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LIMITING SETS AND CONVEX HULLS OF SAMPLES FROM PRODUCT MEASURES[1]

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0. Summary. Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random vectors in R^n (Euclidean n-space). Let the X_i's have a distribution which is a product of n Borel probability measures along an orthogonal set of axes. After sampling m times let H_m be the convex hull of $\{X_1, \cdots, X_m\}$. All possible limiting shapes for H_m are found along with necessary and sufficient conditions that the limit be obtained.

1. Notation. The convex hull of a sample has been studied before by Efron [1], Geffroy [3], [4] and Rényi and Sulanke [6].

Let $X_1, \cdots, X_m, \cdots$ be a sequence of independent identically distributed Borel random vectors in R^d. Let $S(n) = \{X_1, \cdots, X_n\}$. Let $|X_1|$ denote the length of X_1. Let $M(n) = \max\{|X_1|, \cdots, |X_n|\}$ and

$$\begin{aligned} N(n) &= \{X_1/M_n, \cdots, X_n/M_n\} && \text{if} \quad M_n \neq 0, \\ &= \{\vec{O}\} && \text{if} \quad M_n = 0. \end{aligned}$$

$H(n)$ is the convex hull of $N(n)$. For $A \subseteq R^d$, let $|A| = \sup\{|X| : X \varepsilon A\}$.

Thus $N(n)$ is the sample shrunk so that it is contained in the unit ball. $H(n)$ is the convex hull of this shrunk sample.

$S(A, \epsilon)$ denotes the ϵ neighborhood of a set A, that is,

$$S(A, \epsilon) = \{y \mid \exists x \varepsilon A, |x - y| < \epsilon\}.$$

The distance, $d(A, B)$, between two bounded sets A and B is $d(A, B) = \inf\{\epsilon > 0 \mid S(A, \epsilon) \supseteq B \text{ and } S(B, \epsilon) \supseteq A\}$.

If T is a closed set we write $\lim_{n\to\infty} B_n = T$ i.p. if $P(d(B_n, T) < \epsilon) \to 1$ as $n \to \infty$ for each $\epsilon > 0$. $\lim_{n\to\infty} B_n = T$ a.s. if $P(d(B_n, T) \to 0) = 1$.

The set T is required to be closed in order to make the limits unique. It is easy to see that if the closure of N is equal to T then $d(B_n, N) \to 0$ iff $d(B_n, T) \to 0$.

For any distribution function (df) F on R^1 we define

$$L(x) = \min\{y \mid F(y - 0) \leqq 1 - 1/x \leqq F(y)\}.$$

In R^2 we let

$$C(a) = \{(x, y) \mid x \geqq 0, y \geqq 0, x^a + y^a \leqq 1\} \quad \text{for} \quad 0 < a < +\infty,$$

$$C(0) = \{(x, y) \mid x = 0 \quad \text{and} \quad 0 \leqq y \leqq 1 \quad \text{or} \quad y = 0 \quad \text{and} \quad 0 \leqq x \leqq 1\},$$

$$C(\infty) = \{(x, y) \mid 0 \leqq x \leqq 1 \quad \text{and} \quad 0 \leqq y \leqq 1\}.$$

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2. Limits in R^2. The simplest case is contained in the following theorem.

THEOREM 1. *Let each X_i take values in R^2 with a distribution which is a product of identical measures along an orthogonal set of axes. Let the one dimensional measure have a* df *F satisfying $F(0-) = 0$ and $F(x) < 1$ for all x.*

(a) *If* $\lim_{n\to\infty} N(n) = T$ i.p. *or* a.s. *then* $T = C(a)/|C(a)|$, $0 \leqq a \leqq \infty$. *If* $a > 0$ *the limit exists* a.s. $\lim_{n\to\infty} N(n) = C(a)/|C(a)|$ a.s. iff $L(y^x) \cong x^{1/a}L(y)$ *as* $y \to \infty$ *for each* $x \varepsilon (0, 1)$. $\lim_{n\to\infty} N(n) = 2^{-\frac{1}{2}}C(\infty)$ a.s. iff $L(y) \cong L(y^{\frac{1}{2}})$ *as* $y \to \infty$.

(b) *The results of* (a) *hold if* $N(n)$ *is replaced by* $H(n)$, $0 \leqq a \leqq \infty$ *by* $1 \leqq a \leqq \infty$, *and* $a > 0$ *by* $a > 1$.

COMMENTS. Since each of the limits for $a > 1$ exists for $N(n)$ as well as $H(n)$ it appears that the theorem has little to do with convex hulls. However, it is not obvious a priori that this is the case, and much of the proof is concerned with establishing this fact. If $a = 1$ it may be that $N(n)$ does not have a limit while $H(n)$ does.

PROOF. For simplicity we shall assume that F is continuous and strictly increasing for $x > 0$. The more general case follows from routine but tedious approximation of an arbitrary distribution function by distribution functions of this type.

For ease of understanding we break the proof up into a series of lemmas. One direction of the proof is easy. If F behaves nicely it is easy to see that we have the correct limits.

LEMMA 1. *Let* $L(y^x) \cong x^{1/a}L(y)$ *as* $y \to \infty$ *for each* $x \varepsilon (0, 1)$

(i) *if* $a \geqq 1$ *then*

$$\lim_{n\to\infty} N(n) = \lim_{n\to\infty} H(n) = C(a)/|C(a)| \text{ a.s.}$$

(ii) *if* $1 > a > 0$ *then*

$$\lim_{n\to\infty} N(n) = C(a) \text{ a.s.}$$

$$\lim_{n\to\infty} H(n) = C(1) \text{ a.s.}$$

PROOF. It is enough to show the statements involving $N(n)$ since if $N(n)$ has some limit it is clear that$H(n)$ will have a limit which is the convex hull of the limit of the $N(n)$. To show that $N(n)$ has the limit it is enough to show that $S(n)/L(n) \to C(a)$ a.s.

Finally, to show that $S(n)/L(n) \to C(a)$ a.s. it is enough to show (1) and (2) below.

(1) $(x, y) \not\varepsilon C(a)$ implies the existence of a neighborhood B of (x, y) such that

$$\lim_{N\to\infty} P(S(n)/L(n) \cap B = \varnothing, n \geqq N) = 1.$$

(2) $(x, y) \varepsilon C(a)$ implies that for any neighborhood B of (x, y),

$$\lim_{N\to\infty} P(S(n)/L(n) \cap B \neq \varnothing, n \geqq N) = 1.$$

We first show that (1) holds under the hypotheses of the lemma. For each

$(x, y) \varepsilon R^2$, let

$$B(x, y, \epsilon) = \{(u, v): |u - x| \leqq \epsilon, |v - y| \leqq \epsilon\} \quad \text{and}$$
$$Q(x, y, \epsilon) = \{(u, v): u \geqq x - \epsilon, v \geqq y - \epsilon\}.$$

Now $S(n)/L(n) \cap Q(x, y, \epsilon) = \varnothing, n \geqq N$ is equivalent to

$$X_i \not\varepsilon Q(xL(n), yL(n), \epsilon L(n)), \qquad i = 1, \cdots, n; \qquad n \geqq N.$$

Recalling that $L(n) \to \infty$ we see that if $X_n(w) \not\varepsilon Q(xL(n), yL(n), \epsilon L(n))$ for all but a finite number of n then there exists an $N(w)$ such that

$$S(n)(w)/L(n) \cap Q(xL(n), yL(n), \epsilon L(n)) = \varnothing, \qquad n \geqq N.$$

We only need consider $x \geqq 0$ and $y \geqq 0$. Suppose $x^a + y^a > 1$. Choose $\epsilon > 0$ such that $(x - 2\epsilon)^a + (y - 2\epsilon)^a > 1$ then

$$\begin{aligned} &P(X_n \varepsilon Q(xL(n), yL(n), \epsilon L(n)) \\ &\quad = (1 - F((x - \epsilon)L(n)))(1 - F((y - \epsilon)L(n))) <_{\text{large } n} (1 - F(L(n^{(x-2\epsilon)^a}))) \\ &\qquad \cdot (1 - F(L(n^{(y-\epsilon)^a}))) \\ &\quad = n^{-b} \quad \text{where} \quad b = (x - 2\epsilon)^a + (y - 2\epsilon)^a > 1 \end{aligned}$$

where we assumed $x > 0$ and $y > 0$. If $x = 0$ or $y = 0$ the modifications are clear. The inequality for large n follows from the assumption on L. An application of the Borel-Cantelli lemma shows that (1) is true.

To show (2) let $x^a + y^a < 1$. It is sufficient to consider $x > 0$ and $y > 0$. Then we have

$$\begin{aligned} &P(S(n)/L(n) \cap B(x, y, \epsilon) = \varnothing) \\ &\quad = \textstyle\prod_{i=1}^n P(X_i/L(n) \not\varepsilon B(x, y, \epsilon)) \\ &\quad = (1 - [F(x + \epsilon)L(n)) - F((x - \epsilon)L(n))] \\ &\qquad\quad \cdot [F((y + \epsilon)L(n)) - F((y - \epsilon)L(n))])^n \\ &\quad <_{\text{large } n} (1 - [F(L(n^{(x+\frac{1}{2}\epsilon)^a})) - F(L(n^{(x-\frac{1}{2}\epsilon)^a}))] \\ &\qquad \cdot [F(L(n^{y+\frac{1}{2}\epsilon)^a})) - F(L(n^{(y-\frac{1}{2}\epsilon)^a}))])^n \\ &\quad = (1 - [n^{-(x-\frac{1}{2}\epsilon)^a} - n^{-(x+\frac{1}{2}\epsilon)^a}][n^{-(y-\frac{1}{2}\epsilon)^a} - n^{-(y+\frac{1}{2}\epsilon)^a}])^n \\ &\quad <_{\text{large } n} (1 - n^{-b})^n \text{ where } b = (x - \tfrac{1}{2}\epsilon)^a + (y - \tfrac{1}{2}\epsilon)^a + \delta < 1 \end{aligned}$$

for some $\delta > 0$. The Borel-Cantelli lemma gives $S(n)/L(n) \cap B(x, y, \epsilon) = \varnothing$ only finitely often a.s., thus completing the proof.

The following lemma is proved in much the same manner and its proof is omitted.

LEMMA 2. *If $L(y^{\frac{1}{2}}) \cong L(y)$ as $y \to \infty$ then*

$$\lim_{n\to\infty} N(n) = \lim_{n\to\infty} H(n) = 2^{-\frac{1}{2}}C(\infty) \text{ a.s.}$$

The more interesting portion of the proof is involved in showing that we have all possible limits. The following three technical lemmas will be needed.

LEMMA 3. (a) *Let* $x \varepsilon (0, 1)$ *and* $L(y)$ *a positive continuous function that is increasing for large* y. *Suppose that for some* $a > 0$,

$$L(y^x) \leqq x^{1/a} L(y) \quad \text{for large} \quad y. \tag{2.1}$$

Then given $\epsilon > 0$ *there exists a sequence* $y(n) \to \infty$ *such that* $1 \geqq z > \epsilon$ *implies*

$$L(y(n)^z) \leqq z^{1/a} L(y(n)) \quad \text{for} \quad n = 1, 2, \cdots. \tag{2.2}$$

(b) *If the inequality in* (2.1) *is reversed the inequality in* (2.2) *may be reversed.*

PROOF. (a) Clearly it is enough to show that we may find one such $y(n)$ which is arbitrarily large. Let $\epsilon > 0$ be given and choose N such that

$$L(y^x) \leqq x^{1/a} L(y) \quad \text{for} \quad y^x \geqq N.$$

Let $v > N^{1/\epsilon}$ and $v > 1$. We now show that we may find a $u \varepsilon [v, v^{1/x}]$ such that

$$L(u^z) \leqq z^{1/a} L(u) \quad \text{for all} \quad z > \epsilon. \tag{2.3}$$

Let $H(y) = L(y)/(\log y)^{1/a}$. Then

$$H(y^x) \leqq H(y) \quad \text{for} \quad y^x \geqq N \tag{2.4}$$

since $L(y^x)/(\log y^x)^{1/a} \leqq x^{1/a} L(y)/x^{1/a}(\log y)^{1/a}$. Let $u = \min \{y \mid H(y) \geqq H(z)$ for all $z \varepsilon [v, v^{1/x}], y \varepsilon [v, v^{1/x}]\}$. By iterating (2.4) we find that

$$H(y^{(1/x)^m}) \geqq H(y) \quad \text{for} \quad y \geqq N, \qquad m \text{ a positive integer.} \tag{2.5}$$

By our choice of v, $u^z > N$ for $1 \geqq z > \epsilon$. We may find a nonnegative integer n such that $u^{z \cdot (1/x)^n} \varepsilon [v, v^{1/x}]$ since $u^z \leqq u \varepsilon [v, v^{1/x}]$ and $u^{z \cdot (1/x)^n} \to \infty$ as $n \to \infty$. Then by applying (2.5) we have $H(u^z) \leqq H(u^{z \cdot (1/x)^n}) \leqq H(u)$.

But $H(u^z) \leqq H(u)$ implies $L(u^z) \leqq z^{1/a} L(u)$ by going back to the definition of H. Let $y(1) = u$.

(b) is proved in the same manner by reversing the appropriate inequalities.

LEMMA 4. $N(n)$ *has a limit* i.p. *iff* $S(n)/L(n)$ *has a limit in probability.* $H(n)$ *has a limit* i.p. *iff the convex hull of* $S(n)/L(n)$ *has a limit* i.p.

PROOF. From standard theorems on convex sets it is clear that if $H(n)$ has a limit set L i.p. and if (x, y) is an extreme point of L then for each neighborhood B of (x, y), $\lim_{n\to\infty} P(N(n) \cap B \neq \varnothing) = 1$. Let $MX(n) = \max \{X_1, \cdots, X_n\}$ and $MY(n) = \max \{Y_1, \cdots, Y_n\}$. Assume that $\lim_{n\to\infty} N(n) = T$ i.p. or $\lim_{n\to\infty} C(n) = T$ i.p. It is clear that T must be symmetric about $x = y$. In either case, $MX(n)/M(n) \to_p a > 0$ and $MY(n)/M(n) \to_p a$. Thus, $MX(n)/MY(n) \to_p 1$. Since MX and MY are independent random variables it is easy to see that this implies that MX and MY are relatively stable in probability. From Gnedenko [6], $MX(n)/L(n) \to_p a > 0$. Thus, $S(n)/L(n) = (M(n)/L(n)) \cdot (S(n)/M(n))$ converges i.p. if $N(n)$ does. A similar argument holds for $H(n)$ and the convex hull of $S(n)/L(n)$.

The "if" portions of the lemma are obvious completing the proof.

LEMMA 5. *Let* $\lim_{n\to\infty} N(n)$ *and/or* $\lim_{n\to\infty} H(n)$ *exist* i.p. *If* (x, y), $x > 0$ *and* $y > 0$ *is a point such that for each neighborhood* B *of* (x, y) $\lim_{n\to\infty} P(S(n)/L(n) \cap B \neq \emptyset) = 1$ *then the same statement is true for all* (x', y') *with* $0 < x' \leqq x$ *and* $0 < y' \leqq y$.

PROOF. From the proof of the last lemma $M(n)$ is relatively stable in probability. From Gnedenko [6] this is true iff for each $k > 1$,

$$\lim_{x\to\infty} (1 - F(kL(x))(/(1 - F(L(x))) = 0.$$

It follows that if $a > b > 0$

$$F(aL(n)) - F(bL(n)) \cong 1 - F(bL(n)) \quad \text{as} \quad n \to \infty \text{ and}$$
$$(1 - F(bL(n)))/(1 - F(aL(n))) \to \infty \quad \text{as} \quad n \to \infty.$$

Using these observations and $B(x, y, \epsilon)$ defined in Lemma 1, if $x - \epsilon > x'$; $y - \epsilon > y'$;

$$\begin{aligned}
&P(S(n)/L(n) \cap B(x, y, \epsilon) = \emptyset) \\
&\quad = \textstyle\prod_{i=1}^{n} P(X_i \notin B(xL(n)yL(n), \epsilon L(n))) \\
&\quad = (1 - [F((x + \epsilon)L(n)) - F((x - \epsilon)L(n))] \\
&\qquad \cdot [F((y + \epsilon)L(n)) - F((y - \epsilon)L(n))])^n >_{\text{large } n} (1 - [1 - F(x'L(n))] \\
&\qquad \cdot [1 - F(y'L(n))])^n >_{\text{large } n} (1 - [F((x' + \epsilon)L(n)) - F((x' - \epsilon)L(n))] \\
&\qquad \cdot [F((y' + \epsilon)L(n)) - F((y' - \epsilon)L(n))])^n \\
&\quad = P(S(n)/L(n) \cap B(x', y', \epsilon) = \emptyset).
\end{aligned}$$

Since the first probability approaches zero the last does and since the $B(x', y', \epsilon)$ form a base for the neighborhood system of (x', y') as ϵ varies the lemma is proved.

Let $D(n) = S(n)/L(n)$ and $E(n)$ be the convex hull of $D(n)$. The following proposition combined with Lemma 4 and the fact $\lim_{n\to\infty} A(n) = T$ a.s. implies $\lim_{n\to\infty} A(n) = T$ i.p. [$A(n)$ equal to $N(n)$, $H(n)$, $D(n)$ or $E(n)$ gives us all possible limits.]

PROPOSITION 1. (a) *If* $\lim_{n\to\infty} D(n) = N$ i.p. *then* $N = C(a)$ *where* $0 \leqq a \leqq \infty$.
(b) *If* $\lim_{n\to\infty} E(n) = N$ i.p. *then* $N = C(a)$, $1 \leqq a \leqq \infty$

PROOF. Since $MX(n)/L(n) \to_p 1$ it is clear that N contains $(0, 0)$, $(0, 1)$ and $(1, 0)$ under either (a) or (b).

We prove (a) first. If $(1, 1) \in N$ then by the preceding comment $N = C(\infty)$. If $(\epsilon, \epsilon) \notin N$ for every $\epsilon > 0$, by Lemma 5, $N = C(0)$. Thus assume that $N \neq C(0)$ or $C(\infty)$. Set $z = \max \{x: (x, x) \in N\}$. $0 < z < 1$. From the definition of z we see (where $P(x, y) \equiv \{(u, v): u \geqq x, v \geqq y\}$):

(i) if $x > z$, $P(E(n) \cap P(x, x) = \emptyset) \to 1$ as $n \to \infty$, and
(ii) if $x < z$, $P(E(n) \cap P(x, x) = \emptyset) \to 0$ as $n \to \infty$.

Letting $G(x) = 1 - F(x)$, (i) and (ii) are equivalent to

$$\text{(2.6)} \qquad \begin{aligned} &\text{if} \quad x > z, \quad (1 - G^2(xL(n)))^n \to 1 \quad \text{as} \quad n \to \infty. \\ &\text{if} \quad x < z, \quad (1 - G^2(xL(n)))^n \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Taking the logs of (2.6) we see that

$$\text{(2.7)}\qquad \begin{aligned} &\text{if} \quad x > z, \quad nG^2(xL(n)) \to 0 \\ &\text{if} \quad x < z, \quad nG^2(xL(n)) \to \infty. \end{aligned}$$

Now if $xL(n) \leqq L(n^{\frac{1}{2}})$ then $nG^2(xL(n)) \geqq nG^2(L(n^{\frac{1}{2}})) = 1$. Thus, if $x > z$ we have $xL(n) \geqq L(n^{\frac{1}{2}})$. Similarly, if $x < z$ then $xL(n) \leqq L(n^{\frac{1}{2}})$. Combining these results,

$$\text{(2.8)}\qquad zL(n) \cong L(n^{\frac{1}{2}}) \quad \text{as} \quad n \to \infty.$$

Letting $a = -\log 2/\log z$ by (2.8) we have

$$\text{(2.9)}\quad zL(y) \cong L(y^{(z^a)}) \text{ as } y \to \infty \quad \text{or} \quad (z^a)^{1/a}L(y) \cong L(y^{(z^a)}) \text{ as } y \to \infty.$$

We used the fact that $M(n)$ relatively stable i.p. implies L is a slowly varying function to go from n to the continuous variable y.

Let $x > 0$ and $y > 0$ and $x^a + y^a > 1$. Then by (2.9) and Lemma 3 choose $y(n)$ such that $r > \min\,(x, y)/2$ implies $L(y(n)^{r^{a'}}) \leqq rL(y(n))$ where $a' > a$ and $x^{a'} + y^{a'} > 1$. It follows that:

$$\text{(2.10)}\qquad P(D(n) \cap P(x, y) = \varnothing) = (1 - G(xL(n))G(yL(n)))^n$$

$$\text{(2.11)}\qquad P(D_{[y(n)]} \cap P(x, y) = \varnothing) \gtrsim (1 - y(n)^{-(x^{a'}+y^{a'})})^{[y(n)]}$$

which approaches 1 as $y(n) \to \infty$ where $[x]$ is the integer part of x.

From (2.11) it follows that $(x, y) \not\in C(a)$ is not in N since it will have a neighborhood which does not contain points of $D(n)$ with high probability for some large values of n.

Using the opposite inequality of Lemma 3 it follows that if $x^a + y^a < 1$ then $P(x, y)$ has points of $D(n)$ with a probability approaching one for some large n. Thus, some point of $P(x, y) \in N$ and hence by Lemma 5 $(x, y) \in N$. (a) is thus proved.

Proof of (b). Suppose $N \neq C(1)$ or $C(\infty)$. Since N is convex and contains $(0, 0)$, $(1, 0)$ and $(0, 1)$ it follows that $N \supseteq C(1)$. Find z and a as in the proof of (a). The same argument yields that $(x, y) \not\in C(a)$ implies $P(D(n) \cap P(x, y) = \varnothing) \to 1$ for some subsequence of integers n. As mentioned previously each neighborhood of an extreme point of N must intersect $D(n)$ with a probability approaching one. It follows that $N \subseteq C(a)$. Thus, (z, z) is an extreme point of N and any neighborhood contains points of $D(n)$ with a probability approaching one as $n \to \infty$. As in the proof of (a), $zL(n) \cong L(n^{\frac{1}{2}})$ which implies that each neighborhood of a boundary point of $C(a)$ has points of $D(n)$ with a high probability for some large n. Thus, $N \supseteq C(a)$ completing the proof.

To complete the proof of our theorem we need to show that L behaves in an appropriate fashion when $C(a)$ is a limit.

LEMMA 6. *Let* $\lim_{n\to\infty} D(n) = C(a)$ i.p., $0 < a < \infty$, *or* $\lim E(n) = C(a)$ i.p., $1 < a < \infty$. *Then* $x \in (0, 1)$ *implies*

$$\lim_{n\to\infty\; x'(n)\to x} L(n/I(x'(n)L(n)))/L(n) = (1 - x^a)^{1/a}$$

where $I(z) = L^{-1}(z)$.

PROOF. Fix an $x \varepsilon (0, 1)$. The probability that the x coordinate of a sample point is to the right of $xL(n)$ is $G(xL(n))$. The expected number of sample points in the first n with an x coordinate greater than $xL(n)$ is $nG(xL(n)) = nG(L(I(xL(n)))) = n/I(xL(n))$. By Chebyshev's inequality the probability that we have between $\frac{1}{2}$ and 2 times the number of points approaches one. Since this number approaches infinity and the maximum associated with F is relatively stable in probability with a probability approaching one, the maximum y-component of the points of S_n with x-component greater than $xL(n)$ is between $(1 - \epsilon)L(n/2I(xL(n)))$ and $(1 + \epsilon)L(2n/I(xL(n)))$. Since L is slowly varying and we know that

$$\max \{y_i \mid X_i = (x_i y_i), x_i \geqq xL(n), i = 1, 2, \cdots, n\}/L(n) \to (1 - x^a)^{1/a} \text{ i.p.}$$

the lemma is proved.

LEMMA 7. *Let* $\lim_{n\to\infty} D(n) = C(a), 0 < a < \infty$; or $\lim E(n) = C(a), 1 < a < \infty$. *Then if* $x \varepsilon (0, 1) L(y^x) \cong x^{1/a}L(y)$.

PROOF. Let $S = \{x \mid x > 0 \text{ and } L(y^x) \cong x^{1/a}L(y)\}$. Let $b = 1/a$. $1 \varepsilon S$ and $\frac{1}{2} \varepsilon S$, which is shown in the proof of Proposition 1. Note: (a) If $x \varepsilon S$ and $y \varepsilon S$ then $xy \varepsilon S$ (since then $L(Z^{xy}) \cong y^b L(Z^x) \cong y^b x^b L(Z)$). (b) If $x \varepsilon S$ then $1/x \varepsilon S$ (since $L(y^x) \cong x^b L(y)$ implies $x^{-b}L(y^x) \cong L(y)$. Let $Z = y^x$; then $(1/x)^b L(Z) \cong L(Z^{1/x})$). (c) If $x \varepsilon S$ and $y \varepsilon S$ then $x/y \varepsilon S$ (by (a) and (b)).

By induction we show that S contains the set of positive integers. Suppose that $1, 2, \cdots, p - 1 \varepsilon S$ where $p - 1 \geqq 2$. If p is not prime then $p = p_1 \cdot p_2$ where $p_1 \varepsilon S$ and $p_2 \varepsilon S$. By (a), $p \varepsilon S$. If p is prime, then since 2 and $(p + 1)/2 \varepsilon S$, applying (a), $p + 1 \varepsilon S$. From (b), $1/(p + 1) \varepsilon S$. Thus,

$$L(y^{1/(p+1)}) \cong (1/(p + 1))^b L(y). \tag{2.12}$$

Select $x(y)$ such that $L(y^{1/(p+1)}) = x(y)L(y)$. Then by (2.12)

$$x(y) \to (1/(p + 1))^b.$$

Now using Lemma 6,

$$\begin{aligned}(1 - 1/(p + 1))^b &= (1 - (1/(p + 1))^{b \cdot a})^b \\ &= \lim_{n\to\infty} L(y/I(x(y)L(y)))/L(y) \\ &= \lim_{n\to\infty} L(y^{1-1/(p+1)})/L(y).\end{aligned}$$

Thus, $1 - 1/(p + 1) = p/(p + 1) \varepsilon S$. Since $p + 1 \varepsilon S$ by (a), $p \varepsilon S$.

By (c), S contains all positive rational numbers. Since L is monotonic it follows that S contains all positive numbers.

LEMMA 8. *If* $\lim_{n\to\infty} D(n) = C(\infty)$ i.p. *or* $\lim_{n\to\infty} E(n) = C(\infty)$ i.o. *then* $L(y) \cong L(y^{\frac{1}{2}})$.

PROOF. With probability approaching one we have points of $E(n)$ in any neighborhood of $(1, 1)$. As in the proof of Proposition 1, if $x < 1$ $xL(n) \gtrsim L(n^{\frac{1}{2}})$ which gives the stated result.

The totality of the results proved so far prove the theorem.

3. Extension to R^n. From the theorem of Section 2 the most general limit in R^n (for i.i.d. random vectors whose distribution is a product of one dimensional measures along an orthogonal set of axes) may be found. The ideas used in proving the result are: (1) if a limit exists in n-dimensions by projecting onto a lower dimensional subspace a limit still exists; (2) limits are invariant under affine transformations; and (3) in looking at a limit in two-dimensions if we linearly transform the limit so that the planes of support along the product axes are equidistant from the origin we may assume that the one dimensional product measures are the same. The theorem is stated below, but not proved since the proof is tedious but not enlightening.

In n-dimensional Euclidean space let $(x_1, \cdots, x_n)$ be the coordinate representation of a vector with respect to a fixed orthonormal basis for the space. A set Q is a quadrant of R^n if it consists of all points such that each coordinate has a fixed sign (consider 0 as both positive and negative). A set A is in $S(a)$ if the intersection of A with each quadrant Q is of the form

$$\{(x_1, \cdots, x_n) : \textstyle\sum_{i=1}^{n} (|x_i|/b_i)^a \leqq 1, \ (x_1 \cdots, x_n) \ \varepsilon \ Q\}$$

where each $b_i \geqq 0$ and $0/0$ is taken as zero. Let P be a Borel probability measure on R^n which is a product of n one-dimensional measures along the orthogonal set of axes. Let F_i be the distribution function of the ith measure and

$$L_i(x) = \min \{y : F_i(y - 0) \leqq 1 - 1/x \leqq F_i(y)\}$$

$$G_i(x) = \max \{y : F_i(y - 0) \leqq 1/x \leqq F_i(y)\}.$$

THEOREM 2. *Under the conditions and definitions of this section:*

(a) *All possible limits for* $N(n)$ i.p. *or* a.s. *are in* $S(a)$, $0 \leqq a \leqq \infty$, *and have norm one. If* $\lim_{n\to\infty} N(n) =_p A \ \varepsilon \ S(a)$, $0 < a \leqq \infty$, *then the limit exists* a.s.

In order that $\lim_{n\to\infty} N(n)$ *exist* a.s. *and is a set in* $S(a)$, $0 < a < \infty$, *which is not a line segment it is necessary and sufficient that there exist a function* L *such that* $L(y^x) \cong x^{1/a} L(y)$ *as* $y \to \infty$ *for all* $x \ \varepsilon \ (0, 1)$ *and that* $\lim_{n\to\infty} L_i(x)/L(x)$ *and* $\lim_{n\to\infty} |G_i(x)|/L(x)$ *all exist as nonnegative numbers and at least two distinct* i *values correspond to a positive limit.*

In order that $\lim_{n\to\infty} N(n)$ *exist as an* a.s. *limit in* $S(\infty)$ *that is not a line segment it is necessary and sufficient that a function* L *exist such that* $L(y) \cong L(y^{1/2})$ *as* $y \to \infty$ *and the conditions of the previous sentence involving* L_i *and* G_i *hold.*

(b) *All the results of* (a) *hold if* $0 \leqq a$ *is replaced by* $1 \leqq a$ *and* $N(n)$ *by* $H(n)$.

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