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THE GROWTH OF A RANDOM WALK[1]

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1. Introduction. Let $X_1, X_2, \cdots,$ be independent, identically distributed, nondegenerate, real-valued random variables and set $S_n = X_1 + X_2 + \cdots + X_n$, $n \geqq 1$. It is important for some applications to know whether for all $c > 0$

$$T = \min [n \mid S_n > cn^{\frac{1}{2}}]$$

defines a random variable which is finite with probability one. This is equivalent to knowing whether

$$P(\limsup_{n\to\infty} S_n/n^{\frac{1}{2}} = \infty) = 1. \tag{1}$$

Of course if X_1 has mean zero and finite variance, the law of the iterated logarithm implies that (1) holds. Also if X_1 has finite positive mean, (1) is a consequence of the strong law of large numbers.

It was shown by the author in [2] that if X_1 has mean zero, then (1) holds. More generally it was shown that if S_n is recurrent (which is necessarily the case if X_1 has mean zero), then (1) holds.

The author conjectured in [2] that if S_n is nonnegative infinitely often with positive probability, then (1) should hold. In this paper we will show that this conjecture is valid and can even be strengthened. Specifically we will prove the following

THEOREM. *Either* (1) *holds or*

$$P(\lim_{n\to\infty} S_n/n^{\frac{1}{2}} = -\infty) = 1. \tag{2}$$

As an immediate consequence of this theorem, we obtain the conjecture of [2]:

COROLLARY. *If*

$$P(S_n \geqq 0 \text{ i.o.}) > 0, \tag{3}$$

then (1) *holds.*

2. Proof. We first prove the corollary to the theorem. We assume that (3) holds and will prove that (1) holds. Because of the known results summarized in the introduction, we can further assume that S_n is a transient random walk and that $EX_1^+ = EX_1^- = \infty$.

Let μ denote the distribution of X_1 and let $\hat{\mu}$ denote the characteristic function of μ (characteristic functions of other probability measures will be denoted similarly). Let P denote a compact neighborhood of the origin such that $\hat{\mu}(\theta) \neq 1$ for $\theta \varepsilon P - \{0\}$. Then, as is well-known,

$$\int_P \Re(1/(1 - \hat{\mu}(\theta)))\, d\theta < \infty \tag{4}$$

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and

(5) $$\lim_{\theta\to 0} \theta^2/(1 - \Re\hat{\mu}(\theta)) = 0.$$

It follows from (4) and (5) that

(6) $$\int_P |\theta|/|1 - \hat{\mu}(\theta)|\, d\theta < \infty.$$

To verify (6) it suffices to show that

(7) $$\int_P |\theta|\, \Re(1/(1 - \hat{\mu}(\theta)))\, d\theta < \infty$$

and

(8) $$\int_P |\theta|\, |\Im(1/(1 - \hat{\mu}(\theta)))|\, d\theta < \infty.$$

Equation (7) follows immediately from (4). To obtain (8) write $P = P_1 \cup P_2$, where

$$P_1 = \{\theta \,\varepsilon\, P \mid |\theta|\, |\Im\hat{\mu}(\theta)| \leqq 1 - \Re\hat{\mu}(\theta)\}$$

and

$$P_2 = \{\theta \,\varepsilon\, P \mid |\theta|\, |\Im\hat{\mu}(\theta)| > 1 - \Re\hat{\mu}(\theta)\}.$$

Then

(9) $$|\theta|\, |\Im(1/(1 - \hat{\mu}(\theta)))| \leqq \Re(1/(1 - \hat{\mu}(\theta))), \qquad \theta \,\varepsilon\, P_1,$$

(10) $$|\theta|\, |\Im(1/(1 - \hat{\mu}(\theta)))| \leqq |\theta|/|\Im\hat{\mu}(\theta)|, \qquad \theta \,\varepsilon\, P_2,$$

and by (5) and the definition of P_2

(11) $$\sup_{\theta \varepsilon P_2} |\theta|/|\Im\hat{\mu}(\theta)| < \infty.$$

Clearly (4), (9), (10), and (11) yield (8).

We can find probability measures ν and φ on R such that $\mu = (\nu + \varphi)/2$, $\hat{\nu}(\theta) = 1$ if and only if $\hat{\mu}(\theta) = 1$, and φ has mean zero and finite positive variance σ^2. We would like to be able to state that if the random walk induced by μ satisfies (3), then so does the random walk induced by ν. This seems hard to do directly, but a slight detour will yield the desired result.

For the detour, let δ_0 denote the probability distribution that is concentrated at the origin and set $\psi = (\nu + \delta_0)/2$. We will prove that if the random walk induced by μ satisfies (3), then so does the walk induced by ψ. It will be trivial to go from ψ to ν.

By the standard zero-one law, (3) is equivalent to

(12) $$P(S_n \geqq 0 \text{ i.o.}) = 1.$$

It is a well-known result of fluctuation theory (Spitzer [1]) that (12) is equivalent to

(13) $$\sum_{n=1}^{\infty} n^{-1} P(S_n \geqq 0) = \infty.$$

The key to the proof of the theorem is the following

LEMMA. *Let T_n be the random walk generated by ψ. Then*

(14) $$\sum_{n=1}^{\infty} n^{-1} P(T_n \geqq 0) = \infty.$$

PROOF OF LEMMA. Let $F_n(x) = P(S_n \geqq x)$ and $G_n(x) = P(T_n \geqq x)$. Let $k(x)$ denote a symmetric probability density having finite second moment and whose Fourier transform $\hat{k}(\theta)$ has support in P. It is well-known that

$$(F_n(x+1) - F_n(x))/n^{\frac{1}{2}} \quad \text{and} \quad (G_n(x+1) - G_n(x))/n^{\frac{1}{2}}$$

are bounded uniformly in x and n and hence that

$$n^{\frac{1}{2}}(F_n(x) - F_n(0))/(|x|+1) \quad \text{and} \quad n^{\frac{1}{2}}(G_n(x) - G_n(0))/(|x|+1)$$

are bounded uniformly in x and n. Thus (13) and (14) are respectively equivalent to

$$\sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{\infty} F_n(x)k(x)\,dx = \infty \tag{15}$$

and

$$\sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{\infty} G_n(x)k(x)\,dx = \infty. \tag{16}$$

In order to show that (16) holds (knowing (15) holds), it suffices to show that for some $0 \leqq M < \infty$

$$\left|\sum_{n=1}^{\infty} n^{-1}\lambda^n \int_{-\infty}^{\infty} (F_n(x) - G_n(x))k(x)\,dx\right| \leqq M, \quad 0 \leqq \lambda < 1. \tag{17}$$

The proof of (17) proceeds by use of Fourier analysis. Since $\hat{\mu} = (\hat{\nu} + \hat{\varphi})/2$ and $\hat{\psi} = (\hat{\nu} + 1)/2$, it follows that $\hat{\mu} - \hat{\psi} = (\hat{\varphi} - 1)/2$. Set $f(\theta) = \hat{\mu}(\theta) - \hat{\psi}(\theta)$. Then $f(\theta) = 0(\theta^2)$ as $\theta \to 0$. A standard Fourier inversion formula yields that

$$\int_{-\infty}^{\infty} k(x)(F_n(x) - G_n(x) - F_n(x+y) + G_n(x+y))\,dx$$
$$= (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{k}(\theta)(\hat{\mu}^n(\theta) - \hat{\psi}^n(\theta))(i\theta)^{-1}(1 - \exp(-i\theta y))\,d\theta.$$

By letting $y \to \infty$ we see that

$$\int_{-\infty}^{\infty} k(x)(F_n(x) - G_n(x))\,dx = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{k}(\theta)(\hat{\mu}^n(\theta) - \hat{\psi}^n(\theta))(i\theta)^{-1}\,d\theta.$$

An application of dominated convergence justifies summing on n to obtain for $0 \leqq \lambda < 1$

$$\sum_{n=1}^{\infty} n^{-1}\lambda^n \int_{-\infty}^{\infty} k(x)(F_n(x) - G_n(x))\,dx$$
$$= (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{k}(\theta) \log[(1 - \lambda\hat{\psi}(\theta))/(1 - \lambda\hat{\mu}(\theta))](i\theta)^{-1}\,d\theta$$
$$= (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{k}(\theta) \log[1 + (\lambda f(\theta)/(1 - \lambda\hat{\mu}(\theta)))](i\theta)^{-1}\,d\theta.$$

In order to verify (17) it is now enough to know that $\int_P |\theta|/|1 - \hat{\mu}(\theta)|\,d\theta < \infty$, which is just equation (6). Thus (17) holds and the proof of the lemma is now complete.

It follows from the lemma that

$$P(T_n \geqq 0 \text{ i.o.}) = 1. \tag{18}$$

Let U_n be the random walk induced by ν. Since $\psi = (\nu + \delta_0)/2$ induces the random walk T_n, a simple probability argument together with (18) yields

$$P(U_n \geqq 0 \text{ i.o.}) = 1. \tag{19}$$

To proceed from (19) to (1) one need only follow the procedure given in [2]. For completeness we repeat the argument.

Let U_n be a random walk induced by ν. Let V_n be a random walk induced by φ. Let $j(n)$ be the sum of n independent binomial random variables taking on the values 0 and 1 with probability $\frac{1}{2}$ and set $k(n) = n - j(n)$. We can assume that the processes U_n, V_n and $j(n)$ are independent of each other. Set $W_n = U_{j(n)} + V_{k(n)}$. Then the process W_n has the same probabilistic structure as S_n. By the zero-one law in order to obtain (1) it suffices to show that for any N

$$\lim_{m\to\infty} P(W_n/n^{\frac{1}{2}} \geqq N \quad \text{for some} \quad n \geqq m) > 0.$$

Set $\tau(m) = \min [n \geqq m \mid j(n) \leqq 3n/4 \text{ and } U(n) \geqq 0]$. Clearly $\tau(m)$ is finite with probability one. Now $U(\tau(m)) \geqq 0$ and $k(\tau(m)) \geqq \tau(m)/4$. Thus

$$P(W_n/n^{\frac{1}{2}} \geqq N \quad \text{for some} \quad n \geqq m) \geqq P(V_{k(\tau(m))}/(\tau(m))^{\frac{1}{2}} \geqq N)$$
$$\geqq P(V_{k(\tau(m))}/(k(\tau(m)))^{\frac{1}{2}} \geqq 2N)$$

which approaches $1 - \Phi(2N/\sigma)$ as $m \to \infty$ by the central limit theorem. Here Φ denotes the standard normal distribution function. This completes the proof of the corollary.

Similar arguments allow us to strengthen the corollary to the theorem. Suppose (1) does not hold. Then by the corollary

$$P(S_n \geqq 0 \text{ i.o.}) = 0, \tag{20}$$

where S_n is the random walk induced by μ. Let T_n be the random walk induced by $(\mu + \delta_0)/2$. Clearly

$$P(T_n \geqq 0 \text{ i.o.}) = 0. \tag{21}$$

Let φ be a nondegenerate probability measure having mean zero and finite positive variance σ^2. Let U_n be the random walk induced by $(\mu + \varphi)/2$. It follows from (21) by the proof of the above lemma that

$$P(U_n \geqq 0 \text{ i.o.}) = 0. \tag{22}$$

Essentially the same probabilistic construction used above to prove the corollary now shows that (2) holds. This completes the proof of the theorem.

REFERENCES

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