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ON THE MONOTONICITY OF THE OC OF AN SPRT[1]

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1. Introduction. This paper presents a theorem which may be of interest to anyone wishing to establish the monotonicity of the OC function of an SPRT. The same theorem may also be useful in finding bounds on the probability of acceptance when the actual value may be difficult or impossible to obtain.

In a well-known result, Lehmann [5], [6] established a sufficient condition for monotonicity when the observations form a sequence of independent random variables. This condition, simply stated, is that the distributions of the likelihood ratios be stochastically monotone. This is in turn satisfied if the family of densities possesses a monotone likelihood ratio. When the observations are not necessarily independent, Ghosh [3] has given a sufficient condition for monotonicity; namely, that the joint density of the observations possesses a monotone likelihood ratio. For further discussion in this area the reader is referred to the paper by Hall, Wijsman and Ghosh [4].

The theorem which is given in Section 2 of the paper is basically an extension of Lehmann's result to non-independent variables. It enables us to establish monotonicity for some problems in which Ghosh's condition is not met. In Section 3 two such examples of nonparametric type SPRT's are given.

2. Monotonicity of the OC. Let $\mathbf{X}_n = (X_1, \cdots, X_n)$ be a random vector with probability density function $p_{n\theta}(x_1, \cdots, x_n)$ which depends on the real parameter θ. Let

$$Z_i = \log \frac{p_{i\theta_1}(X_1, \cdots, X_i)}{p_{i\theta_0}(X_1, \cdots, X_i)} \qquad i = 1, 2, \cdots$$

be the sequence of log likelihood ratios which with the boundaries a, b define the SPRT of the hypotheses $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$.

We will use the following

Lemma. *Let H be a Lebesgue measurable function defined on the real line which is nonincreasing and nonnegative. If F_1 and F_2 are two distribution functions on the real line such that $F_1(x) \geqq F_2(x)$ for all x then $\int H dF_1 \geqq \int H dF_2$.*

Proof. The result follows easily by considering $\int H d(F_1 - F_2)$ and constructing a sequence of step functions increasing to H and finally by applying dominated convergence.

In the following theorem we consider two sequences of distribution functions $F = \{F_n\}$ and $G = \{G_n\}$ and compare the probability of accepting H_0 when the

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distribution of $(Z_1, \cdots, Z_n)$ is F_n with the probability of acceptance when the distribution of $(Z_1, \cdots, Z_n)$ is G_n.

THEOREM. *Let $F = \{F_n\}$ and $G = \{G_n\}$ be two sequences of absolutely continuous distribution functions, F_n and G_n being defined on n-space, satisfying*

(i) $G_n(z_n \mid z_1, \cdots, z_{n-1}) \geqq F_n(z_n \mid z_1, \cdots, z_{n-1})$ *and*

(ii) $G_n(z_n \mid z_1, \cdots, z_{n-1})$ *is nonincreasing in* $z_1, \cdots, z_{n-1}$

or $F_n(z_n \mid z_1, \cdots, z_{n-1})$ *is nonincreasing in* $z_1, \cdots, z_{n-1}$

for all n. Then $P_G[\text{accept } H_0] \geqq P_F[\text{accept } H_0]$.

In applying this theorem to establish that an SPRT has a monotone OC function, we see that the requirement of a stochastically increasing "conditional" likelihood ratio replaces the usual stochastically increasing condition used in the independence case. An additional requirement, condition (ii), appears to be a natural one and in most cases should not cause much difficulty.

A second application of the theorem is in obtaining bounds on the probability of acceptance when the usual approximation techniques fail. For example, suppose we wish to know the probability of acceptance, $P_F[\text{accept } H_0]$, but can not approximate it directly. It may be possible, however, to find a distribution G for the likelihood ratios for which $P_G[\text{accept } H_0]$ is either known or can be approximated. Further, if the conditions of the theorem are satisfied, we then have $P_G[\text{accept } H_0]$ as an upper bound to the desired probability of acceptance, $P_F[\text{accept } H_0]$.

PROOF. Define the sets

$$A_i = \{(z_1, \cdots, z_i) \mid b < z_j < a \quad \text{for} \quad j = 1, \cdots, i-1 \quad \text{and} \quad z_i \leqq b\}$$
$$C_i = \{(z_1, \cdots, z_i) \mid b < z_j < a \quad \text{for} \quad j = 1, \cdots, i\}$$

and for $i = 1, \cdots, n+1$, let

(1) $$h_i(z_1, \cdots, z_n) = f_{i-1}(z_1, \cdots, z_{i-1}) g_n(z_i, \cdots, z_n \mid z_1, \cdots, z_{i-1})$$

where f_n and g_n are the densities associated with F_n and G_n, respectively. If it can be shown that

(2) $$P_G(\textstyle\bigcup_{i=1}^n A_i) \geqq P_F(\bigcup_{i=1}^n A_i)$$

holds for all n then the theorem will follow. To accomplish this it will suffice to show that

(3) $$P_{h_i}(\textstyle\bigcup_{j=1}^n A_j) \geqq P_{h_{i+1}}(\bigcup_{j=1}^n A_j)$$

is satisfied for $i = 1, \cdots, n$ and all n because $h_1 = g_n$ and $h_{n+1} = f_n$. For $j < i$

(4) $$P_{h_i}(A_j) = P_{h_{i+1}}(A_j);$$

therefore we need only to establish that

(5) $$\textstyle\sum_{j=i}^n P_{h_i}(A_j) \geqq \sum_{j=i}^n P_{h_{i+1}}(A_j).$$

We hereafter hold i fixed and let n be $\geqq i$. Let $I(\cdot)$ denote the indicator function and define

$$(6) \qquad \tilde{A}_j = \{(z_i, \cdots, z_n) \mid b < z_k < a \quad \text{for} \quad k = i, \cdots, j-1, \quad \text{and} \quad z_j \leqq b\}$$

for $j = i, \cdots, n$. This permits us to write

$$(7) \qquad \textstyle\sum_{j=i}^{n} P(A_j) = \int I(C_{i-1}) \{\int \sum_{j=i}^{n} I(\tilde{A}_j)\, dP(z_i, \cdots, z_n \mid \mathbf{z}_{i-1})\}\, dP(\mathbf{z}_{i-1})$$

where P represents either H_i or H_{i+1}, the distribution functions associated with h_i and h_{i+1}, and where $\mathbf{z}_i = (z_1, \cdots, z_i)$.

Next define

$$(8) \qquad K_{n-1} = \textstyle\int \sum_{j=i}^{n} I(\tilde{A}_j)\, dP(z_n \mid \mathbf{z}_{n-1}) \qquad \text{and}$$

$$K_j = \int K_{j+1}\, dP(z_{j+1} \mid \mathbf{z}_j) \quad \text{for} \quad i = i, \cdots, n-2.$$

Now from (1) we may write

$$(9) \qquad h_i(z_1, \cdots, z_n) = f_{i-1}(\mathbf{z}_{i-1}) g_i(z_i \mid \mathbf{z}_{i-1}) g_n(z_{i+1}, \cdots, z_n \mid \mathbf{z}_i)$$

$$h_{i+1}(z_1, \cdots, z_n) = f_{i-1}(\mathbf{z}_{i-1}) f_i(z_i \mid \mathbf{z}_{i-1}) g_n(z_{i+1}, \cdots, z_n \mid \mathbf{z}_i).$$

We shall assume from condition (ii) that $G_n(z_n \mid z_1, \cdots, z_{n-1})$ is nonincreasing in $z_1, \cdots, z_{n-1}$. If instead $F_n(z_n \mid z_1, \cdots, z_{n-1})$ is assumed to be nonincreasing we change the definition of h_i in (1) and proceed in the same fashion. Now, if $j \geqq i$, it follows from the expressions in (9) that

$$h_i(z_{j+1} \mid \mathbf{z}_j) = h_{i+1}(z_{j+1} \mid \mathbf{z}_j) = g_{j+1}(z_{j+1} \mid \mathbf{z}_j)$$

and therefore from condition (ii) that $P(z_{j+1} \mid \mathbf{z}_j)$ is nonincreasing in $z_i, \cdots, z_j$ for $j = i, \cdots, n$.

Define $\tilde{C}_{n-1} = \{(z_i, \cdots, z_n) \mid b < z_k < a \quad \text{for} \quad k = i, \cdots, n-1\}$ and let the set D be the complement of $(\bigcup_{j=i}^{n-1} \tilde{A}_j) \cup \tilde{C}_{n-1}$. Then the three sets $\bigcup_{j=i}^{n-1} \tilde{A}_j$, $\tilde{C}_{n-1}$ and D are mutually exclusive and exhaustive. On these three sets K_{n-1} has the values 1, $P(b \mid \mathbf{z}_{n-1})$ and 0. Furthermore, when z_j increases, $\mathbf{z}_{n-1}$ can never move from $\tilde{C}_{n-1}$ to $\bigcup_{j=i}^{n-1} \tilde{A}_j$ or from D to $\tilde{C}_{n-1}$ or $\bigcup_{j=i}^{n-1} \tilde{A}_j$. It then follows that K_{n-1} is nonincreasing in $\mathbf{z}_{n-1}$. Using this as a first step in an induction proof, it is readily shown by means of the lemma that K_j is nonincreasing in $z_i, \cdots, z_j$.

Returning to expression (7), we may write

$$(10) \qquad \textstyle\sum_{j=i}^{n} P(A_j) = \int I(C_{i-1}) \{\int K_i\, dP(z_i \mid \mathbf{z}_{i-1})\}\, dP(\mathbf{z}_{i-1}).$$

Now from (9) we have

$$(11) \qquad P_{h_i}(z_i \mid \mathbf{z}_{i-1}) = G_i(z_i \mid \mathbf{z}_{i-1})$$

$$P_{h_{i+1}}(z_i \mid \mathbf{z}_{i-1}) = F_i(z_i \mid \mathbf{z}_{i-1})$$

and thus by condition (i) and the lemma we obtain

$$(12) \qquad \int K_i\, dP_{h_i}(z_i \mid \mathbf{z}_{i-1}) \geqq \int K_i\, dP_{h_{i+1}}(z_i \mid \mathbf{z}_{i-1}).$$

Finally, since $P_{h_i}(\mathbf{z}_{i-1}) = P_{h_{i+1}}(\mathbf{z}_{i-1})$ we have from (10) and (12) that (5) is established, which completes the proof.

3. An application. By way of illustration, we apply the theorem to the two-sample sequential rank test given by Parent [7] (see also [1], [2], [4]). Employing Parent's notation we have two sequences of independent random variables $\{X_i\}$, $\{Y_i\}$ and we wish to test $H_0:\theta = 1$ against $H_1:\theta = \theta_1$ ($\theta_1 > 1$), where F is the univariate distribution of the X's, and F^θ is the distribution of the Y's. It is assumed that the observations are taken alternatingly as X_1, Y_1, X_2, $Y_2, \cdots$ and that the SPRT is based on the sequential ranks of the observations. After n observations have been taken and ranked, let J_i equal θ if the observation of rank i is a Y; otherwise J_i is set equal to 1. Parent then shows that

$$
\begin{aligned}
(13) \qquad Z_n &= \log\{n!\,\theta_1^{\frac{1}{2}(n-1)}/\textstyle\prod_{i=1}^n B_i\} && n \text{ odd}\\
&= \log\{n!\,\theta_1^{\frac{1}{2}n}/\textstyle\prod_{i=1}^n B_i\} && n \text{ even}
\end{aligned}
$$

where $B_i = \sum_{j=1}^i J_j$ and $\theta = \theta_1$.

In order to show that this test's OC function is decreasing in θ we will establish conditions (i) and (ii) of the theorem. To begin, it can be seen from (13) that z_n is an increasing function of x_i and a decreasing function of y_i. Also the sequence $\{Z_n\}$ is transitive (see [4] for a discussion of transitivity) with respect to the ranks. Thus condition (ii) can be shown to be satisfied.

Condition (i) is somewhat more involved. For n odd it follows from (13) that the probability that the rank of $Y_{\frac{1}{2}(n+1)}$ is equal to k, conditional on ranks of the first n observations, is equal to

$$(14) \qquad C_k(\theta) = \theta \textstyle\prod_{i=k}^n B_i/\prod_{i=k-1}^n (\theta + B_i).$$

Now since $\{Z_i\}$ is transitive and Z_i is a decreasing function of y_i we have that for n odd condition (i) is equivalent to

$$(15) \qquad C_{jn} = \textstyle\sum_{k=1}^j C_k(\theta)$$

being a decreasing function of θ for all j ($1 \leqq j \leqq n+1$). This can be easily shown, and also the case of n even is obtained by similar arguments.

Another nonparametric SPRT is given by Weed [9]. His test is based upon signed ranks, and by using the same arguments as above, the monotonicity of the OC function can be established. Clearly we do not have monotone likelihood ratios for either of these two tests, and hence Ghosh's method is not applicable here.

Finally, it should be mentioned that we have not established that the two-sample sequential rank test which draws the observations in pairs ([1], [2], [8]), or in other ways [4], possesses a monotone OC function (though of course such a test could be interpreted as if observations came one at a time). One further point is that the above proof of the monotonicity of Parent's sequential rank test does not really depend upon the order in which the observations are taken. They must, however, be taken singly and their order must not be affected by previous data.

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