = C_M \otimes C_S$, one gets $\Delta_N^2 = \Delta_M^2 \otimes \Delta_S^2$ and consequently

$$\Delta_N = \Delta_M \otimes \Delta_S = \ulcorner\, d_{0,M}\, \Delta_S,\, d_{1,M}\Delta_S,\, \cdots,\, d_{M-1,M}\Delta_S \,\lrcorner$$

where the half-brackets denote a direct sum of matrices. Now,

$$\Delta_N \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix},$$

where $\alpha^* = d_{0,M}\Delta_S\alpha$ and $(\beta^*)' = [\beta_{(1)}^{*\prime}, \cdots, \beta_{(M-1)}^{*\prime}]$ with $\beta_{(t)}^* = d_{t,M}\Delta_S\beta_{(t)}$ for $t = 1, 2, \cdots, M-1$.

From the theory of fractional replication (see [4]), the statistical model for the fractional replicate utilizing block X_v is

$$Y_v = C_{v,N} \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} + \epsilon_v, \tag{3.3}$$

where $(C_N^*)' = [C_{0,N}', C_{1,N}', \cdots, C_{M-1,N}']$; $C_{v,N}$ is a block of S rows of C_N^*. From the definition of Kronecker direct multiplication and the fact that $C_N^* = C_M^* \otimes C_S^*$, we have

$$C_{v,N} = [c_{v0,M}C_S^*, H_v^*], \tag{3.4}$$

where

$$c_{vu,M} = d_{u,M}^{-1} c_M(v, u) \tag{3.5}$$

and

$$H_v^* = [H_{v1}^*, \cdots, H_{v,M-1}^*] \quad \text{with} \quad H_{vu}^* = c_{vu,M}C_S^*. \tag{3.6}$$

Thus,

$$\begin{aligned} Y_v &= [M^{-\frac{1}{2}}C_S^*, H_v^*] \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} + \epsilon_v \\ &= [C_S, H_v^*] \begin{pmatrix} \alpha \\ \beta^* \end{pmatrix} + \epsilon_v. \end{aligned} \tag{3.7}$$

It is interesting to note that this transformation has not changed the vector of preassigned parameters but merely changed the nuisance parameters in a one-to-one manner. The model given in (3.7) will be called the "fully orthogonalized model for the fractional replicate utilizing block X_v".

The particular investigation which culminated in the above results was centered around estimation theory (see [4]). A major problem encountered was that $H_v H_v'$ had no known structure when $p \geqq 3$, whereas $H_v^*(H_v^*)' = [(M-1)/M]I_S$ under the fully orthogonalized model.

It was shown in [4] that all results in Sections 3 and 4 of Zacks [6] are valid for general prime p with certain modifications. Specifically, one needs to replace $S^{-1}(C^{(S)})'$ by C_S^{-1}; H_v by H_v^*; $S^{-1}I^{(S)}$ by Δ_S^{-2} and positive-definite by positive semi-definite. Proofs of several of these results can be considerably shortened by use of the following. Let E_ξ denote the expectation operator under probability vector ξ.

LEMMA. *Let F_v be an $S \times S$ matrix satisfying the two conditions of Theorem* 3.2 *of* [6]. *Then,*

$$E_{\xi^*}(F_v \mid H_v^*) = O \quad \textit{for} \quad v = 0, 1, \cdots, M-1.$$

PROOF. $E_{\xi^*}(F_v H_v^*) = E_{\xi^*}(Q_v H_v^*)$, where $Q_v = E_{\xi^*}(F_v \mid H_v^*)$. Then, by the second condition, $QH^* = O$, where $Q = [Q_0, \cdots, Q_{M-1}]$ and

$$(H^*)' = [(H_0^*)', \cdots, (H_{M-1}^*)'].$$

From the condition $E_{\xi^*}(F_v) = O$, it follows that $E_{\xi^*}(Q_v) = O$ or equivalently $\sum_{v=0}^{M-1} Q_v C_S^* = O$. Thus, $Q(1^{(M)} \otimes C_S^*) = O$, which combined with $QH^* = O$ gives $QC_N^* = O$ and the lemma follows.

4. Generalized inverse estimators. Some elementary properties of generalized inverses (g-inverses) from Rao [5] are used in the development. The most general definition of g-inverses is used herein; i.e., letting $\mathcal{G}(A)$ denote the class of g-inverses of an $m \times n$ matrix A, the $n \times m$ matrix A^- belongs to $\mathcal{G}(A)$ if and only if $AA^-A = A$.

Since the coefficient matrix (C_S, H_v^*) of the fully orthogonalized model (3.7) is of order $S \times N$ and rank S, its transpose has left inverses and hence, the normal equations associated with (3.7) have the particularly simple form

$$(4.1) \qquad (C_S, H_v^*)\begin{pmatrix} \hat{\alpha} \\ \beta^* \end{pmatrix} = Y_v.$$

Hereafter, let $C = C_S$ and $G_v = H_v^*$.

Under g-inverse theory, the system (4.1) has solutions

$$(4.2) \qquad \begin{pmatrix} \hat{\alpha} \\ \beta^* \end{pmatrix} = C_{v,N}^- Y_v,$$

where $C_{v,N} = (C, G_v)$ and $C_{v,N}^-$ belongs to $\mathcal{G}(C_{v,N})$.

Letting U^* be an arbitrary $N \times S$ matrix, it can be shown (see [4]) that the set $\mathcal{G}(C_{v,N})$ can be generated by

$$(4.3) \qquad C_{v,N}^-(U^*) = C_{v,N}^- + (I_N - C_{v,N}^- C_{v,N})U^*.$$

The matrix $\left[\begin{smallmatrix} C^{-1} \\ O \end{smallmatrix}\right]$ belongs to $\mathcal{G}(C_{v,N})$, whereby the generator (4.3) reduces to

$$(4.4) \qquad C_{v,N}^-(U) = \begin{bmatrix} C^{-1}(I_S + G_v U) \\ -U \end{bmatrix}$$

where $U^* = \left[\begin{smallmatrix} U_1 \\ U \end{smallmatrix}\right]$ is partitioned so that U is of order $K \times S$ with $K = N - S$. Substitution of (4.4) into (4.2) leads to the following.

DEFINITION. Any solution of the normal equations (4.1) under g-inverse theory will be called a type-g estimator; i.e.,

$$\hat{\alpha}_v(U) = C^{-1}(I_S + G_v U)Y_v \tag{4.5}$$

is a type-g estimator of α for every v and every arbitrary $K \times S$ matrix U. It is interesting to note that $\hat{\alpha}_v(O)$ is precisely the estimator of α obtained under the classical analysis when the nuisance parameters are assumed zero.

5. Unbiasedness in the class of type-g estimators. Because of the importance of RPI, in which a block of treatment combination is chosen at random, (see [1]), we first investigate unbiasedness of type-g estimators under this randomization scheme.

THEOREM 5.1. *The only unbiased type-g estimator of α under* RPI *is $\hat{\alpha}_v(O) = C^{-1}Y_v$, for each $v = 0, \cdots, M - 1$.*

PROOF. Writing $\hat{\alpha}_v(U) = C^{-1}Y_v + C^{-1} G_v U Y_v$, it follows that a necessary and sufficient condition for unbiasedness of $\hat{\alpha}_v(U)$ is $E(G_v U G_v) = O$ since $E(C^{-1}Y_v) = \alpha$, $E(G_v) = O$, and G_v is independent of ϵ_v. Now, $G_v U G_v = (T_v G_{v1}, \cdots, T_v G_{v,M-1})$, where $T_v = G_v U = \sum_{t=1}^{M-1} G_{vt} U_t$, each U_t being of order $S \times S$. Thus, $E(G_v U G_v) = O$ is equivalent to $E(T_v G_{vu}) = O$ for each $u = 1, \cdots, M - 1$; i.e., $E(C_S^* \sum_{t=1}^{M-1} c_{vt,M} c_{vu,M} U_t C_S^*) = O$ since $G_{vu} = c_{vu,M} C_S^*$. But, under RPI, $E(c_{vt,M} c_{vu,M}) = \delta_{tu}$, the Kronecker's delta, and the theorem follows.

We now show that unbiasedness is impossible when blocks X_v are selected according to probability vector $\xi \neq \xi^*$.

THEOREM 5.2. *There exists no unbiased type-g estimators of α under a randomized procedure $\xi \neq \xi^*$.*

PROOF. Since

$$\hat{\alpha}_v(U) = \alpha + C^{-1}G_v U C\alpha + C^{-1}(I_S + G_v U)\epsilon_v + C^{-1}(G_v + G_v U G_v)\beta^*,$$

it follows that necessary and sufficient conditions for unbiasedness of $\hat{\alpha}_v(U)$ are $E_\xi(G_v U) = O$ and $E_\xi(G_v + G_v U G_v) = O$. Now, let $G_v U = C_S^* P_v$ and $G_v + G_v U G_v = (Q_{v1}, \cdots, Q_{v,M-1})C_S^*$, where $P_v = \sum_{t=1}^{M-1} c_{vt,M} U_t$ and $Q_{vu} = c_{vu,M}(I_S + C_S^* P_v)$ for each $u, v = 1, \cdots, M - 1$. Therefore, equivalent conditions are $E_\xi(P_v) = O$ and $E_\xi(Q_{vu}) = O$ for each $u, v = 1, \cdots, M - 1$. Define $\pi_{t,u} = \sum_{v=0}^{M-1} \xi_v c_{vu,M} c_{vt,M}$. Then,

$$E_\xi(P_v) = \sum_{v=0}^{M-1} \xi_v \sum_{t=1}^{M-1} c_{vt,M} U_t = M^{\frac{1}{2}} \sum_{t=1}^{M-1} \pi_{0,t} U_t = O, \tag{5.1}$$

and

$$E_\xi(Q_{vu}) = \sum_{v=0}^{M-1} \xi_v c_{vu,M}(I_S + C_S^* P_v) = M^{\frac{1}{2}} \pi_{0,u} I_S + C_S^* \sum_{t=1}^{M-1} \pi_{u,t} U_t = O \tag{5.2}$$

for each $u = 1, \cdots, M - 1$. Equations (5.1) and (5.2) give a system of M equations in $(M - 1)$ matrices, the $C_S^* U_t$. This system can be written as

$$(\pi^* \otimes I_S) \begin{bmatrix} C_S^* U_1 \\ \vdots \\ C_S^* U_{M-1} \end{bmatrix} = -(\pi_1^* \otimes I_S), \tag{5.3}$$

where

$$\pi^* = [\pi_{t,u}] \quad \text{for} \quad t = 0, \cdots, M-1; u = 1, \cdots, M-1$$
$$(\pi_1^*)' = M^{\frac{1}{2}}(0, \pi_{0,1}, \cdots, \pi_{0,M-1}).$$

It will be shown that system (5.3) is inconsistent by showing that the coefficient matrix $(\pi^* \otimes I_S)$ and the augmented matrix $(\pi^*, \pi_1^*) \otimes I_S$ have different ranks. To this end, it suffices to show that π_1^* is linearly independent of the columns of π^*. On the contrary, suppose there exists $M - 1$ constants: $a_1, \cdots, a_{M-1}$, not all zero, such that

$$\sum_{u=0}^{M-1} a_u \pi_{0,u} = 0 \tag{5.4}$$

and

$$\sum_{u=0}^{M-1} a_u \pi_{t,u} = \pi_{0,t} \quad \text{for} \quad t = 1, \cdots, M-1. \tag{5.5}$$

For each $t = 1, \cdots, M - 1$, equation (5.5) can be written as

$$\sum_{v=0}^{M-1} \left(\sum_{u=1}^{M-1} a_u c_{vu,M}\right) \xi_v c_{vt,M} = \sum_{v=0}^{M-1} \xi_v c_{vt,M}$$

which implies that $\sum_{u=1}^{M-1} a_u c_{vu,M} = 1$; this result substituted into (5.4) gives $\sum_{v=0}^{M-1} \xi_v = 0$, a contradiction. This completes the proof.

6. Decision theoretic framework and optimal strategies. A set of strategies available to the statistician for estimating the subvector α is the class $\{(\xi, U)\}$, where ξ is the vector of probabilities used in selecting the blocks of treatments and U is the matrix used in calculating the type-g estimator. These strategies are now investigated relative to the risk function: the trace of the mean square error matrix; i.e., the risk due to (ξ, U) at $\theta = \begin{pmatrix} \alpha \\ \beta^* \end{pmatrix}$ is

$$R(\xi, U; \theta) = \operatorname{tr} \{E_\xi[\hat{\alpha}_v(U) - \alpha][\hat{\alpha}_v(U) - \alpha]'\}. \tag{6.1}$$

When the parameters θ have prior distribution $\eta(\theta)$, the prior risk is

$$\rho(\xi, U; \eta) = E_\eta[R(\xi, U; \theta)]. \tag{6.2}$$

For convenience, let E_v denote the expectation with respect to v (i.e., over all blocks X_v) under RPI. A workable expression for the risk function is first given.

LEMMA 6.1. *For a type-g estimator of α under* RPI, *the risk function has the form:*

$$\begin{aligned} R(\xi^*, U; \theta) = {} & \sigma^2 \operatorname{tr} (\Delta_S^{-2} + U'DU) + \operatorname{tr} [(C'U'DUC)\alpha\alpha'] \\ & + 2 \operatorname{tr} \{E_v[(G_v UC)'R^{-1}W_v]\beta^*\alpha'\} \\ & + \operatorname{tr} \{E_v[W_v' R^{-1} W_v]\beta^*(\beta^*)'\}, \end{aligned} \tag{6.3}$$

where $R = CC'$; $W_v = G_v + G_v U G_v$; *and*

$$D = [M^{-1}\delta_{ij}\Delta_S^{-2}], \quad \textit{for} \quad i, j = 1, \cdots, M-1.$$

PROOF. Writing $\hat{\alpha}_v(U) = \alpha + C^{-1}G_v UC\alpha + C^{-1}(I_S + G_v U)(G_v\beta^* + \epsilon_v)$, it follows that

$$R(\xi^*, U; \theta) = E_v\{[\alpha'(G_vUC')R^{-1}(G_vUC)\alpha + 2\alpha'(G_vUC)'R^{-1}W_v\beta^* \tag{6.4}$$
$$+ (\beta^*)'W_v'R^{-1}W_v\beta^*$$
$$+ \epsilon_v'(I_S + G_vU)'R^{-1}(I_S + G_vU)\epsilon_v] \mid \alpha, \beta^*\}.$$

Since $G_v'R^{-1}G_v = [c_{vi,M}c_{vj,M}\Delta_S^{-2}]$, for $i, j = 1, \cdots, M - 1$, $E_v(G_v'R^{-1}G_v) = D$ and the first term of (6.4) becomes $\alpha'C'U'DUC\alpha$. Using the fact that the trace of a matrix is invariant under any cyclic permutation of its factors (see [3], p. 7) and by the independence of G_v and ϵ_v, the last term of (6.4) can be written as

$$\operatorname{tr}\{E_v[(I_S + G_vU)'R^{-1}(I_S + G_vU)]E_v(\epsilon_v\epsilon_v')\}$$
$$= \sigma^2 \operatorname{tr}\{E_v(R^{-1} + 2U'G_v'R^{-1} + U'G_v'R^{-1}G_vU)\} \tag{6.5}$$
$$= \sigma^2 \operatorname{tr}(\Delta_S^{-2} + U'DU).$$

Taking the trace of (6.4) and by cyclic permutations, the lemma now follows.

Theorem 6.2. *Relative to risk function* (6.1) *and* RPI, $\hat{\alpha}_v(O) = C^{-1}Y_v$ *is the Bayes type-g estimator of* α *under any prior distribution* $\eta(\theta)$ *such that:*

$$E_\eta(\alpha\alpha') = \mu^2 I_S; \qquad E_\eta(\beta^*\alpha') = O; \qquad E_\eta[\beta^*(\beta^*)'] = \tau^2 I_K. \tag{6.6}$$

Proof. From Lemma 6.1 and equations (6.2) and (6.6), the prior risk function takes the form

$$\rho(\xi^*, U; \eta) = \sigma^2 \operatorname{tr}(\Delta_S^{-2} + U'DU) + \mu^2 \operatorname{tr}(RU'DU) \tag{6.7}$$
$$+ \tau^2 \operatorname{tr}\{E_v[R^{-1}(I_S + G_vU)G_vG_v'(I_S + G_vU)']\}.$$

But, $G_vG_v' = [(M - 1)/M]I_S$ and by manipulations analogous to those used to obtain equation (6.5),

$$\rho(\xi^*, U; \eta) = a^2 \operatorname{tr}(\Delta_S^{-2} + U'DU) + \mu^2 \operatorname{tr}(RU'DU), \tag{6.8}$$

where $a^2 = \sigma^2 + [(M - 1)/M]\tau^2$. Since $\operatorname{tr}(U'DU)$ and $\operatorname{tr}(RU'DU)$ are each nonnegative, $\inf_U \rho(\xi^*, U; \eta) = a^2 \operatorname{tr}(\Delta_S^{-2})$. This infimum is attained only when $U = O$, and the theorem is proved.

Before giving a corollary to Theory 6.2, it should be emphasized that conditions (6.6) assume equal variances for the transformed nuisance parameters β^*; the original nuisance parameters β would have equal variances if and only if $p = 2$.

Corollary 6.2.1. *Relative to risk function* (6.1) *and* RPI, $\hat{\alpha}_v(O) = C^{-1}Y_v$ *is an admissible type-g estimator of* α.

The next two theorems deal with optimality of general strategies; i.e., when both ξ and U are arbitrary. It is convenient to designate the non-randomized probability vectors by $\xi_t' = (\xi_0, \cdots, \xi_{M-1})$ for each $t = 0, 1, \cdots, M - 1$, where $\xi_v = \delta_{tv}$.

Theorem 6.3. *The Bayes type-g strategies, relative to risk function* (6.1) *and a given distribution* $\eta(\theta)$, *are non-randomized.*

PROOF. Let (ξ_η, U_η) by Bayes against η. Then, $\rho(\xi_\eta, U_\eta; \eta) \leqq \inf_U \rho(\xi, U; \eta)$ for every ξ, in particular, when $\xi = \xi_t$, for $t = 0, \cdots, M-1$. Let $U_{t,\eta}$ be defined for each t and η by $\rho(\xi_t, U_{t,\eta}; \eta) = \inf_U \rho(\xi_t, U; \eta)$. Thus, $\rho(\xi_\eta, U_\eta; \eta) \leqq \rho(\xi_{t,\eta}, U_{t,\eta}; \eta)$, for each $t = 0, \cdots, M-1$, and consequently

$$\rho(\xi_\eta, U_\eta; \eta) \leqq \inf_t \rho(\xi_t, U_{t,\eta}; \eta). \tag{6.9}$$

But,

$$\begin{aligned} \rho(\xi_\eta, U_\eta; \eta) &= \inf_\xi \inf_U \textstyle\sum_{v=0}^{M-1} \xi_v \rho(\xi_v, U; \eta) \\ &\geqq \inf_\xi \textstyle\sum_{v=0}^{M-1} \xi_v \inf_U \rho(\xi_v, U; \eta) \\ &= \inf_\xi \textstyle\sum_{v=0}^{M-1} \xi_v \rho(\xi_v, U_{v,\eta}; \eta) \geqq \inf_v \rho(\xi_v, U_{v,\eta}; \eta). \end{aligned} \tag{6.10}$$

Equations (6.9) and (6.10) imply that $\xi_\eta = \xi_k$ for some $k = 0, \cdots, M-1$. This completes the proof.

THEOREM 6.4. *Relative to risk function* (6.1), $(\xi^*, U = O)$ *is a minimax type-g procedure.*

PROOF. Taking the trace of equation (6.4), replacing ξ^* by ξ, and using cyclic permutations, we get

$$\begin{aligned} R(\xi, U; \theta) = {} & \operatorname{tr}[(UC\alpha\alpha'C'U')E_\xi(Z_v)] \\ & + 2\operatorname{tr}[\alpha(\beta^*)'E_\xi(W_v'J_v')UC] + \operatorname{tr}[\beta^*(\beta^*)'E_\xi(W_v'R^{-1}W_v] \\ & + \sigma^2 \operatorname{tr}\{E_\xi[(I_S + G_vU)'R^{-1}(I_S + G_vU)]\}, \end{aligned} \tag{6.11}$$

where $J_v = G_v'R^{-1}$ and $Z_v = G_v'R^{-1}G_v$. For the non-randomized procedure ξ_t, the risk function is

$$\begin{aligned} R(\xi_t, U; \theta) = {} & \sigma^2 \operatorname{tr}(\Delta_S^2 + 2U'J_t + U'Z_tU) + \operatorname{tr}[(C'U'Z_tUC)\alpha\alpha'] \\ & + 2\operatorname{tr}[(W_t'R^{-1}G_tUC)\alpha(\beta^*)'] + \operatorname{tr}[(W_t'R^{-1}W_t)\beta^*(\beta^*)']. \end{aligned} \tag{6.12}$$

Suppose, for each fixed $t = 0, \cdots, M-1$, that θ has a prior distribution $\eta(\theta)$ such that

$$\begin{aligned} E_\eta(\alpha\alpha') &= 2a^2\Delta_S^{-2} \\ E_\eta[\alpha(\beta^*)'] &= -[Ma^2/(M-1)]C^{-1}G_t \\ E_\eta[\beta^*(\beta^*)'] &= \tau^2 I_K. \end{aligned} \tag{6.13}$$

Then, the prior risk for a non-randomized procedure, ξ_t, is

$$\begin{aligned} \rho(\xi_t, U; \eta) &= \sigma^2 \operatorname{tr}(\Delta_S^{-2} + 2U'J_t + U'Z_tU) + 2a^2 \operatorname{tr}(U'Z_tU) \\ &\quad - [2Ma^2/(M-1)]\operatorname{tr}(G_tW_t'R^{-1}G_tU) + \tau^2 \operatorname{tr}(W_t'R^{-1}W_t) \\ &= a^2 + a^2 \operatorname{tr}(U'Z_tU). \end{aligned} \tag{6.14}$$

Because Z_t is positive semi-definite for each $t = 0, \cdots, M-1$, $\inf_U \rho(\xi_t, U; \eta) = a^2$ and is attained only when $U = O$; i.e., $\hat{\alpha}_t(O) = C^{-1}Y_t$ is Bayes against any η satisfying (6.13). Now, let $\{\eta_k(\theta)\}$ be a sequence of prior distributions of θ such that, for each fixed t,

$$E_{\eta_k}(\alpha\alpha') = 2a_k^2\Delta_S^{-2}$$
$$E_{\eta_k}[\alpha(\beta^*)'] = -[Ma_k^2/(M-1)]C^{-1}G_t$$
$$E_{\eta_k}[\beta^*(\beta^*)'] = k\tau^2 I_K,$$

where $a_k^2 = \sigma^2 + k(M-1)\tau^2/M$.
Then, for each $k = 1, 2, \cdots$; $\hat{\alpha}_{t,k} = C^{-1}Y_t$ is Bayes against η_k and $\rho(\xi_t, \hat{\alpha}_{t,k}; \eta_k) = a_k^2 + a_k^2 \operatorname{tr}(U'Z_tU)$, whereby $\lim\sup_{k\to\infty} \rho(\xi_t, \hat{\alpha}_{t,k}; \eta_k) = \infty$. And, from equation (6.3),

$$R(\xi^*, U = O; \theta) = \sigma^2 \operatorname{tr}(\Delta_S^{-2}) + (\beta^*)'D\beta^*$$

whereby $\sup_\theta R(\xi^*, U = O; \theta) < \infty$, assuming β^* is a finite vector, and the theorem is established.

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