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ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the Annual meeting, New York, New York, August 19–22, 1969 Additional abstracts appeared in earlier issues.*)

27. Maximum likelihood histograms. EDWARD J. WEGMAN, University of North Carolina.

Conventional histograms are characterized as being measurable with respect to σ-algebras. They enjoy certain maximum likelihood and consistency properties. The conventional histogram is generalized to the case where all intervals need not be equally long. This generalization is maximum likelihood. The number of such histograms is computed. Consistency is shown to depend on the existence of an optimal histogram. Based on this optimal histogram, asymptotically maximum likelihood histograms are defined. Other types of histograms are discussed in several examples. (Received 21 May 1969.)

28. Ladder variables for a continuous time process (preliminary report). N. U. PRABHU AND MICHAEL RUBINOVITCH, Cornell University.

Ladder processes are introduced and studied for a class of stochastic processes with stationary independent increments. This is the class of processes which are representable as the difference between two independent processes on $[0, \infty)$, where one is a compound Poisson process and the other a process with stationary independent increments. The distribution of the ladder process is obtained via a continuous time version of Feller's combinatorial lemma [The Harold Cramér Volume (1959), 75–91]. A limit theorem for the supremum functional of the underlying process is given and is used to investigate the asymptotic behavior of a dam model introduced by Gani and Pyke [*J. Math. Mech.* **9** (1960) 639–652]. (Received 23 May 1969.)

29. A random walk approach to a shutdown queuing system. PAUL R. MILCH AND MARK H. WAGGONER, Naval Postgraduate School.

A queueing system with two service stations is discussed. Service times are independent exponentially distributed random variables with rates λ and μ, respectively. There are initial queues of sizes M and N, respectively, at the two stations. Operations begin simultaneously at both stations. Customers served by one station join the queue (if any) at the other station. Customers served by both stations leave the system. The results are the non-steady state Laplace transforms of the total operation time of the system; idle time of a station and time spent in the system by an individual customer. These results are obtained by a two-dimensional random walk representation of the service operations. The combinatorial methods used are the Reflection Principle and a new device called the Telescope Principle. (Received 23 May 1969.)

30. On some invariant tests concerning covariance matrices of multivariate normal populations. N. GIRI, University of Montreal. (By title)

Let $N(\xi_i, \Sigma_i)$, $i = 1, \cdots k$ be k independent p-variate normal populations with unknown mean ξ_i and unknown covariance matrix Σ_i. For testing hypothesis (A) $H_{10}: \Sigma_1 = \Sigma_0$ against the alternatives $H_{11}: \Sigma_1 \neq \Sigma_0$, where Σ_0 is a specified positive definite matrix; (B) $H_{20}: \Sigma_1 = \Sigma_2$ against the alternatives $H_{21}: \Sigma_1 \neq \Sigma_2$ and (C) $H_{30}: \Sigma_1 = \cdots = \Sigma_k$ against the alternatives

$H_{31}:\Sigma_1 = l_2\Sigma_2 = \cdots = l_k\Sigma_k$, where $l_2, \cdots, l_k$ are unknown scalar constants; it has been shown that tests based on tr $\Sigma_0^{-1}S_1$, tr $S_2(S_1 + S_2)^{-1}$ and $\text{tr}(\Sigma_2^k S_i(\Sigma_1^k S_i)^{-1}$ are locally best invariant and unbiased, where S_i is the sample covariance matrix of the ith population. (Received 26 May 1969.)

31. On fixed width confidence bounds for the mean of a multivariate normal distribution—cost of not knowing the covariance matrix. M. S. Srivastava and R. P. Bhargava, University of Toronto and The Ontario Institute for Studies in Education.

Srivastava [*J. Roy. Statist. Soc. Ser. B* **29** (1967) 132–40] proposed a spherical confidence region of fixed diameter 2d for the mean vector θ of any p-variate distribution with finite but unknown covariance matrix Σ. It was shown there that (i) the specified coverage probability α is attained in the limit as $d \to 0$ and (ii), the sequential sample size $N(d)$ has the property that $E\,N(d)/C \to 1$ as $d \to 0$ where $[C]$ is the fixed sample size required when Σ is known. In this paper it is shown that for the normal distribution the coverage probability α is attained, independently of d, θ and Σ, by taking a *fixed* number of k (depending on α) additional observations than prescribed by the sequential procedure (op. cit). Some inequalities on the moments of the sample size $N(d)$ are also presented. Specifically, it is shown that $E(N) \leqq C + O(1)$. (Received 29 May 1969.)

32. On a class of rank order tests for regression with partially informed stochastic predictors. Malay Ghosh and Pranab Kumar Sen, University of North Carolina.

Hájek (1962) has obtained asymptotically most powerful rank order tests for simple (linear) regression with non-stochastic predictors. The present paper extends the findings of Hájek to the multiple (linear) regression model with stochastic predictors including the situations where the predictors are partially informed. The proposed tests are shown to be permutationally (conditionally) distribution-free. Their asymptotic properties and efficiencies are studied and the asymptotic optimality is established under the conditions of Wald (1943). (Received 30 May 1969.)

33. Multiplication of polykays using ordered partitions. Edward J. Carney, University of Rhode Island.

Various rules, tables, and functions have been devised for the multiplication of k-statistics, but even with these aids the process is not easy for hand computation, and not easily represented analytically nor algorithmically. Previous work with k-statistics for several factors (generalized polykays) was facilitated by the use of *ordered* partitions in representing the several types of symmetric functions which arise. Further study of the relationships among these symmetric functions, expressed in terms of ordered partitions, leads to results which are attractive for their simplicity and useful as the basis for a multiplication algorithm. (Received 2 June 1969.)

34. A distribution-free test for parallelism. Myles Hollander, The Florida State University.

Consider the linear model $Y_{ij} = \alpha_i + \beta_i X_{ij} + e_{ij}$, $i = 1, 2; j = 1, 2, \cdots, N$ where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are unknown parameters and the X's are known constants. The Y's are observable while the e's are mutually independent unobservable random variables with distribution functions $P(e_{ij} \leqq t) = F_i(t)$. We construct a test of $H_0:\beta_1 = \beta_2$ which is distribution-free under H_0 when F_1, F_2 are continuous. The test, based on a signed rank statistic applied to independent random variables of the form $[(Y_{1s} - Y_{1t})/(X_{1s} - X_{1t})] - [(Y_{2u} - Y_{2v})/(X_{2u} - X_{2v})]$, is similar to a signed rank test of linearity versus convexity proposed by Olshen (*Ann. Math. Statist.* **38** (1967) 1759–69). The signed rank test is compared, via Pitman efficiency and small sample power, with the normal theory t-test of H_0 based on the assumption of equal (unknown) variances for the errors. For the equally spaced design consisting of $N = 2kn$ observations on line i, with n observations at each of the $2k$ points $C_i + 2c_ij$, $j = 0, 1, \cdots, 2k - 1$, the Pitman efficiency of the signed rank test with respect to the t-test, when $F_1 = F_2 = F$ and F is normal, is .955 for $k = 1$, .764 for $k = 2$, and tends to .716 as $k \to \infty$. Monte Carlo power comparisons are given for the t-test, signed rank test, and a conservative nonparametric test of H_0 developed by Potthoff (University of North Carolina Institute of Statistics Mimeo Series No. 445 (1965)). (Received 2 June 1969.)

35. Asymptotically robust tests about covariance matrices. M. W. J. Layard, The University of California at Davis.

The well-known lack of robustness of normal-theory techniques for testing the hypothesis of equality of variances is shown to arise in the multivariate case in testing hypotheses about covariance matrices. There is a well-developed body of normal-theory techniques for testing such hypotheses as equality of two covariance matrices, independence of sets of variates, sphericity, etc. It is shown that many of these techniques are not robust against departures from normality; in particular, the non-robustness of two tests for equality of covariance matrices (the likelihood–ratio test and Roy's test) is demonstrated by deriving the asymptotic distribution of the test statistics. Three asymptotically robust procedures for testing the equality of two covariance matrices are proposed: (1) a "standard error" test, in which the statistic is normalized by an estimate of its asymptotic covariance matrix; (2) a test based on Box's idea of dividing the data into groups, computing certain functions of the second-order moments for each group, and treating the resulting data as approximately normal; (3) a test based on the jackknife procedure. The concepts of Pitman efficiency and Bahadur efficiency are used to make some asymptotic comparisons of these tests. The standard error and jackknife tests are seen to compare favorably with the likelihood-ratio test in the normal case. Variations of the three robust tests are applicable to problems concerning correlation coefficients and the structure of covariance matrices. (Received 2 June 1969.)

36. Optimal designs with polynomial spline regression with a single multiple knot at the centre (preliminary report). V. N. Murty, Purdue University.

Consider the regression of the form $\sum_{i=0}^{n} \theta_i x^i + \sum_{i=k}^{n} \theta_i' x_+{}^i$; $x \varepsilon [-1, 1]$. Then the following theorem is proved. Theorem. (i) Optimal design for estimating θ_0 is unique and is supported on $x = 0$; (ii) Optimal designs for estimating $\theta_{k+1-2\nu}$, $\theta'_{k+2\nu-2}$, $(k \geqq 1)$ and $\theta_{k+\nu-1} (k \geqq 2)$; $\nu = 1, 2, \cdots$ etc. are unique and are supported on the set E. (iii) If $k = 1$, optimal designs for estimating θ_ν; $\nu = 1, 2, \cdots$ etc. are unique and are supported on the full set $E \cap [-1, 0]$. (iv) Optimal designs for estimating $\theta_{k-2\nu}$ and $\theta'_{k+2\nu-1}$ $(k \geqq 1)$ $\nu = 1, 2, \cdots$ etc. are unique and are supported on the full set E_1. The set E consists of $(2n - k + 2)$ points and the set E_1 consists of $(2n - k + 1)$ points.

37. Optimal designs with Tchebycheff polynomials of the first kind as regression functions (preliminary report). V. N. Murty, Purdue University. (By title)

When the regression is of the form $\sum_{i=0}^{n} \theta_i T_i(x)$, $x \varepsilon [-1, 1]$, where $T_i(x)$ is the Tchebycheff polynomial of the first kind and degree i, $i = 0, 1, \cdots, n$, optimal designs of each individual regression coefficient $\theta_i (i = 0, 1, \cdots, n)$ are explicitly obtained. (Received 2 June 1969.)

38. Estimation of the last mean of a monotone sequence. Arthur Cohen and Harold B. Sackrowitz, Rutgers University.

In this paper we study various problems of estimating the largest of a set of ordered parameters, when it is known which populations correspond to each parameter. The observed random variables are either normally distributed or are continuous and characterized by a translation parameter. The main portion of the study is devoted to estimating the larger of two normal means when we know which population has the larger mean. Note that in one result below, the first known example of an estimator which is admissible with respect to a convex loss function, but which is not generalized Bayes is given. Farrell (1968) conjectured that some such example could be found. We proceed to state the models and list the results. Let X_i, $i = 1, 2$, be independent normal random variables with means θ_i, and known variances. Without loss of generality we let the variance of X_1 be τ and the variance of X_2 be 1. Assume $\theta_2 \geqq \theta_1$, and consider the problem of estimating θ_2 with respect to a squared error loss function. Let $\delta(X_2)$ be any estimator based on X_2 alone. Consider only those $\delta(X_2)$ which are admissible for estimating θ_2 when X_1 is not observed. The following results are obtained.

(1) If the risk of $\delta(X_2)$ is bounded, then $\delta(X_2)$ is inadmissible. This result can be generalized in a few directions. In fact if θ_i are translation parameters of identical symmetric densities, then for any non-negative strictly convex loss function $W(\cdot)$, with a minimum at 0, X_2 is an inadmissible estimator. Suitable generalizations for arbitrary sample sizes are given. Another generalization is that if C is any positive constant, then $X_2 \pm C$ is inadmissible as a fixed width confidence interval of θ_2.

(2) Let U_τ be the positive solution to the equation $a^2 + (\tau + 1)a - \tau = 0$. The quantity U_τ will be such that, $0 \leqq U_\tau < 1$. Then the estimators aX_2, for $0 \leqq a < U_\tau$, are admissible. It will be shown that no $\delta(X_2)$, such that $\delta(X_2)$ is unbounded below, can be generalized Bayes. Thus this result provides an example of an estimator which is not generalized Bayes, but admissible for the squared error loss function. The results above are also true for estimating the largest of k ordered means. It is interesting that for some $a > 0$, aX_k is admissible, regardless of the size of k. The proof of admissibility of aX_2 uses the methods of Blyth (1951) and Farrell (1968).

(3) Consider the analogue of the Pitman estimator. That is, the estimator which is generalized Bayes with respect to the uniform prior on the space $\theta_2 \geqq \theta_1$. We prove that this estimator is admissible and minimax. (Received 6 June 1969.)

39. Characterization theorems for stochastic processes. Eugene Lukacs, Catholic University. (Invited)

Let $X(t)$ be a continuous and homogeneous stochastic process with independent increments and let $g(t)$ be a continuous function. Stochastic integrals of the form $\int_A^B g(t)\, dX(t)$ will be defined in the sense of convergence in probability as well as in the sense of convergence in the quadratic mean. Sufficient conditions for the existence of these integrals will be given. Properties of such integrals can be used to characterize certain stochastic processes. The Wiener process can be characterized by one of the following assumptions. (a) Two stochastic integrals are given and it is assumed that one has linear regression and constant scatter on the others. (b) Two stochastic integrals are identically distributed. (c) Two stochastic integrals are independently distributed. (d) A stochastic integral has the same distribution as a constant multiple of the increment. In addition to the assumptions (a)—(d), it is necessary to require that the integrands satisfy certain conditions.

Characterizations of symmetric stable processes as well as general stable processes will also be given. (Received 10 June 1969.)

40. Characteristic functions of functions. I. J. Good, Virginia Polytechnic Institute.

A formal process is given for expressing the characteristic function (ch.f.) of a function $g(X)$ in terms of the characteristic function ϕ of X, together with some examples and deductions. A further formalism relates the Mellin transforms of the ch.f. and probability density (p.d.) of X. When applied to multivariate stable distributions the formalism leads to generalizations of parts of multivariate statistical analysis. Two of the numerous examples are: (i) ch.f. $(X^2) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2}\phi(e^{-\pi i/4}v(2t)^{\frac{1}{2}})\, dv$; (ii) if the vector x has the real stable multivariate characteristic function $\exp\{-\lambda(\mathbf{t}'\mathbf{C}\mathbf{t})^{\frac{1}{2}\alpha}\}$, where $\mathbf{C}$ is non-negative definite, then $\mathbf{x}'\mathbf{C}^{-1}\mathbf{x}$ has a distribution $X^2_{\alpha,\lambda}(m)$ mathematically independent of $\mathbf{C}$. (Received 10 June 1969.)

41. Confidence interval estimation of functions of the normal mean and variance. Charles E. Land, Oregon State University. (Invited)

Given the Gauss-Markov estimate $\hat{\mu}$ of the mean of a normal random variable, and an independent estimate $\hat{\sigma}^2$ of its variance, distributed as a constant times a chi-square variate, the characterization of the (multivariate) normal distributions as exponential families leads to uniformly most powerful unbiased level α tests, against one and two-sided alternatives, of hypotheses of form $H_p:\mu + \lambda\sigma^2 = p$, for arbitrary λ. The confidence sets defined in terms of these tests are shown to be intervals, in the one-sided cases; it also appears to be true in the two-sided case, but no proof has been given. Exact confidence intervals for functions of form $g(\mu, \sigma^2) = h(\mu + \lambda\sigma^2)$, such as the mean of lognormal distribution, can be obtained by the method above. Approximate intervals for more general functions, such as EY when some (non-logarithmic) function of Y is normal, can be obtained by constructing a confidence interval for a linear function of μ and σ^2 that approximates $g(\mu, \sigma^2)$ in the region of interest. (Received 10 June 1969.)

42. On a theorem of Hunt. M. T. Wasan, Queen's University. (By title)

Let $\{W(m), m \geqq 0\}$ be a Wiener Process with $W(0) = 0$ with probability one, drift $c > 0$ (assume $c = 1$ for simplicity) and a variance parameter one. For $t \geqq 0$, let $M(t) = \inf\{m: W(m) \geqq t\}$ the first passage time to t. Then it is shown that $\{M(t), t \geqq 0\}$ has stationary independent increments, it is separable process and the sample paths of the process are right continuous. Under appropriate conditions an analogue of Hunt's Theorem 2.5 (Hunt, G. A. Some theorems concerning brownian motion. *Trans. Amer. Math. Soc.* (1956) 294–318) is proved. (Received 10 June 1969.)

43. Multiplication of $k_{\{\}}$ by polykay products of weight 5. P. N. Nagambal and D. S. Tracy, University of Windsor.

Dwyer and Tracy (*Ann. Math. Statist.* **35** (1964) 1174–1185) presented a combinatorial method for multiplying two polykays and gave general formulae for double products $k_{\{\}}k_{q_1q_2\cdots}$ where $\{\}$ is any set $p_1, p_2, \cdots$ and $\sum q_i \leqq 4$. Tracy (*Ann. Math. Statist.* **39** (1968) 983–998) gave rules to obtain multiple products of polykays and obtained general expressions for multiplying $k_{\{\}}$ by polykay products of weights 2, 3 or 4 (*Ann. Math. Statist.* **40** (1969) 1297–1299). Formulae for double products $k_{\{\}}k_{q_1q_2\cdots}$, $\sum q_i = 5$ are given by Nagambal and Tracy (submitted *Ann. Math. Statist.*). In this paper we obtain the expressions for multiplying $k_{()}$ by polykay products of weight 5. These are presented in a tabular form. Some new rules are encountered in the process. The particular formulae for products of weight $\leqq 5$ of Wishart (*Biometrika* **39** (1952) 1–13) and the double products general formulae have been used to check the expressions. Formulae for specific polykay products and their moments and estimators can be obtained using these general formulae. (Received 10 June 1969.)

44. Bounds and approximations for the moments of order statistics. Prakash C. Joshi, University of North Carolina.

For obtaining bounds and approximations for the moments of order statistics from a continuous parent, a method based upon orthogonal polynomials due to Sugiura (*Osaka Mathematical Journal* **14** (1962) 253–263) requires that the random variable X has a finite variance. By a generalization of his method, it is shown that, even with less stringent conditions, one or more different sequences of bounds for all finite moments can be obtained. These bounds and approximations depend on the distribution function only through certain moments of order statistics in small samples. It is shown that for the Cauchy distribution bounds and approximations for all finite moments can be obtained. Some numerical calculations for normal and Cauchy distributions are also given. (Received 11 June 1969.)

45. Robustness of estimators and tests based on Gauss-Markoff model. C. Radhakrishna Rao, Indian Statistical Institute. (Invited)

We consider estimators and tests of hypotheses based on the Gauss-Markoff model on a vector variable $Y: E(Y) = X\beta + \epsilon, D(Y) = \sigma^2 I$, and Y_i the ith component of Y is normally distributed. Properties of these estimators and tests are examined when deviations from the assumed model occur. In particular the nature of the deviations for which the procedures based on the assumed model remain optimum have been determined. (Received 13 June 1969.)

46. On reducing the n^2 design confounded in n blocks to the $(n - 1)^2$ design confounded in $n - 1$ blocks where a level of each factor is picked at random (preliminary report). R. S. Dick, Social Security Administration).

Suppose a 3^2 design confounded in three blocks must be reduced to a 2^2 design confounded in two blocks by removing a level of each factor where, in both cases, the AB interaction is confounded. Then, two of the remaining four observations are from the same block and the other two are from different blocks and must be combined to form a new second block. The even and odd blocks split 5 and 4 times or 4 and 5 times in 2^2 depending on whether 3^2 was confounded on type I or type J confounding. An interesting pattern is found for reducing 4^2 to 3^2 also. In general, it is hoped reducing $m \times n$ designs to $m \times (n - 1)$ or $(m - 1) \times n$ will yield interesting relationships between the split and unsplit blocks. (Received 16 June 1969.)

47. A characterization of hypergeometric distributions. Morris Skibinsky, University of Massachusetts.

Let n, N be arbitrarily chosen but fixed positive integers such that $n \leqq N$. Theorem. *A family of $N + 1$ probability distributions (indexed say, by $j = 0, 1, \cdots, N$), each supported on a subset of $\{0, 1, \cdots, n\}$ is the hypergeometric family having population and sample size* *parameters N, n respectively (the remaining parameter of the jth member being j), if and only if for each number θ, $0 \leqq \theta \leqq 1$, the mixture of the family with binomial (N, θ) mixing distribution is the binomial (n, θ) distribution.* The above theorem is given in various equivalent forms, and moment space ramifications are discussed. (Received 16 June 1969.)

48. Consistency of linear least squares estimate in a regression model with lagged variable. M. Ahsanullah, Canadian Food and Drug Directorate.

Mann and Wald (*Econometrica* (1943) 173–320) proved the consistency of linear least squares estimates of the model $z_t = \beta_1 z_{t-1} + \cdots + \beta_q z_{t-q} + \beta_0 + \epsilon_t$. Durbin (*J. Roy. Statist. Soc. Ser. B* (1960) 139–153) considered the model $z_t = \beta_1 z_{t-1} + \cdots + \beta_q z_{t-q} + a_1 x_{1t} + \cdots + a_p x_{pt} + \epsilon_t$ and proved that under certain conditions on the x's if the errors $\epsilon_1, \cdots$ are independently normally distributed with finite variance then the least squares estimators $\hat{\beta}_1, \cdots \hat{\beta}_q, \hat{a}_1, \cdots \hat{a}_p$ of $\beta_1, \cdots \beta_q$ and $a_1, \cdots a_p$ are asymptotically best unbiased. He also investigated the case in which $\epsilon_1, \cdots$ are not necessarily normally distributed but assumed all x_{it} $(i = 1, \cdots p, t = 1, 2, \cdots)$ are bounded. In this paper the model $z_t = \beta_1 z_{t-1} + \cdots + \beta_q z_{t-q} + a_0 + a_1 t^p + \epsilon_t$, $p \geqq 0$, is considered. Consistency of the linear least squares estimators $\hat{\beta}_1, \cdots \hat{\beta}_q, \hat{a}_0, \hat{a}_1$ of $\beta_1, \cdots \beta_q, a_0, a_1$ are investigated. (Received 16 June 1969.)

49. Spline functions and stochastic processes. George Kimeldorf and Grace Wahba, University of Wisconsin.

Let $L = \sum_0^m a_i D^i$, $a_m \neq 0$ be a linear differential operator with coefficients $a_i(t) \varepsilon C^m$ and let $G(t, u)$ be a Green's function for L. Let $\{Y(t) = \sum_0^m \theta_i z_i(t) + X(t): 0 \leqq t \leqq 1\}$ be a stochastic process where θ_i are unknown parameters, $\{z_i(t)\}$ spans the null space of L and $\{X(t)\}$ is a zero-mean process with covariance $k(s, t) = \int G(s, \cdot)G(t, \cdot)$. Theorem 1. *Let $\{t_j\}$ be n constants in (0, 1) such that the rank of $[z_i(t_j)]$ is m and let $\hat{Y}(t)$ for any fixed t be the minimum variance unbiased linear predictor of $Y(t)$ based on $\{Y(t_j)\}$. Then the prediction $\hat{y}(t)$, given observations $Y(t_j) = \lambda_j$ and now considered as a function of t, minimizes $\int_0^1 (Ly)^2$ among all suitable functions y for which $y(t_j) = \lambda_j$; hence $\hat{y}$ is the unique L-spline of interpolation to the data $\{(t_j, \lambda_j)\}$.* (An analogous result involving differential operators on $(-\infty, \infty)$ with constant coefficients and Bayesian estimation on stationary processes was presented previously by the present authors [*Ann. Math. Statist.* **39** (1968) 1779–1780, Abstract].) Theorem 1 is extended to more general observations (e.g. derivatives) and to observations involving random error. Proofs use the geometry of reproducing kernel Hilbert spaces. (Received 16 June 1969.)

50. Necessary and sufficient condition for approachability. Tien F. Hou, Bell Telephone Laboratories.

In this paper we present a solution for an unsolved problem proposed by Blackwell (*Proc. Internat. Congress Math.* **3** (1954) 336–338). Consider a two person game with a payoff $r \times s$ matrix $M = \|M(i, j)\|$, where each element of M is a probability distribution with finite αth absolute moment for some $\alpha > 1$ in a Euclidean k-space. We introduce a necessary and sufficient condition for approachability for one of the players. (Received 17 June 1969.)

51. On existence of optimal stopping rules for reward sequence $S_{n/n}$. Adhir Kumar Basu, Purdue University.

Siegmund, Simmons, and Feder (*Ann. Math. Statist.* **39** (1968) 1228–1235) proved if $\{X_n\}$ be i.i.d.r.v. with $E(X) = 0$, and absolute moment of order $\max(2, b)$ is finite then the "functional equation rule" (FER) is optimal for the reward sequence $Z_n = n^{-a}|S_n{}^b|$. S_n is the nth partial sum of X's with $2a > b > 0$. There exists also a $K > 0$ for which the FER σ stops at (n, y) whenever $|y| > Kn^{\frac{1}{2}}$. Let $\{S_n = X_1 + \cdots + X_n, F_n, n = 1\}$ be martingale with $E(|X_n|^{\max(2,b)} F_{n-1}) \leqq C < \infty$. Let $0 < b < 2a$. Using a modification of a method of Chow (unpublished work) and Burkholder's inequality (*Ann. Math. Statist.* **37** (1966) 1494–1504) we proved that the FER is optimal for Z_n in the extended sense, and if moreover for some $K > 2B_b$ then $P(|S_n| = Kn^{\frac{1}{2}}$ i.o.$) = 1$ then $P(\sigma < \infty) = 1$. So our result is the martingale extension of the above mentioned work of Siegmund, Simmons, and Feder. (Received 17 June 1969.)

52. Characteristic functions of functions. I. J. Good, Virginia Polytechnic Institute. (Invited)

A formal process is given for expressing the characteristic function (C.F.) of a function $g(X)$ in terms of the characteristic function ϕ of X, together with some examples and deductions. A further formalism relates the Mellin transforms of the C.F. and probability density (P.D.) of X. When applied to multivariate stable distributions the formalism leads to generalizations of parts of multivariate statistical analysis. For example (i) formally C.F. $(X^2) = 2\pi^{\frac{1}{2}} \int_{\infty}^{\infty} e^{-v^2}\phi(e^{-\pi/4}v(2t^{\frac{1}{2}}))\, dv$; (ii) If X and Y are independent and both have the stable distribution with C.F. $\exp(-\alpha|t|^{3/2})$, corresponding to the distribution of acceleration in a chaotic three-dimensional environment, under an inverse square law, then an explicit formula is given for P.D. (X/Y); (iii) If the vector $\mathbf{x}$ has the real stable C.F. $\exp\{-\lambda(\mathbf{t}'C\mathbf{t})^{\frac{1}{2}\alpha}\}$, where C is non-negative definite, then $\mathbf{x}'C^{-1}\mathbf{x}$ has a distribution $\chi^2_{\alpha,\lambda}(m)$ mathematically independent of C. If V has an m-dimensional Wishart distribution with ν degrees of freedom then $T^2_{\alpha,\lambda}$ defined as $\nu\mathbf{x}'V^{-1}\mathbf{x}$ has the distribution of $\chi^2_{\alpha,\lambda}(m)/\chi^2(\nu - m + 1)$ where the numerator and denominator are statistically independent. $T^2_{\alpha,\lambda}$ is a generalization of Hotelling's T^2. A formula is obtained for C.F. $(\log T^2_{\alpha,\lambda})$ and for C.F. $(T^2_{\alpha,\lambda})$. (Received 18 June 1969.)

53. The partial sequential probability ratio test and its properties. Campbell B. Read, Southern Methodist University.

The Partial Sequential Probability Ratio Test is defined and its properties discussed. In testing the mean of a normal variable with known variance or the parameter of a member of the Koopman-type family, n observations are first taken; beyond stage n, the Wald SPRT procedure operates, the likelihood-ratio lying between boundaries A and B if sampling is to continue. Extensions to Wald's approximations to the OC function and ASN are obtained by considering the conditional test, given the first n observed values, and taking expectations over the distribution of the sufficient statistic for the parameter being tested. Comparisons with other procedures are made. (Received 19 June 1969.)

54. Tandem queues with alternating priorities (preliminary report). Sreekantan S. Nair, Purdue University. (By title)

Alternating priority queues get considerable attention these days: Avi-Itzhak, Maxwell, and Miller [*Operations Res.* **16** (1965) 306–318], Neuts and Yadin [Mimeo Series No. 136 (1968), Dept. of Statistics, Purdue Univ.], Takács [*Operations Res.* **16** (1968) 639–650]. The present paper considers a single server alternating between two service units which are in tandem. It is assumed that the distribution of input is exponential and that of service time is general. Using Laplace transforms, the distributions of virtual waiting time and queue length process are studied and their asymptotic moments are obtained. The cases zero switching and non-zero switching [Neuts and Yadin (1968)] are treated separately. The generalization to more than two units and the possible applications are discussed. (Received 20 June 1969.)

55. The limiting behavior of multiple roots of the likelihood equation. MICHAEL D. PERLMAN, University of Minnesota.

$X_1, X_2, \cdots$ are i.i.d. random variables with generalized pdf $p(x, \theta)$, where θ is a real unknown parameter with range $\Omega = (a, b)$ either open or closed, bounded or unbounded. $S(x_1, \cdots, x_n)$ is the set of all solutions of the likelihood equation, given $X_1 = x_1, \cdots, X_n = x_n$. The results of Huzurbazar (*Ann. Eugenics* **14** (1948)) are extended by showing that (under a weaker regularity condition) with probability one all members of $S(x_1, \cdots, x_n)$ except one are bounded away from the true value θ_0 as $n \to \infty$, and the exceptional solution is a relative maximum and approaches θ_0. Furthermore, if $(d/d\theta)I(\theta_0, \theta) < 0 (> 0)$ for all $\theta < \theta_0$ $(> \theta_0)$, where $I(\theta_0, \theta) = E_{\theta_0} \log [p(X, \theta_0)/p(X, \theta)]$, then all members of $S(x_1, \cdots, x_n)$ but one approach $\partial\Omega$. If in addition $|(d/d\theta)I(\theta_0, \theta)| \geqq A > 0$ as $\theta \to \partial\Omega$ then $S(x_1, \cdots, x_n)$ contains exactly one solution for large n. Under still weaker conditions our methods imply the existence (for large n) and strong consistency of the MLE, providing an alternate derivation of Wald's (1949) result. Applications are made to estimation of location or scale parameters. Special cases considered are the Cauchy density and the bivariate normal density with unknown correlation and known variances. Questions raised by Barnett (*Biometrika* **53** (1966)) in the Cauchy case are discussed. (Received 20 June 1969.)

56. Ordered alternatives: a review. D. J. BARTHOLOMEW, University of Kent at Canterbury. (Invited)

Methods estimating parameters subject to order restrictions. Likelihood ratio tests for the equality of parameters subject to order restrictions. Distribution theory of the tests. Extension to orthogonal and non-orthogonal designs in the analysis of variance. Power of the tests. Approximate methods. Distribution-free tests, Scaling methods. Applications. (Received 20 June 1969.)

57. Age distributions. M. S. BARTLETT, Oxford University. (Invited)

Age is an essential variable in determining the rate of replacement of an article or of Fellows of a learned society, the division of a cell or the reproductive capacities of an individual. The derivation of theoretical age distributions in all these cases is discussed, including stationary distributions appropriate to society recruitment (illustrated by reference to the Royal Society, London), cell division and populations with two sexes. (Received 20 June 1969.)

58. Decision-theoretic tests for linear hypotheses, with replacements for the usual F-ratio test. JAMES M. DICKEY, State University of New York at Buffalo. (Invited)

Two classes of practical Bayesian tests for linear hypotheses are surveyed: (1) weighted likelihood ratio tests, first conceived by Jeffreys and recently adapted for ready exploitation of Raiffa and Schlaifer's conjugate-prior theory; (2) test procedures resulting from an imbedding of the two-decision problem within a prediction or estimation problem having various structures. The loss structures considered in (2) are squared-error, absolute-error, and all-or-nothing loss; leading, respectively, to the posterior points of interest, the mean, the median, and the modes. Again, the conjugate-prior theory is exploited. Stable estimation applies readily in both (1) and (2). Replacements are given for the popular and elegant, yet often misleading, F-ratio test with the usual normal-theory multiple regression model. (Received 20 June 1969.)

59. Identification and input signal synthesis problems in discrete-time stochastic control systems. MASANAO AOKI, IBM Research Laboratory. (Invited)

Estimation of unknown parameters in control systems is considered from noisy measurements. The system is governed by a finite difference equation $x_t + \sum_{i=1}^{k} a_i x_{t-i} + \sum_{i=1}^{k} b_i u_{t-i} = 0$, $t = 1, 2, \cdots$, where a_i and b_i, $i = 1, \cdots, k$, are unknown constant parameters and where u's are the known input signals. The measurements are obtained by $y_t = x_t +$ noise, $t = 1, 2, \cdots$. The asymptotic properties of the estimates depend strongly on the properties of input signals employed and on the properties of the system. The input signal synthesis problem is to select a sequence of inputs to produce estimate with desirable properties. Necessary and sufficient conditions on the inputs and the system for the maximum likelihood estimate to be asymptotically unbiased and efficient are stated. These conditions are shown to give convergence with probability one as well as mean square convergence of the estimate. Other estimates and the dependence of their asymptotic properties on the input sequences are also discussed. (Received 20 June 1969.)

60. A characterization of the exponential distribution. ANDRE G. LAURENT AND RAMESH C. GUPTA, Wayne State University.

Let $Y_1, \cdots, Y_n$ be an ordered sample of a random variable (rv) X, with density function (df) $f(x)$, defined and continuous in $(0, \infty)$. THEOREM. *If Y_i and $Y_k - Y_j$ are independent, for a specific set $i \leqq j < k \leqq n$, then $f(0) \neq 0$ and f is exponential.* Hence, under mild assumptions, given an ordered sample $U_1, \cdots, U_n$ of a rv Z, there exists only one "scale" $h(z)$ such that $h(U_i)$ and $h(U_k) - h(U_j)$ are independent, and the df of Z is $g(z) = \exp(-h(z))h'(z)$. As $i \leqq j < k \leqq n$ are any specific integers the theorem suggests tests of exponentiality. A Poisson process interpretation follows easily. The paper generalizes and (or) corrects results by A. P. Basu (*Ann. Inst. Statist. Math.* **17** (1965)). (Received 24 June 1969.)

61. An application of incomplete block designs to the construction of error correcting codes. K. J. C. SMITH, University of North Carolina.

This paper presents an application of incomplete block designs to the construction of error-correcting codes which may be decoded using a relatively simple majority logic decoding procedure. An identity matrix, I, is adjoined to the incidence matrix, N, of a balanced or partially balanced incomplete block design. The resulting matrix is taken as the parity check matrix of a linear code; a relatively simple majority logic decoding procedure for error correction may be used for the code. (Received 24 June 1969.)

62. A new sequential procedure for selecting the best one of K binomial populations. EDWARD PAULSON, Queens College.

Let p_i denote the probability of a success with category $\pi_i (i = 1, 2, \cdots, k)$, denote the ordered probabilities by $p_{[1]} \leqq p_{[2]} \leqq \cdots \leqq p_{[k]}$, let $\pi_{[j]}$ have probability $p_{[j]}$. A sequential procedure for selecting the best category so that $P[\pi_{[k]}$ is selected $| p_{[k]} - p_{[k-1]} \geqq d] \geqq 1 - \alpha$ was given by the writer (*Ann. Math. Statist.* **38** 117–123). A more complicated procedure is given which substantially reduces the average sample size. Let $\{N_{ir}\}$, $(i = 1, 2, \cdots, k; r = 1, 2, \cdots)$ be a sequence of independent random variables with Poisson distribution with mean J, let S_{ir} and F_{ir} denote the number of successes and failures when N_{ir} measurements are taken with π_i, let $\bar{S}_{i,r} = \sum_{\nu=1}^{r} S_{i\nu}/r$ and $\bar{F}_{i,r} = \sum_{\nu=1}^{r} F_{i\nu}/r$. Let $g_{ij}^{(r)}$ denote the root of the equation $-\log(1 - x^2)/x = 1.5\, dJ[\bar{S}_{i,r-1} + \bar{S}_{j,r-1}]^{-1}$ when $\bar{S}_{i,r-1} + \bar{S}_{j,r-1} > 0$, otherwise $g_{ij}^{(r)} = .5$, and let $h_{ij}^{(r)}$ denote the root of $-\log(1 - x^2)/x$

$= 1.5\, dJ[\bar{F}_{i,r-1} + \bar{F}_{j,r-1}]^{-1}$ for $\bar{F}_{i,r-1} + \bar{F}_{j,r-1} > 0$, otherwise $h_{ij}^{(r)} = .5$. Let $T_r(i, j, \alpha) = A_r/B_r$, where $A_r = \log [(k-1)/\alpha] - \sum_{\nu=2}^{r} \{S_{i\nu} \log (1 - g_{ij}^{(\nu)}) + S_{j\nu} \log (1 + g_{ij}^{(\nu)}) + F_{i\nu} \log (1 + h_{ij}^{(\nu)}) + F_{j\nu} \log (1 - h_{ij}^{(\nu)})\}$ and $B_r = J \sum_{\nu=2}^{r} [g_{ij}^{(\nu)} + h_{ij}^{(\nu)}]$. At the first stage take $N_{11}, N_{21}, \cdots, N_{k1}$ measurements from $\pi_1, \pi_2, \cdots, \pi_k$ respectively. At the rth stage $(r = 2, 3, \cdots)$ take N_{jr} measurements from each category π_j not yet eliminated, then eliminate each category π_i for which $T_r(i, j, \alpha) < d$ for some $j \neq i$. When $(k-1)$ categories are eliminated the experiment terminates and the remaining category is selected. (Received 24 June 1969.)

63. Relations between Pitman efficiency and Fisher information.

M. H. DeGroot and M. Raghavachari, Carnegie-Mellon University.

Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables, each with probability density $f(x, \theta)$, where θ is a real-valued parameter. Let $P_n = \prod_{i=1}^{n} f(x_i, \theta)$ and $Z_n(\theta) = (\partial/\partial\theta) \log P_n$. Consider the problem of testing $H: \theta = \theta_0$ against the alternative $K: \theta > \theta_0$. Let T_n be a measurable function of $X_1, \cdots, X_n$, and let $\varphi_n(t, \theta)$ be the density of T_n. Define $Y_n(\theta) = (\partial/\partial\theta) \log \varphi_n$. It is well known that the locally most powerful test based on T_n rejects H when $Y_n(\theta_0)$ is too large. If T_n and T_n^* are two test statistics satisfying the usual assumptions required for the existence of Pitman's asymptotic relative efficiency $\mathcal{E}(T_n, T_n^*)$ of T_n relative to T_n^*, then it is shown that $\mathcal{E}(T_n, T_n^*)$ is given by the limit of the ratio $\rho^2[T_n, Z_n(\theta_0)]/\rho^2[T_n^*, Z_n(\theta_0)]$ of squared correlations. It is also shown that $\mathcal{E}(Y_n(\theta_0), Y_n^*(\theta_0))$ is given by the limit of the ratio $i_{T_n}(\theta_0)/i_{T_n^*}(\theta_0)$ of Fisher information values. Other results are developed relating the efficacy of a test statistic for any finite sample size and Fisher information. These results throw new light on the concept of Pitman efficiency. (Received 25 June 1969.)

64. Curve crossings by random processes.

Raoul D. LePage, Columbia University.

In deriving his expression for the expected number of zero-crossings $E(N)$ exhibited by a sample continuous random process over a finite interval, Leadbetter (*Ann. Math. Statist.* **37** 260–267) has so arranged things that his useful particularization of the general expression involves an assumed uniform limit of the joint density of $Y(t)$ and $(Y(t+\beta) - Y(t))/\beta$ as β tends to zero, where $Y(t)$ is the coordinate random variable at time t. The thrust of the present work is to develop expressions for $E(N)$, and the higher moments as well as certain related expectations, which do not assume sample continuity. The formulae developed are quite simple and apply to such processes as the Poisson, in addition to the normal and other continuous-type processes for which the densities assumed by Leadbetter are well defined. This work much extends that reported by the author in *Ann. Math. Statist.* **39** 1782, in that different methods have been required to extend to the case where N may be infinite. (Received 26 June 1969.)

65. Unbiased estimation of scale parameter of Cauchy density.

Rajinder Singh, University of Saskatchewan.

Let X be a Cauchy random variable having density $f(x; \sigma) = \sigma[\pi(\sigma^2 + x^2)]^{-1}$, $-\infty < x < \infty$ where $\sigma > 0$ is an unknown scale parameter. It is shown that there is no unbiased estimator $\varphi(x)$ of σ for any measurable function φ. The proof consists in showing that the family of densities $\{g(y; \sigma) = 2\sigma[\pi(\sigma^2 + y^2)]^{-1}$ if $y > 0$ and $= 0$ if $y \leq 0\}$ is complete and then using Theorem 2.2 of Ghosh and Singh [*Ann. Math. Statist.* **37** (1966) 1671]. (Received 26 June 1969.)

66. The two-armed bandit with time-invariant finite memory. MARTIN E. HELLMAN AND THOMAS M. COVER, Stanford University.

Let Θ be an infinite family of experiments. The outcome Y of the experiment $\theta \varepsilon \Theta$ is an r.v. drawn according to a probability measure P_θ or Q_θ accordingly as hypothesis H_1 or H_2 is true. Let the payoff $J(Y)$ result. Suppose that one is allowed to select experiments θ_n, $n = 1, 2, \cdots$, sequentially subject to the constraint that $\theta_n = \theta(T_n)$, θ: $\{1, 2, \cdots, m\} \rightarrow \Theta$, is a function solely of the current value of an m-valued statistic T_n, where T_n is updated according to the rule $T_n = f(T_{n-1}, e_{n-1}, Y_{n-1})$. It is desired to maximize $E\{\lim (1/n) \sum_1^n J(Y_i)\}$ over all rules (f, θ). This supremum is found as a simple function of $\{P_\theta\}$ $\{Q_\theta\}$; and a family of ϵ-optimal (f, θ)'s is exhibited. (Received 26 June 1969.)

67. Further comments on the distribution of the multiple correlation coefficient. JOHN GURLAND AND ROY MILTON, University of Wisconsin.

In a previous article, one of the authors (*J. Roy. Statist. Soc. Ser. B* (1968)) presented three series expansions for the general distribution of the multiple correlation coefficient. For $N - p$ even (N is the sample size and p is the dimension of the relevant normal random vector) one of these series is finite with weighting coefficients which are binomial probabilities. In the present paper a general series expansion for the distribution is presented which contains the three aforementioned series as special cases. The rapidity of convergence is investigated and various approximations to the distribution are considered. (Received 26 June 1969.)

68. Asymptotically optimal tests in Markov processes. RICHARD A. JOHNSON AND GEORGE G. ROUSSAS, University of Wisconsin.

Let $\{X_n, n \geqq 0\}$ be a discrete parameter Markov process defined on $(\mathfrak{X}, \mathfrak{A}, P_\theta)$ and taking values in the Borel real line $(R, \mathfrak{B})$. The parameter space Θ is an open subset of the k-dimensional Euclidean space $\mathcal{E}_k$. Under suitable conditions on the process, the following results are derived. Let θ_0 be any fixed point in Θ and let $\Delta_n(\theta_0)$ be related to the quadratic mean derivative of the likelihood based on $X_0, X_1, \cdots, X_n$; $\Delta_n^*(\theta_0)$ denotes a certain truncated version of $\Delta_n(\theta_0)$. A probability measure $R_{n,h}$ is defined in terms of $\Delta_n^*(\theta_0)$ and $h \epsilon \mathcal{E}_k$, for each $n \geqq 0$. Then it is shown that the sequences of measures $\{P_{n,\theta}\}$ and $\{R_{n,h}\}$ are differentially equivalent at θ_0 and $\{\Delta_n^*(\theta_0)\}$ is differentially sufficient for $\{P_{n,\theta}\}$ at θ_0, where $h = n^{\frac{1}{2}}(\theta - \theta_0)$.

In hypotheses testing problems having alternatives of the form $\theta = \theta_0 + n^{-\frac{1}{2}}h_n$, where $\{h_n\}$ is a bounded sequence in $\mathcal{E}_k$, Theorem 6.1 allows one to restrict attention to tests based on $\Delta_n(\theta_0)$ for asymptotic power considerations. In the real-valued parameter case, Theorem 7.1.1 covers asymptotically most powerful tests of $\theta = \theta_0$ versus one-sided alternatives and Theorem 7.1.2 treats asymptotically most powerful unbiased tests for two-sided alternatives. (Received 26 June 1969.)

69. Tests for bivariate stochastic ordering. RICHARD A. JOHNSON AND G. K. BHATTACHARYYA, University of Wisconsin.

We consider an extension to the bivariate case of the problem for testing for ordered alternatives. Let $F(x, y)$ and $G(x, y)$ be continuous bivariate cdf's. Three definitions of ordered alternatives are considered. These are: (i) marginals stochastically ordered, (ii) $F \not\equiv G$ with $F(x, y) \geqq G(x, y)$ or $\bar{F}(x, y) \leqq \bar{G}(x, y)$, and (iii) $F \not\equiv G$ with $F(x, y) \geqq G(x, y)$ and $\bar{F}(x, y) \leqq \bar{G}(x, y)$ where $\bar{F}(x, y) = 1 - F(x, \infty) - F(\infty, y) + F(x, y)$. Each class contains the preceding class. Examples of the last class include positive translations and bivariate Lehmann-type alternatives. A permutation test based on layer ranks is proposed and shown to be unbiased. A large sample unconditional test is obtained which approximates the original test. It is shown to be consistent and an expression is given for the Pitman efficacy. (Received 26 June 1969.)

70. On the bivariate doubly non-central *t*-distribution. S. A. PATIL AND J. L. KOVNER, Rocky Mountain Forest and Range Experiment Station.

In this paper the joint distribution of $t_i = Z_i[Yn^{-\frac{1}{2}}]^{-1}$ $(i = 1, 2)$ is considered when Z_1, Z_2 are independent of Y and have a nonsingular bivariate normal distribution with means μ_1, μ_2, equal variances σ^2 and correlation coefficient ρ. σ^2Y^2 has a non-central chi-square distribution with n degrees of freedom and non-centrality parameter λ. The doubly non-central t-distribution considered by Krishnan (1967) is the marginal distribution of the present distribution. The distribution function is reduced to a doubly infinite series containing a single integral, for large n it can be expressed in a series containing powers of n involving the incomplete moments of the bivariate normal distribution. For $\mu_1 = \mu_2 = 0$, the distribution function can be expressed in a double series containing an incomplete beta function. It is also shown that when $\mu_1 = \mu_2 = 0$ the probabilities in the rectangular region is a monotone function of ρ. The product moments of order k_1, k_2 are considered. (Received 26 June 1969.)

71. Multiple comparison of regression functions. EMIL SPJØTVOLL, University of California at Berkeley.

The problem of multiple comparison of all regression functions in a certain subset of regression functions is studied. It is shown that it reduces to multiple comparison of quadratic functions in an unknown parameter vector. An analogous, but technically simpler, problem, is the problem of multiple comparison of linear functions of an unknown parameter vector, which is solved by e.g. the S-method and T-method of multiple comparison in the analysis of variance. In the paper is given a solution of the problem of multiple comparison of quadratic functions, using the fact that one has a confidence ellipsoid for the parameter vector. The solution is numerically more complicated than the corresponding solutions for linear functions. In the situation with regression analysis some simplification is obtained. (Received 26 June 1969.)

72. Notes on the distribution of quadratic forms in singular normal variables. GEORGE P. H. STYAN, University of Minnesota.

Necessary and sufficient conditions are obtained for a quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ to be distributed as χ^2, where $\mathbf{x}$ follows a multivariate normal distribution with covariance matrix $\mathbf{C}$, possibly singular, and where $\mathbf{A}$ is a symmetric matrix, not necessarily semi-definite. In the central case it is shown that the necessary and sufficient condition $\mathbf{CACAC} = \mathbf{CAC}$ reduces to $\mathbf{AC} = (\mathbf{AC})^2$ if and only if rank $(\mathbf{AC})$ = tr $(\mathbf{AC})$ or rank $(\mathbf{AC})$ = rank $(\mathbf{CAC})$, where r = tr $(\mathbf{AC})$ = rank $(\mathbf{CAC})$ is the number of degrees of freedom. In the non-central case additional restrictions on the mean vector $\boldsymbol{\mu}$ are needed when $\mathbf{C}$ is singular. These reduce to $\boldsymbol{\mu}'\mathbf{ACA}\boldsymbol{\mu} = \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$ if and only if rank $(\mathbf{AC})$ = tr $(\mathbf{AC}) = r$, or rank $(\mathbf{AC})$ = rank $(\mathbf{CAC}) = r$. For independence of two quadratic forms extra conditions are also required for $\boldsymbol{\mu}$ when $\mathbf{C}$ is singular, but these are not needed when the forms are semi-definite. Finally, Cochran's theorem is generalized to the case of singular covariance matrix $\mathbf{C}$, with $\boldsymbol{\mu} = \mathbf{0}$ or the quadratic forms positive semi-definite. (Received 26 June 1969.)

73. Uncertainty functions and orderings of experiments. DORIAN FELDMAN, Michigan State University.

Let Ω be a set of states, Ξ the class of prior distributions on Ω. Following DeGroot (*Ann. Math. Statist.* **33** (1962) 404–419), a nonnegative, continuous concave function on Ξ is called an uncertainty function and if $X = (\mathfrak{X}, \mathfrak{A}, P_\theta, \theta \varepsilon \Omega)$ and $Y = (\mathfrak{Y}, \mathfrak{B}, Q_\theta, \theta \varepsilon \Omega)$ are two experiments, X is called more informative than Y with respect to U if $U(\xi \mid X) \leqq U(\xi \mid Y)$ for all $\xi \varepsilon \Xi$, where $U(\xi \mid X)$ is the expected posterior uncertainty for an observation on X when the prior is $\xi \varepsilon \Xi$. Any such function U induces a partial ordering on the class of all experiments. It is shown here that the ordering induced by U is in fact total if and only if U is a geometric mean. In the finite state case the comparison amounts to comparing values of the Laplace transforms of the experiments. (Received 26 June 1969.)

74. A note on the asymptotic normality of the distribution of the number of empty cells in occupancy problems. B. HARRIS AND C. J. PARK, University of Wisconsin and University of Nebraska.

Assume that we have a random sample of n observations from a multinomial distribution with N equiprobable cells. Let s be the number of empty cells. In this note we present two new proofs of the asymptotic normality of s when n and N tends to infinity so that $n/N \to c > 0$, $n/N^{\frac{1}{2}} \to \infty$, or $n/N - \frac{1}{3}\log N \to -\infty$. We accomplish this by estimating the factorial cumulants of s. (Received 26 June 1969.)

75. Robust procedures for variance component problems using the jackknife. JAMES N. ARVESEN, Purdue University.

In a recent article (*Ann. Math. Statist.* **39** (1968) 567–582) Miller showed that the jackknife technique can be used to obtain asymptotic tests or confidence intervals for variances. Unlike the standard procedures involving use of chi-square or F-tests, the procedure described was shown to be robust against non-normality. In the present paper, these results are extended to problems involving variance components. In particular, the model $Y_{ij} = \mu + a_i + e_{ij}$, $i = 1, \cdots, I$, $j = 1, \cdots, J_i$ where $\{a_i\}$ i. i. d. with variance τ^2 and 4th moment, $\{e_{ij}\}$ i. i. d. with variance σ^2 and 4th moments is discussed. Since the theoretical results deal only with asymptotic behavior, a monte carlo simulation was also made between the jackknife technique and the F-test. Finally it is shown the jackknife technique can be extended to more complicated and higher way layouts. (Received 26 June 1969.)

76. A characteristic property of the multivariate normal distribution and some of its applications. G. P. PATIL AND M. T. BOSWELL, Pennsylvania State University.

This paper generalizes and regionizes the work of Price, Popoulis, Brown and Plackett. THEOREM 1 (an extension of Price's theorem). *Let* $\mathbf{X} \equiv (X_1, \cdots, X_n)$ *have an n-variate normal distribution with density function f and let g be an n-dimensional function of bounded variation possessing partial derivatives of second order satisfying* $x_i - g(\mathbf{x})f(\mathbf{x}) \to 0$ *as* $|x_i| \to \infty$, $i = 1, 2, \cdots$. *If there exists function* $\{h_i\}$ *such that* $|x_i - g(\mathbf{x})f(\mathbf{x})| \leqq h_i(x_i)$ *and* $\int h_i(x)\,dx < \infty$, *then* $\partial E[g(\mathbf{X})]/\partial\sigma_{jk} = E[\partial^2 g(\mathbf{X})/\partial X_j \partial X_k]$ if $j \neq K$; $\partial E[g(\mathbf{X})]/\partial\sigma_{jk} = \frac{1}{2}E[\partial^2 g(\mathbf{X})/\partial X_j^2]$ if $K = j$. THEOREM 2. *If Theorem* 1 *holds, then* $\mathbf{X}$ *has an n-variate normal distribution.* Other related characterizations are available. Applications using the former theorem to evaluate specific expectations are given. Also, some conditions on moments which imply independence in the normal case are given. (Received 27 June 1969).

77. Construction of some partially balanced weighing designs with two associate classes. K. V. SURYANARAYANA, University of North Carolina.

Partially Balanced Weighing Designs (PBWD) have arisen as some extensions of balanced weighing designs developed by Bose and Cameron (*J. Res. Nat. Bur. Standards Sec. B.* **69** (1965) 323-32 and **71** (1967) 149–60.) A PBWD $(v, b, r, p, \lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22})$ with a given two associate class is an arrangement of v objects to be weighed in b weighings, with p objects on each pan, the frequencies for an ith associate to appear together or in opposite pans being respectively λ_{1i}, λ_{2i} $(i = 1, 2)$. The construction of these designs is considered in this paper for the cases of the following association schemes: (a) group divisible scheme, (b) triangular scheme and (c) Latin square scheme. Apart from some general results of constructing these designs from PBIB designs and some existing PBWD, methods are developed to construct these by making use of the structure of the particular association schemes. One of the general results can be described thus. THEOREM: *If there exists a resolvable PBIB design* $(v, rt, r, k, \lambda_1, \lambda_2)$ *then a PBWD* $(v, r\binom{t}{2}, r(t-1), k, \lambda_1(t-1), (r-\lambda_1), \lambda_2(t-1), (r-\lambda_2))$ *can be constructed from it.* (Received 27 June 1969.)

78. On a selection procedure concerning the t best populations. JERONE N. DEVERMAN and SHANTI S. GUPTA, Sandia Laboratories and Purdue University.

A set (subset, collection, etc.) is an n-set (n-subset, n-collection, etc.) if it has exactly n elements. We consider k populations and a binary relation which yields a complete ordering so that for any integer t $(1 \leqq t \leqq k)$, there is a unique t-subset comprising precisely the t best of the k populations. For fixed integers t, s $(1 \leqq s \leqq k)$, and c $(\max[1, s+t+1-k] \leqq c \leqq \min[s, t])$ we define an interesting s-subset to be any s-subset having at least c of the t best populations. Given a fixed probability P*, we require a procedure R_s that will select a (non-empty) subcollection of the $\binom{k}{s}$-collection of s-subsets such that at least one interesting s-subset is selected with probability P^*. That such a (trivial) procedure exists is clear since selecting the entire $\binom{k}{s}$-collection suffices. However, for completely ordered classes of populations admitting statistics which are stochastically ordered in the same way, a general non-trivial procedure R_s possessing certain optimal properties is proposed and investigated. Procedures previously proposed by Gupta and others for selecting a subset to contain the best population are shown to be special cases of the generalized procedure R_1. (Received 27 June 1969.)

79. A class of non-eliminating sequential multiple decision procedures. AUSTIN BARRON and SHANTI S. GUPTA, Ohio State University and Purdue University.

This paper is concerned with the ranking and selection of k independent normal populations having means $\theta_1, \cdots, \theta_k$ and a common known variance σ^2. In addition, a function of the differences of the means $\theta_i - \theta_j$, $i \neq j$, is also assumed known. A class of sequential and multi-stage procedures using the "subset selection" approach is defined and investigated. The class consists of rules of a non-eliminating type; a rule belonging to this class selects and rejects populations at various stages but continues taking samples from all populations until the procedure terminates. Exact and approximate expressions for the probability of selecting a population and the expected number of stages to reach a decision are found using a random walk approach. An approximate minimax rule for choosing a specific procedure that minimizes the maximum number of samples needed to make a decision on each population, among those procedures guaranteeing certain probability conditions, is discussed. Finally, some comparisons to a fixed-sample size procedure are offered. (Received 27 June 1969.)

80. Robustness of estimators and tests based on the Gauss-Markoff model. C. Radhakrishna Rao, Indian Statistical Institute. (Invited)

We consider estimators and tests of hypotheses based on the Gauss-Markoff model on a vector variable Y: $E(Y) = X\beta + \epsilon$, $D(Y) = \sigma^2 I$, and Y_i, the ith component of Y, is normally distributed. Properties of these estimators and tests are examined when deviations from the assumed model occur. In particular the nature of the deviations for which the procedures based on the assumed model remain optimum have been determined. (Received 30 June 1969.)

81. Central limit theorems for dependent random variables and some applications. Aryeh Dvoretzky, Hebrew University. (Invited)

The limit laws of sums of large numbers of 'small' independent random variables have been exhaustively studied and necessary and sufficient conditions for convergence to a given law have been established. It is natural to try to weaken the independence assumption. Most of the available results are, however, of a somewhat fragmentary nature. We attempt a more systematic study and show, among other things, how to carry over the sufficiency part mentioned above to rather general situations. A sample result is the following: Let $x_{n,i}$ $(n = 1, 2, \cdots; i = 1, \cdots, k_n)$ be random variables and put $s_{n,i} = x_{n,1} + \cdots + x_{n,i}$. If $\sum_i E(x_{n,i} \mid s_{n,i-1}) \rightarrow_P 0$, $\sum_i (E(x_{n,i} \mid s_{n,i-1}))^2 \rightarrow_P 0$, $\sum_i E(x_{n,i}^2 \mid s_{n,i-1}) \rightarrow_P 1$ and $\sum_i \int_{|x_{n,i}|>\epsilon} x_{n,i}^2 \rightarrow 0$ for every $\epsilon > 0$, then s_{n,k_n} is asymptotically normal N (0, 1). Many extensions of the central limit theorem are special caes of this result. It also has numerous applications, e.g., a version of the three series theorem for dependent random variables, extensions of results on optimal stopping rules and various convergence theorems. (Received 30 June 1969.)

83. Some application of the method of structural inference to problems involving several unknown parameters. D. A. S. Fraser and F. Streit, University of Toronto.

An advantage of the theory of fiducial inference, a method of obtaining inference statements about unknown parameters, is that it does not rely on any knowledge of a priori distributions. However, the fiducial argument leads to inconsistent results for certain problems involving several unknown parameters. The method of structural inference (D. A. S. Fraser. *The Structure of Inference.* Wiley, New York (1968)) provides fiducial results, which are based on a more comprehensive statistical model (including an internal error variable and a group of transformations), but it avoids the above-mentioned paradoxical situations. For models with several parameters the marginal distributions of a single parameter and the distributions of combinations of these parameters are obtained by an appropriate integration of the over-all structural distribution and are unique within the framework of the chosen model. In this paper several applications of this technique are demonstrated: generalizations of the Behrens-Fisher problem are analyzed. For the bivariate analogue with normal error variable a single mean and also the difference of the two means are shown to follow a rescaled and relocated t-distribution. The Creasy-Fieller problem is also discussed and a structural distribution of a value x yielding a given response level of a quadratic regression-equation is derived. (Received 30 June 1969.)

84. ϵ-Bayes and infinite action results for the compound decision problem. ALLAN OATEN, Michigan State University.

The set version of the compound decision problem involves simultaneous consideration of N statistical decision problems, called the component problems, with identical generic structure: state space Ω, action space A, sample space $\mathfrak{X}$ and non-negative loss function L defined on $\Omega \times A \times \mathfrak{X}$. With $\mathbf{x} = (x_1, x_2, \cdots, x_N)$ distributed according to $\prod_{i=1}^{N} P_{\theta(i)} = P_\theta$, a compound procedure is a vector, $\varphi = (\varphi_1, \cdots, \varphi_N)$ such that, for each $i, \varphi_i : \mathfrak{X}^N \to A$. The risk of the procedure φ is $R(\theta, \varphi) = N^{-1} \sum_{r=1}^{N} \int L(\theta_r, \varphi_r(\mathbf{x}), x_r) P_\theta(dx)$ and the modified regret is $D(\theta, \varphi) = R(\theta, \varphi) - R(G_N)$ where G_N is the empirical distribution of $\theta(1)$, $\theta(2)$, $\cdots$, $\theta(N)$ and $R(G)$ is the Bayes risk against G in the component problem. In this paper we consider quite wide classes of procedures, φ, which consist of using $\mathbf{x}$ to estimate G_N and then playing ϵ-Bayes against the estimate in each component problem. For the $m \times n$ problem (i.e. Ω has m elements, A has n) we establish results analogous to those of Hannan and Robbins (*Ann. Math. Statist.* **26** (1955) 38–50, Theorem 3) for one class of procedures, and these results are used to get results of the form $D(\theta, \varphi) = o(1) + \epsilon$ as $N \to \infty$ for a still wider class of procedures in them $m \times n$ and $\times\ \infty$ problems. (Received 30 June 1969.)

85. Some robust selection procedures. RONALD H. RANDLES, University of Iowa.

Let $\mathbf{X} = \{X_{it}\}_{t=1,\cdots,n\ i=1,\cdots,k}$ be comprised of independent observations from k populations with respective cdf's $F(x - \theta_i)$. Lehmann (*Math. Ann.* **150** (1963)) proposed procedures for selecting the population with largest θ-value based on the ranks of the observations. However, he later showed that the slippage configuration used to find the A.R.E. was not least favorable. Puri and Puri (*Ann. Math. Statist.* **40** (1969)) restricted the set of θ-values to prevent this difficulty. If $\mathbf{x}_i = \{x_{it}\}_{t=1,\cdots,n}$ and $\Psi(\mathbf{x}_i, \mathbf{x}_j)$ is an estimate of $\theta_i - \theta_j$, selection procedures are considered which select the population corresponding to max $\{\Psi_1(\mathbf{x}), \cdots, \Psi_k(\mathbf{x})\}$ where $\Psi_i(\mathbf{x}) = k^{-1} \sum_j \Psi(\mathbf{x}_i, \mathbf{x}_j)$. The ith population is defined to be good if θ_i is within $\Delta > 0$ of the largest θ-value. Conditions are given on $\Psi(\mathbf{x}_i, \mathbf{x}_j)$ under which the slippage configuration is least favorable for the selection of a good population. For any $\Delta > 0$ and $\Psi(\mathbf{x}_i, \mathbf{x}_j)$, the Hodges-Lehmann (*Ann. Math. Statist.* **34** (1963)) estimator corresponding to the two sample F_0 scores test, the A.R.E. of the resulting procedure relative to the procedure based on the sample means is shown to be the Pitman efficiency of the F_0 scores test relative to the t-test. (Received 30 June 1969.)

86. A unified approach for a class of multivariate hypergeometric models. K. G. JANARDAN and G. P. PATIL, Pennsylvania State University.

A unified approach is introduced for the study of a class of multivariate hypergeometric models which includes multivariate hypergeometric, multivariate negative hypergeometric, multivariate inverse hypergeometric, multivariate negative inverse hypergeometric, multivariate Polya and the multivariate inverse Polya distributions. (Received 30 June 1969.)

87. Simple and robust estimation of the location parameter of a symmetric distribution. JAMES J. FILLIBEN, Princeton University.

The problem of determining (i) simple, and (ii) robust linear estimators of the location of a symmetric distribution is studied. 34 distributions are considered in the analysis: the rectangular, Gaussian, logistic, double exponential, and Cauchy distributions plus 29 members (viz., $\lambda = .9\ (-.1).\ 1(-.2) - .1(-.1) - 1.0\ (-.2) - 3.0$) of the λ distribution family (defined by the percent point function $R(p) = (p^\lambda - (1-p)^\lambda)/\lambda$). These 34 distributions are continuous, symmetric, and have tail lengths ranging from short to extremely long (100 times longer than Cauchy). The order statistic means and covariances are computed for the λ distribution for all 29 λ values and all sample sizes $n \leqq 20$. Complete and incomplete sample BLUE's are obtained for all 34 distributions and all $n \leqq 20$. Three near-optimal simple estimation procedures are introduced and evaluated: NBLUE, DJBLUE, and DJT. The optimal amount h (and optimal percent p) to be trimmed and Winsorized from each end of the ordered sample is determined for all 34 distributions, for all $n \leqq 20$, and for $n = \infty$. For the robustness analysis, the set of 34 distributions is decomposed into 21 subsets of distributions of varying tail length. For each of the 21 subsets, the optimal trimmed mean robust estimator, the optimal Winsorized mean robust estimator, and the optimal linear robust estimator are determined and compared. (Received 30 June 1969.)

88. Estimation in exponential distribution after preliminary tests of significance. M. M. DESU and PETER ENIS, State University of New York at Buffalo.

There are two two-parameter (negative) exponential distributions with location parameters μ and η and with common scale parameter σ. On the basis of two independent samples, we are interested in estimating μ. The proposed estimator is based on a preliminary test, T, of significance for the hypothesis $\mu = \eta$. The bias and the mean squared error of this estimator are obtained. The bias is always no greater than the bias of the usual estimator (sample minimum). The disadvantage coefficient (Mosteller, *J. Amer. Statist. Assoc.*, 1948, 231–242) in using the proposed estimator has been found to be a function of $\delta = (\mu - \eta)/\sigma$. The problem of estimating σ, after the preliminary test, T, of significance has also been considered. These results have been generalized to the case where complete samples are not available, the situation often encountered in life testing. (Received 30 June 1969.)

89. A limit theorem for conditioned recurrent random walk attracted to a stable law. BARRY BELKIN, Cornell University.

Consider an ensemble of independent particles whose motion describes a random walk on Z^d, the d-dimensional lattice of integers. If A is an arbitrary subset of Z^d and the random walk is assumed recurrent (consequently $d \leqq 2$), then as time passes it becomes increasingly unlikely that any given particle has avoided A. Suppose, however, that at each stage attention is restricted only to those particles whose past history is such that A has been avoided and one considers the possible distortive effects of this conditioning on the asymptotic behavior of the particle motion. In particular, let A be finite and assume that the transition distribution is attracted to a stable law G_α of index α. Under the additional hypothesis that $\tilde{g}_A(0) \neq 0$, it is shown that the conditional distribution of the particles whose past motion has avoided the set A is also attracted (with the same norming constants) to a limit distribution H_α. (The condition $\tilde{g}_A(0) \neq 0$ is shown to be equivalent to requiring that a particle starting at the origin can reach the exterior of any finite set along a sample path which avoids A.) For $d = 1$ with G_α Cauchy and for $d = 2$ with G_α normal we show that $H_\alpha = G_\alpha$. For $d = 1$ and $\alpha = 2$, under certain further restrictions on A, G_α turns out to be a two-sided Rayleigh distribution. (Received 30 June 1969.)

90. Comparison of approximations for classical occupancy distributions. WILLIAM L. HARKNESS, Pennsylvania State University

Properties of the three-parameter family of discrete distributions having pdf of the form $P_x(r, n, p) = \binom{n}{x} \sum_{\nu=0}^{n-x} (-1)^\nu \binom{n-x}{\nu}[1 - (\nu + x)p/n]^r$, $x = 0, \cdots, n$, are described. This family, which includes the well-known classical occupancy distribution (see, for example, Feller, **1**, 3rd ed., 101–102) as a special case (corresponding to $p = 1$ and r a positive integer), has important applications in such areas as biology, epidemiology, sampling theory, etc. It is shown that $P_x(r, n, p)$ is a pdf for real numbers r and p and a positive integer n if either (i) $r > 0$ and $0 \leqq p \leqq 1$ or if (ii) $r < 0$ and $p < 0$. A binomial approximation for $P_x(r, n, p)$ is given and shown (for small values of r and n) to be vastly superior (in typical cases) to the usual Poisson approximation described, e.g., by Feller, 103–105. (Received 30 June 1969.)

91. An iterated logarithm theorem for some weighted averages of independent random variables (preliminary report). R. JAMES TOMKINS, Purdue University.

Let $X_1, X_2, \cdots$ be independent random variables, each with mean zero and variance one. Assume the existence of a sequence $0 < c_n = o((\log\log n)^{-\frac{1}{2}})$ and of a number $N > 0$ such that if $n \geqq N$ and $0 < |t|c_n \leqq 1$, then, for all integers $k \leqq n$, $\exp((t^2/(2n))(1 - |t|c_n)) < E \exp(tX_k n^{-\frac{1}{2}}) < \exp((t^2/(2n))(1 + |t|c_n/2))$. In particular, this condition is satisfied if the X_n-sequence is (i) identically distributed with normal distribution, or (ii) bounded such that $|X_n| \leqq M_n = o((n/\log\log n)^{\frac{1}{2}})$, or (iii) uniformly bounded. Let f be a real-valued continuous function on $[0, 1]$ with the property that, for some $a > 0$, the set $\{0 < x < a \mid f(x) = 0\}$ has Lebesgue measure zero. Then $\lim\sup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} \sum_{k=1}^{n} f(m/n)X_m \geqq \|f\|$ a.e., where $\|f\|$ denotes the L_2-norm of f on $[0, 1]$. Furthermore, equality is obtained in this relation if f has a power series representation. The method of proof is similar to that of Gaposhkin (*Theor. Probability Appl.* **10** (1965) 411–420). (Received 30 June 1969.)

92. On a characterization of the Poisson distribution. R. C. SRIVASTAVA and A. B. L. SRIVASTAVA, Ohio State University and National Council of Educational Research and Training, New Delhi.

Recently Rao (*Intern. Symp. Discrete Dist.* (1963) 320–332) has considered discrete models where an original observation produced by nature is subjected to a destructive process and we observe the undestroyed part of the original observation. Suppose the original observation produced by nature is distributed according to a Poisson distribution with parameter λ and the probability that the original observation n is reduced to r due to the destructive process is $\binom{n}{r}\pi^r(1 - \pi)^{n-r}, 0 < \pi < 1$. Now if Y denotes the resulting random variable, then it is easily seen that $P(Y = r) = P(Y = r \mid \text{undamaged}) = P(Y = r \mid \text{damaged}) = e^{-\lambda\pi}(\lambda\pi)^r/r!$. Later Rao and Rubin (*Sankhyā, Ser. A* (1964) 295–298) proved that this is a characterizing property of the Poisson distribution. In this paper, an elementary characterization of the binomial distribution is obtained: THEOREM. *If a discrete* r.v. X *follows a Poisson distribution with parameter* λ *and if* $s(r \mid n)$ *denotes the probability that an observation* $X = n$ *on* X *is reduced to* r *during the destructive process and, further, if* Y *denotes the resulting* r. v. *which takes values* $0, 1, \cdots$ *and* $P(Y = r) = P(Y = r \mid \textit{damaged}) = P(Y = r \mid \textit{undamaged})$, *then* $s(r \mid n) = \binom{n}{r}\pi^r(1 - \bar{\pi})^{n-r}$. We also consider a multivariate discrete model and based on this model, some interesting properties of the binomial and the bivariate Poisson distribution are proved. (Received 30 June 1969.)

93. Nonorthogonal fractions of the 4^3 factorial. PETER W. M. JOHN, University of Texas.

The 4^3 factorial design can be made into a 2^6 factorial by replacing each of the four level factors, P, Q, R by pairs of two level factors A and B, C and D, E and F. The three degrees of freedom for the main effect of P correspond to A, B and AB. The $3(2^{n-k})$ series of fractions may then be adapted to give useful fractions of the 4^3 design. Two examples are considered. A resolution III design, $3(4^{3-2})$, with twelve points is given by a $3(2^{6-4})$ fraction obtained as follows: we omit from the 2^4 in A, B, C and D the quarter defined by $I = AC = BD = ABCD$ and let $E = ABD$, $F = ACD$. This design may be run in three blocks of four points. A 48 point design, $3(4^{3-1})$ of resolution V is obtained by omitting from the complete 2^6 factorial the quarter defined by $I = ACE = BDF = ABCDEF$. (Received 30 June 1969.)

94. Infinite divisibility and reproductivity of certain probability distributions with an application to the mixtures and randomly stopped sums.
A. V. GODAMBE and G. P. PATIL., Pennsylvania State University.

Functional forms are discussed for certain reproductive and infinitely divisible discrete distributions. It is shown that the power series distribution is infinitely divisible if and only if the logarithm of its series function is also a series function. The equivalence of a Poisson stopped sum and a mixture of the Poisson distribution is studied and it is shown that a Poisson mixture is representable as a Poisson stopped sum if and only if the mixing distribution is infinitely divisible. (Received 30 June 1969.)

95. Efficiencies of a robust estimator of location.
VALERIE MIKE, Cornell University Graduate School of Medical Sciences.

For the problem of efficiency-robust estimation of location approximate versions of the optimally robust Pitman-type estimators given in Birnbaum and Laska (*J. Amer. Statist. Assoc.* **62** (1967) 1230–1240) were developed by Mike (Doctoral dissertation (1967), New York University). Consider the family of continuous, symmetric pdf's $\sigma_\lambda^{-1}\ ((x - \theta)\sigma_\lambda^{-1} \mid \lambda)$, with unknown parameters $\theta \varepsilon \Theta = \{\theta \mid -\infty < \theta < \infty\}$, $0 < \sigma_\lambda < \infty$, and $\lambda \varepsilon \Lambda$, a specified set of "shapes." The proposed estimators of θ, based on k sample quantiles and defined in the context of the model representing their asymptotic normal distributions, were shown to approach full efficiency, with increasing k, for each $\lambda \varepsilon \Lambda$. A Monte Carlo study has now been carried out to study their performance for moderate sample sizes. Efficiencies were obtained for $n = 30, 40, 50, 100$ and $k = 2, 6, 10, 20$, under the following shapes: normal, logistic, double-exponential, and contaminated normal (1%, 5%, 10%). Over all these shapes, the efficiencies obtained are approximately 88% or more for $n = 30$; these rise to approximately 91% or more for $n = 100$. Discussion of the method and results includes comparison with other estimators proposed for robustness. (Received 30 June 1969.)

96. A quantitative approach to robustness.
LAWRENCE O. HATCH and HARRY O. POSTEN, United States Department of Defense, Fort George G. Meade and University of Connecticut.

The research provides a method which may be used to rigorize and quantify the concept of statistical robustness. The technique is based upon the notion of "regions of robustness". Such regions are regions in the space of assumption alternatives within which the user of a statistical procedure is assured that he will make an error in his "criterion function" of a magnitude no more than a prespecified level. Certain concepts associated with regions of robustness are discussed and various properties of such regions are developed. Finally, four investigations of statistical procedures for testing or estimating population means are presented. Tables are available which enable one to identify regions of robustness of various appropriate levels for each of the procedures investigated. (Received 30 June 1969.)

97. On the admissible estimators for certain fixed sample binomial problems. BRUCE MCK. JOHNSON, The University of Connecticut. (By title)

Let $X_1, \cdots, X_q$ be independent binomial random variables, $b(n_i, p_i)$, $i = 1, \cdots, q$. Let f_i, $i = 1, \cdots, q$, be continuous real valued functions defined on [0, 1] and f a continuous real valued function defined on $[0, 1]^q$. The problems of estimating all the $f_i(p_i)$, with summed squared error loss, and of estimating $f(p_1, \cdots, p_q)$, with squared error loss, are considered (the squared error loss may be somewhat relaxed). The main result of this note is that for these problems, the classes of admissible estimators are closed in the topology of pointwise convergence of the estimators. Also, a procedure is given for constructing these classes. In the first problem it is shown that there is no Stein effect. That is, $(\delta_1, \cdots, \delta_q)$ is admissible if for each i, δ_i is admissible for estimating $f_i(p_i)$ based only on X_i. The same results regarding closure and construction are also established for the analogues of the above problems when the $X_1, \cdots, X_q$ are multinomial distributed. (Received 30 June 1969.)

98. On the loss of information in the recovery of interblock information. K. R. SHAH, University of Waterloo.

In any procedure of recovery of inter-block information some loss is incurred due to the fact that the ratio of between block variance to within block variance is not known but is estimated. For the procedures used by Stein (Research papers in Statistics ed. F. N. David (1966) 351–366) and by the present author (*Ann. Math. Statist.* **35** (1964) 1064–1078) the magnitude of this loss is examined for four balanced incomplete block designs. Since both the procedures ignore some between block degrees of freedom one might suspect that the loss of information with these procedures might be severe. However, when the ratio of these two variances is not too large the loss of information turns out to be very small with either procedure. This indicates that either procedure would not be very much inferior to the customary procedure due to Yates for which this loss has not yet been evaluated. Some comments are made about the model assumed by Stein. (Received 1 July 1969.)

99. Dilution and bioassay. Pure error and the structural model. D. A. S. FRASER and R. L. PRENTICE, University of Toronto.

Many of the statistical problems analyzed by Fisher are found on closer examination to be more thoroughly discubed by structural models (Fraser, *The Structure of Inference.* Wiley (1968)). The dilution series method of estimating the density of living organisms in a solution seems to be an outstanding exception eluding such description. On reexamining the dilution problem the authors find that it has perhaps the most clearcut and purest form of error variable describing the reaction of known units before the experimenter. The transformation in the structural model developed turns out to be no more than a label on one unit in an open ended sequence of units under investigation. An analogous experimental description is presented for the bioassay problem. The paper develops structural distributions for the concentration of organisms in a given and for the LD50 amount of an administered drug, and it derives marginal likelihood for secondary parameters. (Received 1 July 1969.)

100. Some multivariate classification procedures (preliminary report). ELIZABETH H. YEN, Grumman Corporation.

The problem of classifying a new observation **Z** as coming from one of the two populations $F(\mathbf{X})$ and $G(\mathbf{Y})$, from which two samples have been drawn, has been discussed by various authors. In the special case that F and G are known except some parameters, the problem can be solved by the likelihood ratio criterion. We assume that F and G are continuous and different. However, their distributions are not known. From the samples, we form statistical equivalent blocks according to the values of the quadratic forms. (For example, in the bivariate case, the statistical equivalent blocks are a set of concentrated ellipses centered at its mean). Our quadratic forms are based on actual values or ranks of the observations. (Ranked on each axis separately). Then we compute the probabilities of exceedance of **Z** belong to F and G respectively. Classification is determined by these probabilities. Some simulated data shows that in the multivariate normal case, the results are as good as the likelihood ratio method. In other arbitrary populations, the results are better than the likelihood ratio method falsely assuming that F and G are normal. (Received 1 July 1969.)

101. Empirical priors are often flat (preliminary report). JAMES BONDAR, Indiana University.

In the structural models of Fraser (Fraser, *The Structure of Inference*), the distribution of the sample is a transformation of a fixed distribution. Using appropriate identifications, this is written $x = \theta \cdot E$, where x is an element in a group G of transformations, indexing a point in the sample space; θ in G indexes the unknown distribution of x; the dot indicates group multiplication, and E is a fixed random variable with values in G. In some situations θ is the product of many small independent randomly chosen transformations in G. Therefore, if G is such that the appropriately normalized distribution of $\theta = \theta_1 \cdot \cdots \cdot \theta_n$ converges to invariant measure (e.g. compact groups, translation group, scale group), then application of the Empirical Bayes method to our model will give an empirical prior distribution close to this invariant measure in the region of interest. This argument justifies Bayesian use of invariant priors in a way similar to using the Central Limit Theorem to justify using normal distribution methods when x is a sum of small independent random variables. (Received 1 July 1969.)

102. The form of marginal likelihood. WINSTON KLASS and D. A. S. FRASER, University of Toronto.

Fraser, (*The Structure of Inference*, Wiley (1968), Chapters 1, 4), defines families of statistical models, which are structural, conditional on the specification of certain additional parameters. The principle of marginal likelihood is then applied to estimate these parameters, given a response value. This paper elaborates on this principle, and relates it to the problem of data transformations. Four marginal likelihood functions are developed, three of which are based on a marginal probability element for the error orbit, given an arbitrary additional parameter value. The filament likelihood function, (proposed initially), and the sectional likelihood function, (corresponding to likelihood analysis conditional on the sufficient statistic), represent two interpretations of this probability element. The fibre likelihood function incorporates a standard classical treatment of the unknown error position, and the saddle likelihood function is a re-interpretation of a generalized 'Box and Cox' problem, (*J. Roy. Statist. Soc. Ser. B* **26** 211–252). The filament and sectional likelihood functions, in application to data transformations with a group structure, are modified to satisfy an 'invariance' requirement. Theoretical and numerical analysis of the family of power transformations, acting on selected regression models, favor the filament likelihood function. (Received 2 July 1969.)

103. A multiple-decision approach to the selection of the best set of predictor variates: II—Correlated predictors. JOHN S. RAMBERG, University of Iowa.

A multiple-decision formulation of a prediction problem where the goal is to select the "best" variate from a set of k predictor variates $(X_1, X_2, \cdots, X_k)$ is considered. The "best" predictor variate is defined to be that variate for which the predictand X_0 has the smallest population conditional variance. A $(k + 1)$-variate normal model is assumed and consideration is limited to single-stage procedures. Sample size requirements are obtained by using the asymptotic joint distribution of the transformed statistics. An exact asymptotic result is obtained for $k = 2$, while for $k > 2$ the Bonferroni inequality is used to obtain conservative sample size approximations. (Received 2 July 1969.)

104. Efficient detection of a change in distribution (preliminary report). GARY LORDEN, California Institute of Technology.

Suppose independent random variables $X_1, X_2, \cdots$, are observed sequentially with $X_1, \cdots, X_m$ having distribution function F and $X_{m+1}, X_{m+2}, \cdots$ having distribution function G (m unknown). If we want to take a certain action as soon as possible after the mth observation then it is reasonable to seek a stopping time N which minimizes the supremum over m of $E(N - m \mid N \geqq m)$ subject to a condition of the form $EN \geqq B$ when $m = \infty$. The problem is solved asymptotically as $B \to \infty$ by using stopping times based on repeated sequential probability ratio tests. Other formulations are considered, including the problem where G is replaced by a class of possible alternatives to F. The applicability of these considerations to a wide variety of practical problems is illustrated by examples. (Received 2 July 1969.)

105. A Bayesian analysis of some nonparametric functions. THOMAS S. FERGUSON, University of California.

A random probability measure is said to be selected by a Dirichlet process if the joint distribution of the values it gives to disjoint sets is Dirichlet. If a probability measure is chosen by a Dirichlet process, and if a sample is chosen from this probability measure, then the posterior distribution of the measure given the observations is also Dirichlet. This fact may be used to treat certain nonparametric statistical problems from a Bayesian point of view. Applications are given to the estimation of the mean, median, variance, and covariance, yielding results comparable to the classical theory. Certain other problems are treated, including a two-sample problem in which the Wilcoxon rank-sum statistic appears naturally. (Received 2 July 1969.)

106. On the efficiency of some median tests. THOMAS P. HETTMANSPERGER and GERALD SIEVERS, Pennsylvania State University and Western Michigan University.

Let S be Mathisen's two-sample median statistic (*Ann. Math. Statist.* **14** (1943) 188–194), let M be Mood's two-sample median statistic (*Ann. Math. Statist.* **25** (1954) 514–522) and let W be the Mann-Whitney statistic (*Ann. Math. Statist.* **18** (1947) 50–60). These tests are used for one-sided shift alternatives. In this paper we make exact Bahadur efficiency comparisons of these statistics when the underlying distributions are normal, Laplace, or Cauchy. In all cases M is preferable to S. For Laplace and Cauchy, S is locally more efficient than W, but the reverse is true for large shifts of one distribution with respect to the other. M is more efficient than W for all shifts in the Cauchy case and for small shifts in the Laplace case. For the normal case W is more efficient than both S and M for all shifts, although M becomes as efficient as W for large shifts. It is also interesting to note that the curve of the approximate Bahadur efficiency of S with respect to W lies below and follows the exact efficiency curve. Hence the approximate efficiency provides the same kind of information for the relative behavior of the two statistics as the exact efficiency. (Received 2 July 1969.)

107. A sequential Wilcoxon test (preliminary report). W. J. HALL, University of North Carolina and University of Rochester.

We consider testing whether two populations are identical (H_0) vs. the hypothesis (H_Δ) of a location shift of $\Delta \cdot \int f^2$ units, where f is the density. A sequential rank test, with decisions at each stage of sampling based on the ranks in a combined ranking of all the accumulated data, is described. The test is based on two statistics, the *Wilcoxon statistic* and the conditional variance (under H_0) of the Wilcoxon statistic given the ranks at the previous stage. Since the second statistic converges a.s. to a constant, an approximate version of the test is a bona fide Wilcoxon test. The following asymptotic (as $\Delta \downarrow 0$) results are obtained: (i) the true error probabilities α_Δ and β_Δ are bounded above by $\alpha + O(\Delta)$ and $\beta + O(\Delta)$, respectively (for given α, β); (ii) asymptotic formulas for the ASN are derived; (iii) asymptotically, the test has the *Wald optimal property* as a test of H_0 vs. *logistic* shift alternatives. General methodology, utilizing embedding techniques [Hall, *Ann. Math. Statist.* **39** (1968) 1777], is described which can be applied to other sequential rank test problems, such as normal scores and rank correlation. Some empirical studies are also reported. (Received 2 July 1969.)

108. An application of variational methods to experimental design. PAUL I. FEDER and REIJI MEZAKI, Yale University. (By title)

Consider the regression model $Y_i = f(x_i; \theta) + e_i$, $i = 1, \cdots, n$ where f is a known function, x is a k-dimensional controlled variable, θ is a p-dimensional unknown parameter to be estimated by least squares, and e_i are i.i.d. disturbances with mean 0 and unknown variance σ^2. It is desired to select the design variables $x_1, x_2, \cdots, x_n$ so as to minimize various functionals of the asymptotic variance covariance matrix. It is shown via several examples that an asymptotic version of the problem can sometimes be transformed into a form suitable for solution by the calculus of variations. Several examples are presented in which locally optimal designs relative to the determinant and trace criteria are derived by this technique. An example is considered which illustrates that even in cases too complicated for analytical solution, variational methods can reduce the problem to such a degree that it is suitable for solution by computer search. (Received 2 July 1969.)

109. A sequential procedure for ranking multivariate normal populations. M. S. SRIVASTAVA and V. S. TANEJA, University of Toronto and New Mexico State University.

Consider k p-variate normal populations π_i with means μ_i and common covariance matrix Σ, i.e., $\pi_i : N(\mu_i, \Sigma)$. The problem is to design a sequential procedure to rank these populations with respect to some distance function. We consider two distance functions $\mu_i'\mu_i$ and $\mu_i'\Sigma^{-1}\mu_i$. Procedures on the lines of Chow and Robbins [*Ann. Math. Statist.* (1965)], Paulson [*Ann. Math. Statist.* (1964)] and Hoel and Majumdae [*Ann. Math. Statist.* (1968)] are obtained. (Received 2 July 1969.)

110. Some asymptotic properties of U-statistics. RAYMOND N. SPROULE, University of California at Davis.

Hoeffding (Univ. of North Carolina, Inst. of Statist., Mimeo Series, No. 302 (1961)) showed that a U-statistic, U_n, may be expressed as an average of independent and identically distributed random variables plus a remainder term. We develop a Kolmogorov-like inequality for this remainder term as well as examine some of its (a.s.) convergence properties. We then relate these properties to the U-statistic. In addition, using a result of Anscombe (*Proc. Cambridge Philos. Soc.* **48** (1952) 600–607), the asymptotic normality of a U-statistic, U_N, where N is a positive integer-valued random variable, is established under certain conditions. In a later paper these results are applied to the development of a fixed-width sequential confidence interval for the mean of U_n of prescribed coverage probability α, in the manner of Chow and Robbins (*Ann. Math. Statist.* **36** (1965) 457–462). (Received 10 July 1969.)

(Abstract of a paper to be presented at the Eastern Regional meeting, Chapel Hill, North Carolina, Spring 1970. Additional abstracts will appear in future issues.)

1. On bounds useful in symmetrical factorial designs and error correcting codes.

BODH RAJ GULATI and E. G. KOUNIAS, University of Missouri and McGill University.

Let $m_t(r + 1, s)$ denote the maximum number of distinct points in finite projective space, $PG(r, s)$, of r-dimensions based on Galois field of order $s = p^h$, where p is a prime, so that no t of the chosen points are conjoint (a set of t points are conjoint if they lie on a flat space of dimensions not greater than $t - 2$). It is well known that $m_t(r + 1, s)$ also symbolises the maximum number of factors that can be accommodated in a symmetrical factorial design in which each factor operates at $s = p^h$ levels, blocks are of size s^{r+1} so that no main effect or t-factor $(t > 1)$ or lower order interaction is confounded. These bounds, which have been lately useful in error correcting codes and information theory, were introduced by Bose, Barlotti, Seiden, Qvist, Segre and many others. Their investigations are restricted to $t = 3$ and no general methods are yet available for $t \geqq 4$. For $t = 4$, the problem reduces to the investigation of maximum number of points in $PG(n, s)$, $n \geqq 3$, so that no four of chosen points lie on the same plane. In this paper, we have established the following bounds: (i) $m_4(4, s) = s + 1$ for $s > 3$; (ii) $m_4(5, s) \leqq s(s - 1)$ for $s > 3$; and (iii) $m_4(r + 1, s) < s^{r-2} - (s + 1) \sum_{j=0}^{r-4} s^j + 1$ for $r > 4$, $s > 3$. (Received 24 June 1969.)

(Abstracts of papers not connected with any meeting of the Institute.)

1. On the species problem with an information-theoretic approach to smoothing.

ROBERT D. PHILLIPS, International Business Machines Corporation.

The sampling of species of animals or of vocabulary often gives rise to multinomial estimation situations in which the sample size is effectively small and the number of classes is large. In 1953, I. J. Good gave a solution to this general problem, which is referred to here as the species problem. The minimum discrimination information statistic and its limiting forms are used to extend I. J. Good's solution to include cases in which population knowledge of a partial nature exists. The information-theoretic approach to partial knowledge inclusion is displayed primarily in the smoothing of the frequency of frequencies distribution. The performance of the derived procedures is investigated by computer-simulated sampling experiments. Random observations from Dirichlet distributions serve as the generated probability distributions. Adjustment of the parameters of the Dirichlet distributions permit varying amounts and kinds of partial knowledge to enter into the simulations. (Received 27 May 1969.)

2. The domain of attraction of the double exponential distribution.

LAURENS DE HAAN, Mathematisch Centrum, Amsterdam.

To any distribution function F we define $x_0 = \sup \{x \mid F(x) < 1\} \leqq \infty$. The sequence $\{F^n\}$ is in the domain of attraction of the double exponential law iff $\lim_{x \uparrow x_0} \{1 - F(x)\} \cdot$

$\{\int_x^{x_0} \int_y^{x_0} [1 - F(t)]\, dt\, dy\}/\{ \int_x^{x_0} [1 - F(t)]\, dt\}^2 = 1$. More generally: if $x_0 < \infty$ $\{F^n\}$ is in the domain of attraction of a non-degenerate distribution function iff the lefthand member of the equation has a limit c with $\frac{1}{2} < c \leqq 1$. If $x_0 = \infty$ $\{F^n\}$ is in the domain of attraction of a non-degenerate distribution function iff $\lim_{x\to\infty} \{1 - F(x)\}\{\int_x^\infty \int_y^\infty \{1 - F(t)\}\, t^{-2}\, dt\, dy\} \cdot [x^2 \{\int_x^\infty \{1 - F(t)\}\, t^{-2}\, dt\}^2]$ with $\frac{1}{2} < c \leqq 1$. For $\frac{1}{2} < c < 1$ the parameter α of the limit distribution is equal to $(1 - c)^{-1} - 2$. (Received 6 June 1969.)

3. Sequential estimation of an integer mean. Herbert E. Robbins, Columbia University.

If $x_1, x_2, \cdots$ are i.i.d. $N(\theta, 1)$ where θ is an unknown *integer* $= 0, \pm 1, \pm 2, \cdots$, a reasonable integer estimate of θ based on a sample of *fixed size* n is $\hat{\theta}_n =$ integer nearest the sample mean $\bar{x}_n$; the error probability is $P(\hat{\theta}_n \neq \theta) \sim (8/\pi n)^{\frac{1}{2}} e^{-n/8}$ as $n \to \infty$. We consider the following *sequential* estimator of θ: choose an $\alpha > 1$, let $N =$ first $n \geqq 2 \log \alpha$ such that for some integer i, $|\bar{x}_n - i| \leqq \frac{1}{2} - n^{-1} \log \alpha$, and put $\theta^* = 1$. It is easy to show, as in Wald's SPRT, that $P(\theta^* \neq \theta) < 2/(1 + \alpha)$, and $EN \sim 2 \log \alpha$ as $\alpha \to \infty$. Putting $\alpha = (\frac{1}{2}\pi n)^{\frac{1}{2}} e^{n/8}$ we see that as $n \to \infty$, lim sup $[P(\theta^* \neq \theta)/P(\hat{\theta}_n \neq \theta)] \leqq 1$ and lim $[EN/n] = \frac{1}{4}$. The sequential procedure can also be adapted to the case of an unknown variance. Some other examples of estimation of a discrete parameter can be treated similarly. (Received 6 June 1969.)

4. On group sequential analysis. Sylvain Ehrenfeld, New York University.

The problem of choosing between two simple hypotheses H_0 and H_1, in terms of iirv, by means of group sequential analysis is studied. The problem, at any stage, is whether to stop sampling and choose between H_0 and H_1, or to continue and decide how large a group of observations to take. Various cost assumptions are made and Bayes procedures investigated. The information at any time is described by π, the current posterior probability of H_0. The problem is to find an optimal stopping and group size rule. In general, it is shown the optimal stopping rule is of SPRT type, namely stop as soon as the inequality $\pi_1 < \pi < \pi_2$ is violated where π_1 and π_2 are specified constants. The special, but important, case where the log likelihood ratio is bounded and can take on only integer multiple values of a fixed constant is studied for any specified SPRT stopping rule. It is shown that the optimal group size, as it depends on π, can be formulated as a Markov sequential decision problem and solved in terms of linear programming. (Received 26 June 1969.)

5. Estimation of variance components. R. R. Hocking and F. M. Speed, Texas A & M University and NASA, Houston, Texas.

Given the classical component of variance models, it is possible to estimate the components of variance by analyzing the model $Y = Xu + e$ subject to $\theta^T u = \xi$. Let S_1 and S_2 denote the sums-of-squares obtained by testing H_0: $\Lambda_1^T u = \nu_1$ and H_0: $\Lambda_2^T u = \nu_2$. Algorithms are given for computing $E[S_1]$, $E[S_2]$, $V[S_1]$, $V[S_2]$, and Cov $[S_1, S_2]$. Examples of this approach are given. (Received 30 June 1969.)

6. Another look at the analysis of linear models. R. R. Hocking and F. M. Speed, Texas A & M University, and NASA, Houston, Texas.

Suppose that we have sampled p populations which have different means but the same variance. Let us further suppose that linear relations among the means may be known. These assumptions can be expressed as $Y = Xu + e$ subject to $\theta^T u = \xi$. Topics such as point estimation, interval estimation, and tests of hypotheses are reviewed for the above model.

It is shown that by proper choice of θ^T and ξ, there is a one-to-one correspondence between this model and the classical linear models. A discussion of the relevance of this model is given. (Received 30 June 1969.)

7. Estimation of the fixed effects in the mixed model. R. R. Hocking and F. M. Speed, Texas A & M University, and NASA, Houston, Texas.

Given the classical mixed model, it is possible to estimate the fixed effects by using the model $Y = Xu + e$ subject to $\theta^T u = \xi$. It is shown that the recovery of interblock information is just a special case of the general technique described in this paper. This general technique can be used for the classical mixed models by finding the optimum linear combination of a number of unbiased estimates. The resulting estimate has the property of unbiasedness. (Received 30 June 1969.)

8. A multiple decision approach to the selection of the best set of predictor variates: I—Uncorrelated predictors. John S. Ramberg, University of Iowa.

A multiple-decision formulation of a prediction problem where the goal is to select the "best" set of a preassigned number t of variates from a set of k uncorrelated predictor variates $(X_1, X_2, \cdots, X_k)$ is considered. The "best" set of predictor variates is defined to be that set of t variates for which the predictand X_0 has the smallest population conditional variance (or equivalently the largest population multiple correlation coefficient). Our formulation falls within the framework of the Bechhofer "indifference zone" approach in which the ultimate objective is to determine the sample size which will satisfy a specific probability requirement. A $(k + 1)$-variate normal model is assumed and consideration is limited to single-stage procedures. Sample size requirements are obtained from the asymptotic joint distribution of the logarithmic transformation of the sample residual variances (Monte Carlo sampling studies indicate these results are valid for relatively small sample sizes.) An exact asymptotic result is obtained for $k = 2$, while for $k > 2$ the Bonferroni and Slepian inequalities are used to obtain conservative sample size approximations. (Received 2 July 1969.)

9. A note on first passages for $S_n/n^{\frac{1}{2}}$. Alberto Ruiz-Moncayo, Inst. Politecnico Nacional.

Let $X_1, X_2, \cdots$ be independent random variables with mean 0 and variance 1. For $n \geqq 1$ let $S_n = X_1 + \cdots + X_n$, $F_n = \sigma(X_1, \cdots, X_n)$ and T_n = class of all stopping rules with respect to (F_n) which are $\geqq n$ a.s. For each $c > 0$, let $\tau(c)$ = first $n \geqq 1$ such that $|S_n| > cn^{\frac{1}{2}}$, $= \infty$ if no such n exists. $f_n = \text{ess sup}_{t \varepsilon T_n} E(|S_t|/t \mid F_n)$. If for each $c > 0$, $P(\tau(c) < \infty) = 1$ then $P(|S_n| = nf_n \text{ i.o.}) = 1$. (Received 2 July 1969.)

10. Test criteria for the equality of several covariance matrices. K. C. S. Pillai and D. L. Young, Purdue University.

Consider the following tests of hypotheses concerning k p-variate normal populations: (a) $\mathbf{\Sigma}_1 = \cdots = \mathbf{\Sigma}_k = \mathbf{\Sigma}_0$ and (b) $\mathbf{\Sigma}_1 = \cdots = \mathbf{\Sigma}_k$. Pillai has suggested for (a) and (b) the following criteria. For (a) the test is in two steps. (i) Use for test of (b) the criterion $R_1 = \max_i \text{trace } (\mathbf{\Sigma}_0^{-1}\mathbf{S}_i/n_i)/\min_{i'} \text{trace } (\mathbf{\Sigma}_0^{-1}\mathbf{S}_{i'}/n_{i'})$ where $\mathbf{S}_i$ is the S.P. matrix from the ith independent p-variate sample of size $n_i + 1$, $i = 1, \cdots, k$, and (ii) use any of the k traces to test whether the common matrix equals $\mathbf{\Sigma}_0$. Alternately for test of (b) let $k = 2r$. (When

k is odd, two samples can be tested for equality of covariance matrices and pooled, and $k - 1 = 2r$.) First, test pairs of samples for equality of covariance matrices, performing r tests. Then a criterion for (b) has been defined by Pillai as $R_2 = \max_j \operatorname{tr}(n_{j2}\mathbf{S}_{j2}^{-1}\mathbf{S}_{j1}/n_{j1})$ $\min_{j'} \operatorname{tr} (n_{j'2}\mathbf{S}_{j'2}^{-1}\mathbf{S}_{j'1}/n_{j'1})$ or a second one as $R_3 = \max_j \operatorname{tr} ((n_{j1} + n_{j2})(\mathbf{S}_{j1} + \mathbf{S}_{j2})^{-1}\mathbf{S}_{j1}/n_{j1})/\min_{j'1} \operatorname{tr} ((n_{j'1} + n_{j'2})(\mathbf{S}_{j'1} + \mathbf{S}_{j'2})^{-1}\mathbf{S}_{j'1}/n_{j'1})$ where $(\mathbf{S}_{j1}, \mathbf{S}_{j2})$, $j = 1, \cdots, r$, are the homogenous pairs of S.P. matrices. For (a) since $\operatorname{tr} \mathbf{\Sigma}_0^{-1}\mathbf{S}_i$ is distributed as a chi-square variable, R_1 is the ratio of the largest to the smallest of k independent mean squares. For equal sample sizes, tables of upper percentage points of R_1 are available to test against either one-sided or two-sided hypotheses. For unequal sample sizes, tables are under preparation. For R_2 or R_3, tables for the ratio of the largest to the smallest of independent beta variables are useful, since Pillai has shown that in moderate samples Lawley-Hotelling trace is approximately distributed as an F and Pillai trace as a beta variable. Preparation of these tables is in progress. Further, power studies of these criteria are also attempted. (Received 7 July 1969.)