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OPTIMALLY TIMING THE SALE OF STOCK WHEN THE TAX MAN IS BREATHING DOWN YOUR NECK[1]

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1. Introduction. It is January 2. You have just finished filling out your income tax return on last year's earnings. With a shudder, you find that the year was a good one; so good in fact that your withholdings are not adequate to pay your tax bill. During the course of the year, you prudently invested a portion of your wealth in common stock, at the time intending a long-term investment. Now, your stock takes on a speculative character: Some time between now and April 15 the stock will have to be sold in order to pay your debt to the Government. Naturally, you want to sell when the stock is at its high point. Lacking a crystal ball, you seek some rationale for picking the propitious moment.

Intuitively, it seems clear that the stock should be held if it exhibits an upward trend and there is a long time to go before the April 15 deadline. Conversely, as the deadline approaches, a downward trend should motivate immediate sale. The other cases are not so clear-cut and we hope to cast light on them in what follows.

2. A Brownian motion model for stock prices. Assume that the decision whether to sell or not is made at certain discrete instants which are multiples of a constant δ. (δ might be one trading day on the NYSE or a fraction thereof.) Denote the price of the stock at the nth instant by p_n and assume that

$$p_{n+1} - p_n = x_{n+1} + \delta^{\frac{1}{2}} \sigma_p \eta_{n+1} \qquad n = \cdots, 1, 0, 1, 2, \cdots \tag{2.1}$$

where

$$x_{n+1} - x_n = \delta^{\frac{1}{2}} \sigma_x \xi_n \tag{2.2}$$

and

$$\{\xi_n; -\infty < n < \infty\} \quad \text{and} \quad \{\eta_n; -\infty < n < \infty\} \tag{2.3}$$

are independent sequences of i.i.d., $N(0, 1)$ rv's.

If the x_n's were all zero, the model would have the prices behave as samples from a driftless Wiener process with variance σ_p^2 per unit time.

The model which we have presented here, would have prices behave like samples from a "Wiener process" with a time varying drift (x_n) which is itself a zero mean Wiener process. If σ_x is small compared to σ_p, the drift will change slowly compared to the gross price changes. It is assumed that σ_p and σ_x are known. The decision problem can be thus stated:

At times $n\delta$ ($n = 1, 2, \cdots$) you must decide whether or not to sell enough of your stock to pay your taxes (selling in dribs and drabs is not allowed). When

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you sell, you receive an immediate payoff, p_n, the then current price of the stock. If you don't sell, you will be faced with the same decision at time $(n+1)\delta$. In any event, you *must* sell by time $M\delta$ (April 15) or else mandatory liquidation takes place at that time. You wish to sell at such a time as to maximize the (expected) value of p_N (N is the selling instant).

There is some precedence for such a model for stock price fluctuations (c.f. [6]) although many of the contributors to [6] take the option of postulating that the logarithms of prices, instead of the prices themselves, behave like a driftless Wiener process. We feel that the addition of the drift equation is a reasonable generalization to either model. If there is no drift, large sample estimates for σ_x^2 will tend to be close to zero and our model will reduce to the "classical ones". If you are more inclined toward the second model mentioned above (in which the logarithm of prices behaves like a sample from a Wiener process with slowly varying drift), the methods and results of the sequel all remain valid, provided (2.1)–(2.3) is assumed and that p_n is interpreted as the *logarithm* of the stock price at time $n\delta$. In this case, maximizing $\mathscr{E}p_N$ amounts to maximizing utility, if utility is assumed to be logarithmic in money. However, it is much more convenient, from the discursive point of view, to stick to the original interpretation (p_n = stock price at time $n\delta$). With this understood, we proceed with the analysis:

At each instant, δn, the decision to sell or not is made on the basis of the stock price history $\{p_j;\ -\infty < j \leqq n\}$. The posterior distribution of (p_n, x_n), given these prices, is a sufficient statistic for the present problem, and this distribution depends on the data only through p_n and $x_{n\,|\,n}$, the conditional expectation of x_n, given these data up through time $n\delta$. In [2], Bather proved the following relevant facts about $x_{n\,|\,n}$:

LEMMA 2.1. (a) $x_{n+1\,|\,n+1} = qx_{n\,|\,n}+(1-q)(p_{n+1}-p_n)$

where
$$q = \frac{\sigma_x^2}{2\sigma_p^2}\left\{1+\frac{2\sigma_p^2}{\sigma_x^2}-\left(1+\frac{4\sigma_p^2}{\sigma_x^2}\right)^{\frac{1}{2}}\right\}.$$

(b) *The conditional distribution of* $x_{n+1\,|\,n+1}-x_{n\,|\,n}$, *given the data up till time* $n\delta$, *is* $N(0, \delta\sigma_x^2)$.

Part (a) shows how to compute the drift estimate in a convenient recursive fashion (see also Kalman [7]). The estimate is an exponentially weighted sum of price differences, as one might expect. Part (b) will be used in the dynamic programming recursion which will be developed in the next section. For the sake of notational simplicity, we will assume that our unit of money has been chosen so that $\sigma_x^2 = 1$.

3. The relevant functional equation. A policy pursued from time $n\delta$ onward is optimal if it maximizes the conditional expectation (given the data up through time $n\delta$) of the difference between the price of the stock at the time of sale and the then current price (call it p). At first, it would appear that the decision to sell or not should depend upon the current price, p, the drift estimate, x, and the length of time to go before mandatory liquidation, t. Using the familiar optimality principle of dynamic programming, it is straightforward to show by induction on t, that the optimal policy can be chosen independent of p. This is so because the probabilistic behavior of future increases in price, given the data at hand, will depend only on the current drift estimate, not the price. Thus, we are led to define

(3.1) $g_\delta(x, t) =$ The expected difference between the current price and the price at the time of sale under an optimal policy, if the current drift estimate is x, if forced liquidation will occur t units of time hence (t a multiple of δ) and selling is allowed at times which are multiples of δ.

The optimality principle leads us to the following functional equation for g_δ:

$$g_\delta(x, t) = \max(0,\, x\delta + \mathscr{E} g_\delta(x + \delta^{\frac{1}{2}}\xi,\, t - \delta)) \tag{3.2}$$

$$g_\delta(x, 0) = 0\,. \tag{3.3}$$

The expectation in (3.2) is taken with respect to the $N(0, 1)$ rv, ξ. Equation (3.2) says that the optimal policy chooses the better of two alternatives. Sell now or wait one period and proceed optimally thereafter. The former action results in an increase of zero. The latter has an immediate expected increase of $x\delta$ when the drift estimate is x. Thereafter, there will be $t-\delta$ units of time to go and the drift estimate will change by an amount $\delta^{\frac{1}{2}}\xi$ where ξ is $N(0, 1)$ by virtue of Lemma 2.1(b). If we proceed optimally thereafter, the additional expected increase will be

$$\mathscr{E} g_\delta(x + \delta^{\frac{1}{2}}\xi,\, t - \delta) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} g_\delta(x + \delta^{\frac{1}{2}}\xi,\, t - \delta)\, e^{-\frac{1}{2}\xi^2}\, d\xi\,.$$

(3.3) follows from the fact that liquidation is mandatory when $t = 0$. A routine induction argument can be used to establish each part of

THEOREM 3.1. *If t is a multiple of δ, then*

(a) *$g_\delta(\cdot, t)$ is a nonnegative, nondecreasing convex function.*

(b) *$g_\delta(x, t) \leqq g_\delta(x, t+\delta)$ with strict inequality holding whenever the right side is positive.*

(c) *There is a unique scalar, $a_\delta(t)$ such that $g_\delta(\cdot, t)$ vanishes at $a_\delta(t)$, is zero to the left and is strictly increasing to the right of $a_\delta(t)$.*

(d) *$a_\delta(t+\delta) < a_\delta(t) \leqq a_\delta(\delta) = 0$.*

(e) *$g_\delta(x, t) = xt + H_\delta(x, t)$ where $H_\delta(x, t)$ is convex in x, strictly decreasing and approaches zero as $x \to +\infty$.*

(f) *$g_\delta(x, t) = \delta^{\frac{3}{2}} g_1(x\delta^{\frac{1}{2}}, t/\delta)$ and $a_\delta(t) = \delta^{\frac{1}{2}} a_1(t/\delta)$.*

The only part of the theorem which needs any kind of hint is part (b). This says that one is better off as the time to liquidation increases, provided the drift estimate stays the same. For, if one considers the optimal policy within the restricted class of those which sell on or before the next to last allowable moment when there are $t+\delta$ units of time to go, the expected price increase for this policy is exactly $g_\delta(x, t)$. The optimal policy does at least as well and has an expected price increase of $g_\delta(x, t+\delta)$.

Part (c) tells us that the optimal policy compares the current drift estimate to the threshold $a_\delta(t)$, which depends on how much time remains till mandatory liquidation. If x is less than or equal to $a_\delta(t)$, the expected return under the optimal policy is zero. This is achieved by selling immediately. If x exceeds $a_\delta(t)$, the expected price increase is positive. Therefore, it pays to wait a period and then review our position. Part (d) tells us that the threshold grows more stringent as the time to mandatory liquidation approaches. Part (e) asserts that the expected price increase under the optimal policy (conditional on x, the current drift estimate) is approximately bilinear in x and t if x is large and positive. Part (f) is proved by induction on (t/δ) and shows that the computations for g_δ and a_δ need only be carried out for the case $\delta = 1$.

The recursion (3.2) can be used in a straightforward way to generate the optimal selling rule (i.e., thresholds $a_1(t)$ $t = 1, 2, \cdots$) and associated expected price increases, $g_1(x, t)$. The computation becomes difficult and lengthy as t/δ grows large, however. Numerical results are feasible for moderate values of t/δ only.

Computations have been carried out for values of t/δ between 1 and 14. The corresponding values of $a_\delta(t)/\delta^{\frac{1}{2}}$ are given in Table 1, and plotted in Figure 1.

TABLE 1

t/δ	1	2	3	4	5	6	7
$a_\delta(t)/\delta^{\frac{1}{2}}$	0	−.30	−.52	−.70	−.86	−1.0	−1.12
t/δ	8	9	10	11	12	13	14
$a_\delta(t)/\delta^{\frac{1}{2}}$	−1.25	−1.35	−1.47	−1.56	−1.65	−1.75	−1.83

Because of the computational difficulties which arise when t/δ is large, it is of interest to explore asymptotic approximations to a_δ and g_δ for such cases. This will be done in the next four sections. The main results are that if t is a multiple of δ, then

$$\beta t^{\frac{1}{2}} \leqq a_\delta(t) \leqq \beta t^{\frac{1}{2}} + (\delta/t)^{\frac{1}{2}} + O(\delta/t) \quad \text{as} \quad \delta/t \to 0, \tag{3.4}$$

where

$$\beta = -.6388332\cdots \tag{3.5}$$

$$G(x, t) + \delta\beta t^{\frac{1}{2}}\left[\frac{\varphi(u) - u\Phi(-u)}{\varphi(\beta) - \beta\Phi(-\beta)}\right] \leqq g_\delta(x, t) \leqq G(x, t) \tag{3.6}$$

where

$$u = x/t^{\frac{1}{2}}, \tag{3.7}$$

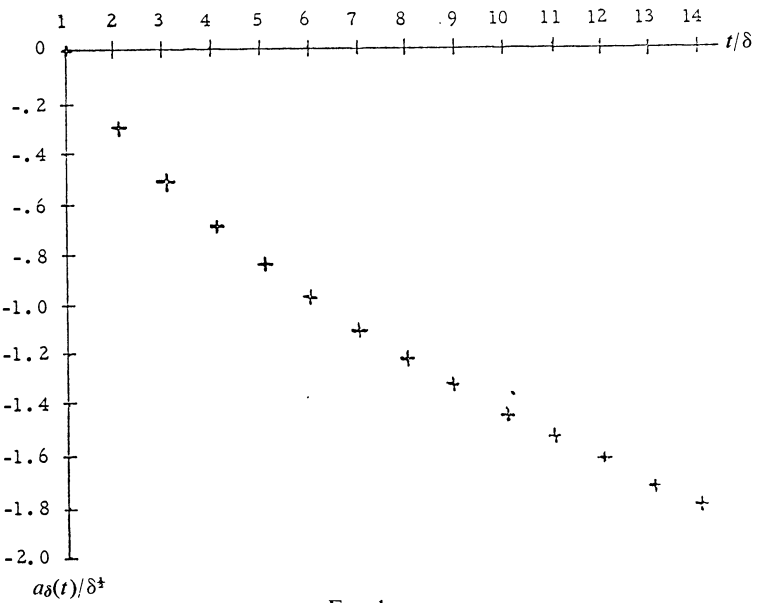

FIG. 1.

$$(3.8)\qquad G(x,t) = t^{\frac{3}{2}}\{u+c[(u^2+2)\varphi(u)-(u^3+3u)\Phi(-u)]\} \quad \text{if} \quad u > \beta,$$
$$= 0 \quad \text{otherwise,}$$

$$(3.9)\qquad c = .267147\cdots.$$

$$(3.10)\qquad \varphi(u) = (2\pi)^{-\frac{1}{2}}\exp -u^2/2 \quad \text{and} \quad \Phi(u) = \textstyle\int_{-\infty}^{u} \varphi(s)\,ds.$$

Thus, for large values of t/δ, we see that $a_\delta(t)$ is approximately $\beta t^{\frac{1}{2}}$ and $g_\delta(x, t)$ is approximately equal to $G(x, t)$. In Figure 2, we plot $G(x, t)$ as a function of x for various (integral) values of t. A plot of

$$\psi(u) = [\varphi(u)-u\Phi(-u)]/[\varphi(\beta)-\beta\Phi(-\beta)]$$

for $u \leqq \beta$ is given in Figure 3.

We now turn to the business of establishing (3.4) ff. This is accomplished by viewing the stock sale problem as a discrete time approximation to a certain continuous time stopping problem whose solution can be obtained in closed form as the result of solving the heat equation with a free boundary on which boundary conditions are specified. This technique has been employed in the past by Chernoff [3], [5], Bather [1], and McKean [8]. (The paper of McKean is actually a mathematical appendix to a paper by Samuelson [9].)

4. A related continuous time stopping problem. Let $\{X(s), -t \leqq s \leqq 0\}$ be a Wiener process with unit variance per unit time (no drift) and consider the stopping problem with payoff

$$\int_{-t}^{-T} X(s)\,ds$$

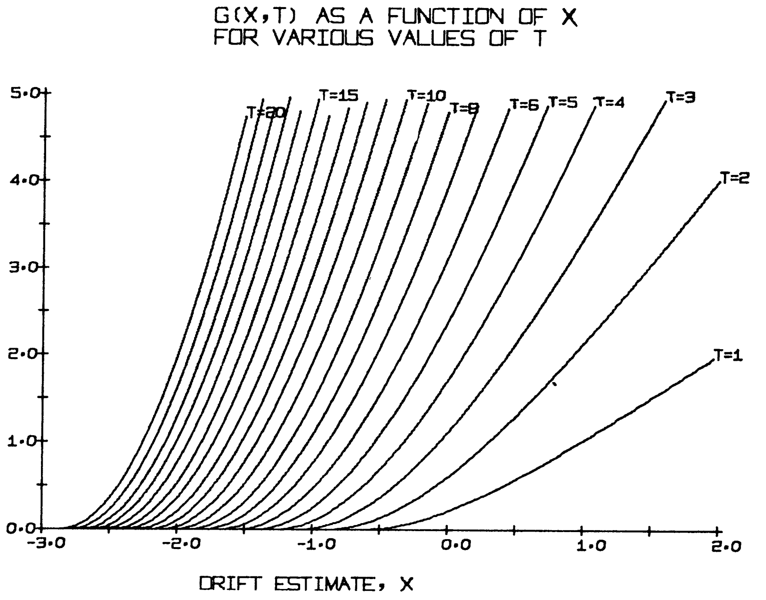
G(X,T) AS A FUNCTION OF X
FOR VARIOUS VALUES OF T
T=20
T=15
T=10
T=8
T=6
T=5
T=4
T=3
T=2
T=1
5.0
4.0
3.0
2.0
1.0
0.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
DRIFT ESTIMATE, X

FIG. 2.

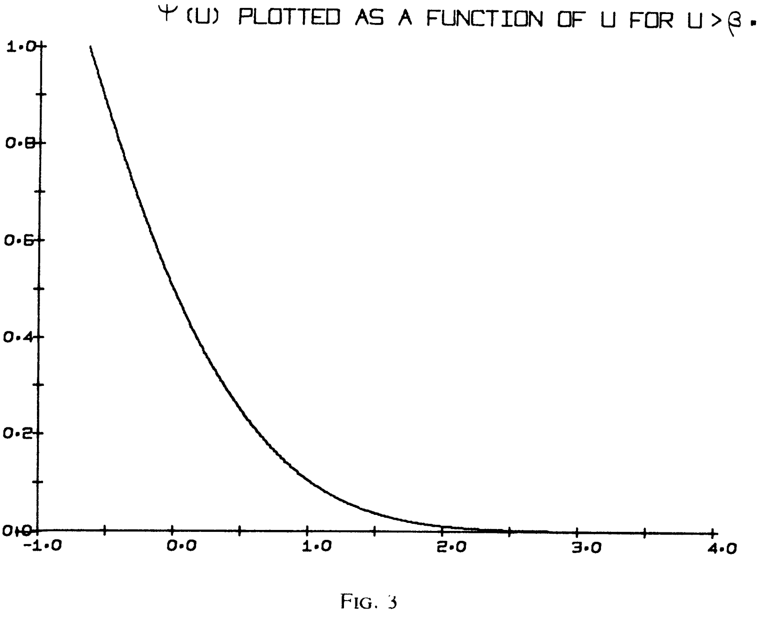
Ψ(U) PLOTTED AS A FUNCTION OF U FOR U > β.
1.0
0.8
0.6
0.4
0.2
0.0
-1.0
0.0
1.0
2.0
3.0
4.0

FIG. 3

if stopping occurs at time $-T \leqq 0$. If stopping has not occurred by time $s = 0$, the "game" is stopped and a payoff in the amount

$$\int_{-t}^{0} X(s)\, ds$$

is received. We study this stopping problem because the discrete version of the present problem is intimately related to the stock sale problem. Specifically, suppose the problem is modified so that stopping is only allowed at times which are multiples of $\delta > 0$. If the starting time, $-t$, is a multiple of δ, we can define

$G_\delta(x, t) =$ the expected payoff under the optimal stopping rule, given that $X(-t) = x$.

The usual dynamic programming argument is then applied and we find

$$G_\delta(x,t) = \max\left(0, \mathscr{E}\left\{\int_{-t}^{-t+\delta} X(s)\, ds + G_\delta(X(-t+\delta), t-\delta) \,\middle|\, X(-t) = x\right\}\right) \tag{4.1}$$

with

$$G_\delta(x, 0) = 0. \tag{4.2}$$

Since $X(s)$ is a Wiener process with unit variance per unit time,

$$\mathscr{E}\left\{\int_{-t}^{-t+\delta} X(s)\, ds \,\middle|\, X(-t) = x\right\} = x\delta \quad \text{and}$$

$$\mathscr{E}\{G_\delta(X(-t+\delta), t-\delta) \,|\, X(-t) = x\} = \mathscr{E} G_\delta(x+\delta^{\frac{1}{2}}\xi, t-\delta)$$

where $\xi \sim N(0, 1)$.

Thus (4.1), (4.2) take the form

$$G_\delta(x, t) = \max(0, x\delta + G_\delta(x+\delta^{\frac{1}{2}}\xi, t-\delta)) \tag{4.3}$$

$$G_\delta(x, 0) = 0 \tag{4.4}$$

which is identical to the recursion (3.2), (3.3) which uniquely determines g_δ and a_δ. Therefore $g_\delta(x, t) = G_\delta(x, t)$ if t is a multiple of δ and the optimal policies coincide: "Stop at the first value of t for which $X(-t) \leqq a_\delta(t)$".

It is reasonable to expect G_δ and the corresponding optimal discrete policy to be close to the expected payoff and optimal policy in the unrestricted (continuous time) stopping problem provided an optimal policy exists for the latter problem. Accordingly, we temporarily assume the existence of such an optimal policy and define

(4.5) $G(x, t) =$ the conditional expected payoff under the optimal continuous time stopping rule, given that $X(-t) = x$.

(Lest the reader be troubled by an apparent lapse in constructive rigor, we hasten to point out that the above-mentioned assumption, along with others which will be stated as needed, will be used to guide us to a plausible candidate for the title of "Optimal Stopping Rule". Later on we will properly prove that said candidate is indeed worthy of the title. So take heart and read on.)

At any instant $s(-t \leqq s \leqq 0)$, the decision whether or not to continue should depend only on $X(s)$ and not on any previous values of X, since they exert no effect on the value of

$$\int_s^{-T} X(u)\,du\,.$$

Therefore, any stopping rule worthy of consideration can be viewed as a partition of the (x, t) plane into an open continuation region, $\mathscr{C}$, and its complement, $\tilde{\mathscr{C}}$. If the process starts with $X(-t) = x$, we stop as soon as $(X(-s), s) \in \tilde{\mathscr{C}}$. Owing to the mandatory stopping proviso at $s = 0$,

$$\{(x, s) : s = 0\} \subset \tilde{\mathscr{C}} \quad a\ priori\,.$$

Associated with the continuation region, $\mathscr{C}$, is a stopping time $-T(x, t)$, where $T(x, t)$ is the first value of $\tau \leqq t$, such that $(X(-\tau), \tau) \in \tilde{\mathscr{C}}$ if $(X(-t), t) = (x, t)$. (Notice that T is bounded below by zero since $\tilde{\mathscr{C}} \supset \{(x, s) : s = 0\}$, and that G necessarily vanishes on $\tilde{\mathscr{C}}$.)

Following the familiar techniques of Chernoff and Bather ([3], [5], [1]), it is straightforward to show that the optimal solution $(G, \mathscr{C})$ to the continuous time stopping problem satisfies the partial differential equation

$$\frac{1}{2}\frac{\partial^2 G}{\partial x^2} + x = \frac{\partial G}{\partial t} \quad \text{inside} \quad \mathscr{C} \tag{4.6}$$

with the boundary conditions

$$G(x, t) = 0, \tag{4.7}$$

$$\frac{\partial G}{\partial x}(x, t) = 0 \qquad \text{on the boundary of } \mathscr{C}, \tag{4.8}$$

provided that an optimal procedure is assumed to exist and provided the associated payoff function, G, is assumed to be appropriately smooth.

Our attack will now proceed as follows: In Section 5 we will show that there is only one possible solution to (4.6)–(4.8) which qualifies as a potential solution to the continuous time stopping problem. We will exhibit it. In Section 6, we will prove that this procedure is indeed the optimal stopping rule, using an argument which is basically due to Chernoff [5]. In Section 7, we will establish the results described in (3.4)–(3.10), which link the solution of the original stock market problem to the solution of the continuous time problem described above.

5. Solution to the free boundary value problem. The continuous time stopping problem, which we posed in Section 4, can be phrased in the following form:

$\{X(s), -t \leqq s \leqq 0\}$ is a driftless Wiener process with unit variance per unit time. Choose the stopping time, $-T$, (or equivalently, the continuation region $\mathscr{C}$) so that

$$\mathscr{E}\left\{\int_{-t}^{-T} X(s)\,ds \,\middle|\, X(-t) = x\right\}$$

is maximized, and denote the associated optimal value by $G(x, t)$.

If X is measured in dollars and t is measured in days, the units of G are dollars × days. If the units were changed to francs and weeks, the stopping rule should be exactly the same (once the units are adjusted) and the ratio of expected payoffs should be in the ratio of dollars × days/francs × weeks.

More generally, let

$$t^* = at, \qquad X^*(s^*) = bX(s^*/a) \quad \text{and} \quad x^* = bx,$$

and consider the problem of choosing the stopping time, $-T^*$, so that

$$\mathscr{E}\{\int_{-t^*}^{-T^*} X^*(u)\,du \mid X^*(-t^*) = x^*\}$$

is maximized. If the maximal value is denoted by $G^*(x^*, t^*)$, it is clear that

$$G^*(x^*, t^*) = abG(x, t). \tag{5.1}$$

For the special choice $b = a^{\frac{1}{2}}$, $\{X^*(s^*); -t^* \leqq s^* \leqq 0\}$ is a Wiener process with unit drift per unit time, so for this choice of a and b

$$G^*(x, t) = G(x, t). \tag{5.2}$$

Combining (5.1) and (5.2) when $a^{\frac{1}{2}} = b$,

$$G(a^{\frac{1}{2}}x, at) = a^{\frac{3}{2}}G(x, t) \tag{5.3}$$

for all $a \geqq 0$, $t \geqq 0$ and all x.

Taking $a = 1/t$ yields

$$G(x, t) = t^{\frac{3}{2}}G(x/t^{\frac{1}{2}}, 1). \tag{5.4}$$

(Note the similarity to Theorem 3.1f.)

We now make the (plausible) assumption that the intersection of $\mathscr{C}$ with the line $[s = 1]$ is a semi-infinite interval, originating at some point $(\beta, 1)$ and extending to the "point" $(+\infty, 1)$. By virtue of (5.4), there is a function of a real variable, $f(\cdot)$, such that

$$G(x, t) = t^{\frac{3}{2}}f(x/t^{\frac{1}{2}}). \tag{5.5}$$

Since G vanishes at $t = 0$, f must be such that

$$\lim_{t\to 0} t^{\frac{3}{2}}f(x/t^{\frac{1}{2}}) = 0. \tag{5.6}$$

Since $(\beta, 1)$ is a boundary point of $\mathscr{C}$, (4.7) and (4.8) dictate that

$$f(\beta) = 0 \tag{5.7}$$

and

$$\dot{f}(\beta) = 0. \tag{5.8}$$

Since G satisfies (4.6) in $\mathscr{C}$, f satisfies the ordinary differential equation

$$\ddot{f}(u) + u\dot{f}(u) - 3f(u) + 2u = 0 \quad \text{if} \quad u > \beta. \tag{5.9}$$

(Again, we stress that we are making guesses and assumptions at this point, which allow us to generate an attractive procedure. Optimality is *proven* in Section 6.)

The general solution to (5.9) is of the form

$$f(u) = u + c[(u^2+2)\varphi(u) - (u^3+3u)\Phi(-u)] + d(u^3+3u) \tag{5.10}$$

where c and d are arbitrary constants.

Condition (5.6) dictates that $d = 0$ while (5.7) and (5.8) serve to specify c and β as the (unique) solution to

$$\begin{aligned} f(\beta) &= c[(\beta^2+2)\varphi(\beta) - (\beta^3+3\beta)\Phi(-\beta)] + \beta = 0 \\ \dot{f}(\beta) &= 3c[(\beta\varphi(\beta) - (\beta^2+1)\Phi(-\beta)] + 1 = 0\,. \end{aligned} \tag{5.11}$$

These equations are easily de-coupled and solved numerically; β is given by (3.5) and c is given by (3.9).

It is easy to show that $f(\cdot)$ is monotone increasing for $u > \beta$; hence, $G(\cdot, t)$ is monotone nondecreasing. If $u > \beta$, $f(u) > 0$ so that $G(x, t)$ is positive if $t > 0$ and $x > \beta t^{\frac{1}{2}}$.

Since G vanishes on $\tilde{\mathscr{C}}$ (c.f. Section 4),

$$\{(x, t): t > 0 \quad \text{and} \quad x/t^{\frac{1}{2}} > \beta\} \subset \mathscr{C}\,.$$

On the other hand, if $t > 0$ and $x/t^{\frac{1}{2}} < \beta$, G must vanish since $G(\cdot, t)$ is monotone and $G(\beta t^{\frac{1}{2}}, t) = 0$. Therefore, G and its derivatives are identically zero in this region, thereby making it impossible for (4.6) to be satisfied anywhere in this region. Since (4.6) holds for all $(x, t) \in \mathscr{C}$, the implication is that

$$\{(x, t): t > 0 \quad \text{and} \quad x/t^{\frac{1}{2}} < \beta\} \subset \tilde{\mathscr{C}}\,. \tag{5.12}$$

The boundary of $\mathscr{C}$ is therefore

$$\{(x, t): t > 0 \quad \text{and} \quad x/t^{\frac{1}{2}} = \beta\} \cup [t = 0]\,, \tag{5.13}$$

and the prime candidate for the optimal stopping rule is to "Continue if and only if $t > 0$ and $X(-t) > \beta t^{\frac{1}{2}}$."

The expected return from this policy, given that $X(-t) = x$ is

$$\begin{aligned} G(x, t) &= t^{\frac{3}{2}} f(x/t^{\frac{1}{2}}) \quad \text{if} \quad x/t^{\frac{1}{2}} > \beta \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{5.14}$$

where $f(\cdot)$ satisfies (5.10) with $d = 0$ and c given by (3.9).

6. Optimality. We will now prove that the policy described above is indeed optimal. Toward this end, let $\mathscr{C}$ be the continuation region $\{(x, t): t > 0$ and $x > \beta t^{\frac{1}{2}}\}$, with associated stopping time $-T(x, t)$, and let $-T'$ be the generic stopping time associated with some other policy. For any $\delta > 0$, define

$$\begin{aligned} T_\delta' &= T' \quad \text{if} \quad T'; \quad \text{is divisible by } \delta \\ &= \delta[T'/\delta] \quad \text{otherwise}\,. \end{aligned} \tag{6.1}$$

$-T_\delta'$ is a stopping time which agrees with $-T'$ if T' is a multiple of δ. Otherwise, $-T_\delta'$ waits till the first instant thereafter which is divisible by δ, then stops. We prove optimality via two lemmas which we state now and prove later.

LEMMA 6.1. *Let $G'(x, t)$ be the conditional expected payoff given that $X(-t) = x$, associated with the stopping time $-T'$. Let $G_\delta'(x, t)$ be the conditional expected payoff associated with the stopping time T_δ'. Then for each x and each $t > 0$*

$$G'(x, t) - G_\delta'(x, t) = O(\delta) \quad \textit{as} \quad \delta \to 0.$$

LEMMA 6.2. *Let $G(x, t)$ be as defined in* (3.8) *and suppose $\xi \sim N(0, 1)$. Then for any $\delta > 0$,*

$$\sup_{\tau \geqq \delta, y} \mathscr{E} G(y + \delta^{\frac{1}{2}}\xi, \tau - \delta) + y\delta - G(y, \tau) \leqq 0. \tag{6.2}$$

(In particular, (6.2) asserts that the policy described in Section 5 cannot be improved upon by the policy which samples for an additional time δ when $X(-t)/t^{\frac{1}{2}}$ is a boundary point of $\mathscr{C}$, and then proceeds according to the stopping rule induced by $\mathscr{C}$ thereafter. As Chernoff puts it, "the procedure cannot be trivially improved upon.")

THEOREM 6.3. *The policy which continues sampling so long as $t > 0$ and $X(-t) > \beta t^{\frac{1}{2}}$ is optimal.*

PROOF OF THEOREM 6.3. Let $G'(x, t)$ be the conditional expected return for any other policy (given that $X(-t) = x$). We will show that $G'(x, t) \leqq G(x, t)$:

Let $x, t > 0$ and $\alpha > 0$ be given. Choose $\delta > 0$ in accordance with Lemma 6.1 so that δ divides t and

$$|G'(x, t) - G_\delta'(x, t)| < \alpha. \tag{6.3}$$

By Lemma 6.2

$$\sup_{y, \tau \geqq \delta} \mathscr{E} G(y + \delta^{\frac{1}{2}}\xi, \tau - \delta) + y\delta - G(y, \tau) \leqq 0. \tag{6.4}$$

The optimal procedure among those which allow stopping only at times which are multiples of δ has conditional payoff function $G_\delta(x, t)$ satisfying (4.3), (4.4). In particular, $G_\delta(y, \delta) = y\delta$ and by (6.4), $G(y, \delta) \geqq y\delta$ so that for $k = 1$,

$$G_\delta(y, k\delta) \leqq G(y, k\delta) \quad \text{for all } y. \tag{6.5}$$

(6.4) and (4.3) can be combined easily, to prove by induction that (6.5) holds for all k and all y. In particular, since t is a multiple of δ:

$$G_\delta(y, t) \leqq G(y, t). \tag{6.6}$$

Since G_δ is the maximal-discrete-time-policy-payoff and since G_δ' is the payoff for such a policy,

$$G_\delta(y, t) \geqq G_\delta'(y, t) \quad \text{for all } y. \tag{6.7}$$

(6.3), (6.6), and (6.7) combine to yield

(6.8) $$G(x, t) \geqq G'(x, t) - \alpha \quad \text{where } \alpha \text{ is arbitrarily small,}$$

which proves the theorem.

PROOF OF LEMMA 6.1. The conditional expected risk under $-T'$ is

$$G'(x, t) = \mathscr{E}\left[\int_{-t}^{-T'} X(s)\, ds \,\middle|\, X(-t) = x\right]$$

whereas

$$G_\delta'(x, t) = \mathscr{E}\left[\int_{-t}^{-T_\delta'} X(s)\, ds \,\middle|\, X(-t) = x\right]$$

so that

$$\left|G_\delta'(x, t) - G'(x, t)\right| \leqq \mathscr{E}\left\{[T' - T_\delta'] \sup_{-t \leqq s \leqq 0} |X(s)| \,\middle|\, X(-t) = x\right\}.$$

Since $T' - T_\delta' \leqq \delta$ with certainty, and since

$$\mathscr{E}\left\{\sup_{-t \leqq s \leqq 0} |X(s)| \,\middle|\, X(-t) = x\right\} < \infty,$$

the conclusion follows.

PROOF OF LEMMA 6.2. Let

$$H_\delta(y, \tau) = G(y, \tau) - \mathscr{E}G(y + \delta^{\frac{1}{2}}\xi, \tau - \delta) - y\delta$$

and let

$$\mathscr{C}_\delta^* = \{(y, \tau): y > \beta\tau^{\frac{1}{2}} \quad \text{and} \quad \tau > \delta\}.$$

It suffices to show that H_δ is nonnegative on the boundary of $\mathscr{C}_\delta^*$. For then, since H_δ satisfies the heat equation inside $\mathscr{C}_\delta^*$ (by virtue of (4.6) and the fact that $\mathscr{C}_\delta^* \subset \mathscr{C}$), the maximal principle guarantees that values of H_δ inside $\mathscr{C}_\delta^*$ are weighted averages of values on the boundary, so that H_δ is nonnegative inside as well (c.f. Chernoff [5], for more details). On the other hand, if $y < \beta\tau^{\frac{1}{2}}$ and $\tau > \delta$, then since G vanishes at $(\beta\tau^{\frac{1}{2}}, \tau)$ and is nondecreasing in its first argument,

$$H_\delta(y, \tau) \geqq G(y, \tau) - G(\beta\tau^{\frac{1}{2}} + \delta^{\frac{1}{2}}\xi, \tau - \delta) - \beta\tau^{\frac{1}{2}}\delta = H_\delta(\beta\tau^{\frac{1}{2}}, \tau).$$

We will therefore show that

(6.9) $$H_\delta(y, \tau) > 0 \quad \text{if} \quad \tau = \delta \quad \text{and} \quad y \geqq \beta\tau^{\frac{1}{2}} \quad \text{or} \quad \tau > \delta \quad \text{and} \quad y = \beta\tau^{\frac{1}{2}}.$$

If $\tau = \delta$ and $y \geqq \beta\tau^{\frac{1}{2}}$, then

$$H_\delta(y, \tau) = G(y, \delta) - y\delta = \delta^{\frac{3}{2}}[(G(y, \delta)/\delta^{\frac{3}{2}}) - (y/\delta^{\frac{1}{2}})],$$

and from (5.14)

$$H_\delta(y, \delta) = c\delta^{\frac{3}{2}}[(u^2 + 2)\varphi(u) - (u^3 + 3u)\Phi(-u)]$$

where $u = y/\delta^{\frac{1}{2}}$ and $c > 0$. The expression in square brackets is a decreasing positive function of u for $u \geqq \beta$, so that $H_\delta(y, \delta) > 0$ if $y \geqq \beta\delta^{\frac{1}{2}}$.

Next, consider the case where $y = \beta\tau^{\frac{1}{2}}$ and $\tau > \delta$: By (5.5),

$$G(x, \tau) = 0 \quad \text{if} \quad u = x/\tau^{\frac{1}{2}} \leqq \beta \tag{6.10}$$
$$= \tau^{\frac{3}{2}} f(u) \quad \text{if} \quad u > \beta$$

where

$$\ddot{f}(u) = 6c[\varphi(u) - u\Phi(-u)], \tag{6.11}$$

$$f(\beta) = 0$$
$$\dot{f}(\beta) = 0 \tag{6.12}$$
$$\ddot{f}(\beta) = -2\beta.$$

Since $\ddot{f}$ is a decreasing positive function, a second order Taylor series expansion shows that $f(u) \leqq \ddot{f}(\beta)\frac{1}{2}(u-\beta)^2$ if $u \geqq \beta$, so that

$$G(x, \tau) = 0 \quad \text{if} \quad u \leqq \beta \tag{6.13}$$
$$\leqq \tau^{\frac{3}{2}}|\beta|(u-\beta)^2 \quad \text{if} \quad u > \beta.$$

Therefore

$$\mathscr{E}G(y+\delta^{\frac{1}{2}}\xi, \tau-\delta) = (\tau-\delta)^{\frac{3}{2}} \int_{[y+\delta^{1/2}\xi/(\tau-\delta)^{1/2} > \beta]} f((y+\delta^{\frac{1}{2}}\xi)/(\tau-\delta)^{\frac{1}{2}})\varphi(\xi)\, d\xi$$

and if $y = \beta\tau^{\frac{1}{2}}$,

$$G(y+\delta^{\frac{1}{2}}\xi, \tau-\delta) \leqq (\tau-\delta)^{\frac{1}{2}}|\beta|\delta \int_v^\infty (\xi-v)^2\varphi(\xi)\, d\xi \tag{6.14}$$

where $v = |\beta|[(t/\delta)^{\frac{1}{2}} - (t/\delta - 1)^{\frac{1}{2}}]$. Since

$$\int_v^\infty (\xi-v)^2\varphi(\xi)\, d\xi \leqq \tfrac{1}{2} \quad \text{if} \quad v \geqq 0$$

(6.14) implies

$$G(y+\delta^{\frac{1}{2}}\xi, \tau-\delta) + \delta y \leqq \beta\delta[\tau^{\frac{1}{2}} - (\tau-\delta)^{\frac{1}{2}}] \quad \text{if} \quad y = \beta\tau^{\frac{1}{2}}. \tag{6.15}$$

Since β is negative, and $G(y, \tau)$ is zero in the present case, the second part of (6.9) is proved.

7. Asymptotic approximations to the optimal stock sale policy. The conditional expected return function, g_δ, for the optimal stock sale problem of Sections I–III can be computed via equations (3.2)–(3.3), the optimal policy being to sell as soon as the conditional expectation of the drift, given the data at hand, falls below the threshold $a_\delta(t)$ when t (a multiple of δ) is the time left until mandatory liquidation and $a_\delta(t)$ is the largest zero of $g_\delta(\cdot, t)$. The computations for $a_\delta(t)$ and $g_\delta(x, t)$ become complex and lengthy as t/δ grows large and this motivated the continuous time stopping problem of Section 4, since, for this problem, the optimal stopping rule amongst those with stopping times restricted to multiples of δ coincides exactly with a_δ (stop and accept payment if $t = 0$ or if $X(-t) \leqq a_\delta(t)$) and the resulting conditional payoff function (given $X(-t) = x$) coincides exactly with $g_\delta(x, t)$.

Whereas $g_\delta(x, t)$ and $a_\delta(t)$ cannot be expressed in closed form, the conditional payoff and optimal stopping rule for the unrestricted continuous time stopping problem can be: "Stop if $X(-t) \leqq \beta t^{\frac{1}{2}}$ or $t = 0$" and the associated conditional payoff function is given by (3.8). It is reasonable to expect $a_\delta(t)$ to be close to $\beta t^{\frac{1}{2}}$ and G_δ (i.e., g_δ) to be close to G when δ is small. This assertion is made precise in

THEOREM 7.1. *If t is a positive multiple of δ, then*

(a) $$0 \leqq \frac{G(x, t) - g_\delta(x, t)}{t^{\frac{3}{2}}} \leqq \left(\frac{\delta}{t}\right)|\beta|\left[\frac{\varphi(u) - u\Phi(-u)}{\varphi(\beta) - \beta\Phi(-\beta)}\right]$$

where $u = x/t^{\frac{1}{2}}$, and

(b) $$0 \leqq \left(\frac{a_\delta(t)}{t^{\frac{1}{2}}} - \beta\right) \leqq \left(\frac{\delta}{t}\right)^{\frac{1}{2}}\left(1 + O\left(\frac{\delta}{t}\right)^{\frac{1}{2}}\right) \quad \textit{as } \delta/t \to 0,$$

where β is given by (3.5).

PROOF. Let $\delta > 0$ be given and suppose t is a multiple of δ. Let $-T(x, t)$ be the stopping time associated with the optimal stopping procedure, let

$$\begin{aligned} \tilde{T}_\delta(x, t) &= T \quad \text{if } T \text{ is a multiple of } \delta \\ &= [T/\delta]\delta \quad \text{otherwise} \end{aligned}$$

and let $\tilde{G}_\delta(x, t)$ be the conditional expected payoff (given $X(-t) = x$) associated with the modified stopping time $-\tilde{T}_\delta$. Clearly,

$$\begin{aligned} 0 \leqq G(x, t) - \tilde{G}_\delta(x, t) &= \mathscr{E}\{\textstyle\int_{-\tilde{T}_\delta}^{-T} X(s)\,ds \mid X(-t) = x\} \\ &= \mathscr{E}\{\mathscr{E}[\textstyle\int_{-\tilde{T}_\delta}^{-T} X(s)\,ds \mid X(-T)] \mid X(-t) = x\} \\ &= \mathscr{E}[(\tilde{T}_\delta - T)X(-T) \mid X(-t) = x]. \end{aligned}$$

If $-T = -t$, then since t is a multiple of δ, $\tilde{T}_\delta = t = T$.

In any event, $0 \leqq T - \tilde{T}_\delta \leqq \delta$ and $X(-T) = \beta T^{\frac{1}{2}} \leqq 0$, so if we define

(7.1) $$\begin{aligned} d(y, s) &= -y\delta \quad &\text{if } s > 0 \\ &= 0 \quad &\text{if } s = 0, \end{aligned}$$

it is clear that $\mathscr{E}[(\tilde{T}_\delta - T)X(-T) \mid X(-t) = x] \leqq \mathscr{E}[d(X(-T), T) \mid X(-t) = x]$. Thus

(7.2) $$0 \leqq G(x, t) - \tilde{G}_\delta(x, t) \leqq q(x, t) =_{\text{def}} \mathscr{E}[d(X(-T), T) \mid X(-t) = x].$$

If (x, t) is a boundary point of $\mathscr{C} = \{(y, s): s > 0 \text{ and } y > \beta s^{\frac{1}{2}}\}$, then $T = t$ and $q(x, t) = d(x, t)$. On the other hand, $q(x, t)$ satisfies the heat equation inside $\mathscr{C}$ ([5]):

(7.3) $$\tfrac{1}{2}q_{xx} = q_t.$$

The boundary value problem $\frac{1}{2}q_{xx} = q_t$ inside $\mathscr{C}$, $q = d$ on the boundary of $\mathscr{C}$ (where d is given by (7.1)) has the unique solution

(7.4) $$q(x, t) = \delta|\beta|t^{\frac{1}{2}}\left[\frac{\varphi(u) - u\Phi(-u)}{\varphi(\beta) - \beta\Phi(-\beta)}\right], \qquad u = \frac{x}{t^{\frac{1}{2}}}.$$

$\tilde{G}_\delta(x, t)$, being the conditional expected payoff for a particular discrete procedure, cannot exceed $G_\delta(x, t)$, the conditional expected payoff from the optimal discrete procedure. Since $G_\delta = g_\delta$, there follows

$$G(x, t) \geqq g_\delta(x, t) \geqq \tilde{G}_\delta(x, t), \tag{7.5}$$

so from (7.2)

$$0 \leqq G(x, t) - g_\delta(x, t) \leqq q(x, t) \tag{7.6}$$

which proves part (a).

To prove part (b), we note that

$$q(x, t) \leqq \delta|\beta| t^{\frac{1}{2}} \quad \text{if} \quad x/t^{\frac{1}{2}} > \beta. \tag{7.7}$$

By virtue of (5.5), (5.9), (5.10), and (5.11),

$$G(x, t) = t^{\frac{3}{2}} f(u) \quad \text{if} \quad u = x/t^{\frac{1}{2}} > \beta$$

where

$$\begin{aligned} &f(\beta) = \dot{f}(\beta) = 0 \\ &\ddot{f}(\beta) = -2\beta, \qquad \dddot{f}(u) = 6c[\varphi(u) - u\Phi(-u)] > 0 \end{aligned} \tag{7.8}$$

and

$$\dddot{f}(\beta) = -6c\Phi(-\beta).$$

Since $\dddot{f}$ is an increasing function, $f(u) \geqq \frac{1}{2}\ddot{f}(\beta)(u-\beta)^2 + \frac{1}{6}\dddot{f}(\beta)(u-\beta)^3$ when $u \geqq \beta$, so

$$G(x, t) \geqq G_\delta(x, t) \geqq G(x, t) - q(x, t) \tag{7.9a}$$

$$\geqq t^{\frac{3}{2}}[\tfrac{1}{2}\ddot{f}(\beta)(u-\beta)^2 + \tfrac{1}{6}\dddot{f}(\beta)(u-\beta)^3 + \beta(\delta/t)] \quad \text{if} \quad u = x/t^{\frac{1}{2}} > \beta. \tag{7.9b}$$

From (7.9a), the largest root of $G_\delta(\cdot, t)$ lies to the right of $G(\cdot, t)$'s largest root. Therefore

$$\beta t^{\frac{1}{2}} \leqq a_\delta(t). \tag{7.10}$$

From (7.9b), we see that $G_\delta(\cdot, t)$ is nonnegative (positive) whenever the expression in square brackets is nonnegative (positive), provided $t > 0$. The cubic

$$\tfrac{1}{2}\ddot{f}(\beta)z^2 + \tfrac{1}{6}\dddot{f}(\beta)z^3 + \beta(\delta/t)$$

has three real roots when δ/t is small. Two roots are positive, the smallest positive root occurring at

$$\tilde{z} = (\delta/t)^{\frac{1}{2}} + O(\delta/t) \quad \text{as} \quad \delta/t \to 0. \tag{7.11}$$

(We will prove this later.) Furthermore, said cubic is positive in a small right-hand neighborhood of $\tilde{z}$. Therefore, $G_\delta(x, t)$ is positive if $(x/t^{\frac{1}{2}}) - \beta > \tilde{z}$. We use the monotonicity of $G_\delta(\cdot, t)$ which is established in part (a) of Theorem 3.1.) From this we conclude that

$$(a_\delta(t)/t^{\frac{1}{2}}) - \beta \leqq \tilde{z} = (\delta/t)^{\frac{1}{2}} + O(\delta/t) \quad \text{as} \quad \delta/t \to 0. \tag{7.12}$$

Equations (7.10) and (7.12) prove part (b).

To prove (7.11), we will show that the smallest root of the equation

$$\xi + b\xi^{\frac{3}{2}} = C \qquad (b < 0) \tag{7.13}$$

is

$$\tilde{\xi} = C + O(C^{\frac{3}{2}}) \quad \text{as} \quad C \to 0. \tag{7.14}$$

(7.11) follows if we let $\xi = (z)^2$ and $C = \delta/t$.

As for (7.14), denote the smallest root of (7.13) by

$$\tilde{\xi}(C) = C + \varepsilon(C). \tag{7.15}$$

Inserting (7.15) into (7.13), we find that $\varepsilon(C) + b(C + \varepsilon(C))^{\frac{3}{2}} = 0$, or equivalently,

$$[\varepsilon(C)/C]/[(1 + \varepsilon(C)/C)^{\frac{3}{2}}] = -bC^{\frac{1}{2}}.$$

The smallest root of the equation $y/(1+y)^{\frac{3}{2}} = t$ is $O(t)$ as $t \to 0$, so that

$$\varepsilon(C)/C = O(C^{\frac{1}{2}}) \quad \text{as} \quad C \to 0. \tag{7.16}$$

(7.14) follows from (7.15) and (7.16).

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