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APPROXIMATION OF AGE DEPENDENT, MULTITYPE BRANCHING PROCESSES[1]

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1. Introduction. Let $X_n(t)$ be a sequence of k-type branching processes. Let $G_n^{\,i}(t)$ and $h_n^{\,i}(z)$ denote, respectively, the distribution function for the lifetime and the generating function for the distribution of the offspring, of a particle of type i. We shall assume $G_n^{\,i}(t)$ is right continuous, $G_n^{\,i}(0) = 0$ and $h_n^{\,i}(\mathbf{1}) = 1$. The following notation will be useful:

$$Z_0^{\,k} = \{l = (l_1, l_2, \cdots l_n): l_i \text{ nonnegative integers}\};$$

$$S^k = \{z = (z_1, z_2, \cdots, z_n): z_i \text{ complex}, |z_i| \leqq 1\};$$

for $l \in Z_0^{\,k}$ and $z \in S^k$

$$z^l = \textstyle\prod_{i=1}^k z_i^{\,l_i};$$

$e(i)$ = vector with ith component 1 and other components 0; $\mathbf{0}, \mathbf{1}$ denote the vectors with all components 0 and all components 1;

If A and B are either vectors or matrices, $A \leqq B$ means the inequality holds for the corresponding elements; for $z \in S^k$

$$|z| = (|z_1|, |z_2|, \cdots, |z_n|) \qquad \text{and}$$

$$\|z\| = \textstyle\sum_{i=1}^k |z_i|.$$

The generating function

$$F_n^{\,i}(z, t) = \textstyle\sum_{l \in Z_0^k} P\{X_n(t) = l \mid X_n(0) = e(i)\} z^l$$

satisfies

$$F_n^{\,i}(z, t) = z_i(1 - G_n^{\,i}(t)) + \textstyle\int_0^t h_n^{\,i}(F_n(z, t-s))\, dG_n^{\,i}(s)$$

where

$$F_n(z, t) = (F_n^{\,1}(z, t) \cdots F_n^{\,k}(z, t)).$$

To simplify notation further, let

$$h_n(z) = (h_n^{\,1}(z) \cdots h_n^{\,k}(z))$$

and let $G_n(t)$ denote the diagonal matrix with diagonal elements $G_n^{\,1}(t) \cdots G_n^{\,k}(t)$. Then

$$F_n(z, t) = z(I - G_n(t)) + \textstyle\int_0^t h_n(F_n(z, t-s))\, dG_n(s),$$

where the meaning of the integration is obvious.

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We are interested in the behavior of F_n under the assumption that there exist right continuous distribution functions $G^i(t)$ with $G^i(0) = 0$ and generating functions $h^i(z)$ with $h^i(\mathbf{1}) = 1$ such that

$$\lim_{n\to\infty} G_n^{\,i}(t) = G^i(t) \quad \text{at all points of continuity} \qquad \text{and}$$

$$\lim_{n\to\infty} h_n^{\,i}(z) = h^i(z) \qquad \text{for all} \quad z \in S^k.$$

We observe that these conditions imply

$$\lim_{n\to\infty} G_n(t) = G(t) \quad \text{almost everywhere} \qquad \text{and} \tag{1.1}$$

$$\lim_{n\to\infty} \sup_{|z| \leqq \mathbf{1}} ||h_n(z) - h(z)|| = 0. \tag{1.2}$$

$F(z, t)$ will denote the vector of generating functions for the branching process corresponding to $G(t)$ and $h(z)$. To avoid unnecessary complications in the proof we shall assume $F(0, t) < \mathbf{1}$ for all $t < \infty$. In particular, we shall prove the following:

THEOREM 1.3. *For every $t \geqq 0$ and $\rho < \mathbf{1}$*

$$\lim_{n\to\infty} \int_0^t \sup_{|z| \leqq \rho} ||F(z, s) - F_n(z, s)||\, ds = 0.$$

THEOREM 1.4. *Suppose $G(t)$ is continuous. Then for every $t \geqq 0$ and $\rho < \mathbf{1}$*

$$\lim_{n\to\infty} \sup_{s \leqq t} \sup_{|z| \leqq \rho} ||F(z, s) - F_n(z, s)|| = 0.$$

THEOREM 1.5. *Suppose*

$$G_n(t) = G(l/n) \quad \textit{for} \quad l/n \leqq t < (l+1)/n.$$

Then for every $t \geqq 0$ and $\rho < \mathbf{1}$

$$\lim_{n\to\infty} \sup_{|z| \leqq \rho} ||F(z, t) - F_n(z, t + n^{-\frac{1}{2}})|| = 0.$$

2. Proofs of the theorems. Let

$$P = \{H(z, t) = \textstyle\sum_{l \in Z_0^k} b_l(t) z^l : b_l(t) \in S^k, b_l(t) \geqq \mathbf{0},$$

$H(\mathbf{1}, t) \leqq \mathbf{1}$, and $H(z, t)$ is a right continuous function of $t \geqq 0\}$. Define $K_n: P \to P$ by

$$K_n H(z, t) = z(I - G_n(t)) + \int_0^t h_n(H(z, t-s))\, dG_n(s).$$

Similarly define $K: P \to P$ with G_n and h_n replaced by G and h.

LEMMA 2.1. *Let H_n, $H \in P$. For every $t \geqq 0$ and $\rho < \mathbf{1}$*

$$\lim_{n\to\infty} \int_0^t \sup_{|z| \leqq \rho} ||H(z, s) - H_n(z, s)||\, ds = 0 \tag{2.2}$$

implies

$$\lim_{n\to\infty} \int_0^t \sup_{|z| \leqq \rho} ||KH(z, s) - K_n H_n(z, s)||\, ds = 0.$$

PROOF. We observe that

$$\begin{aligned}\int_0^t \sup_{|z|\leqq\rho} ||KH(z,s)-K_n H_n(z,s)||\,ds &\leqq \int_0^t \sup_{|z|\leqq\rho} ||z(G_n(s)-G(s))||\,ds \\ &+\int_0^t \sup_{|z|\leqq\rho} ||\int_0^s [h_n(H_n(z,s-u))-h(H_n(z,s-u))]\,dG_n(u)||\,ds \\ &+\int_0^t \sup_{|z|\leqq\rho} ||\int_0^s [h(H_n(z,s-u))-h(H(z,s-u))]\,dG_n(u)||\,ds \\ &+\int_0^t \sup_{|z|\leqq\rho} ||\int_0^s h(H(z,s-u))\,d(G_n(u)-G(u))||\,ds.\end{aligned}$$

The first term on the right-hand side converges to zero by condition (1.1), the second by condition (1.2) and the third by (2.2). Letting

$$H(z,t)=\sum_{l\in Z_0^k} b_l(t)z^l,$$

to show that the fourth term converges to zero it is sufficient to show that

$$\lim_{n\to\infty}\int_0^t |\int_0^s b_l(s-u)\,d(G_n(u)-G(u))|\,ds=0.$$

This can be done by approximating $b_l(t)$ by continuous functions $\gamma_l(t)$ in L^1 and observing that

$$\lim_{n\to\infty}\int_0^s \gamma_l(s-u)\,d(G_n(u)-G(u))=0 \qquad \text{for all } s.$$

Using a similar argument we have

LEMMA 2.3. *Let H_n, $H\in P$ and suppose $H(z,t)$ is a continuous function of t. If $G(t)$ is continuous, then $KH(z,t)$ is continuous and for every $t\geqq 0$ and $\rho<1$*

$$\lim_{n\to\infty}\sup_{s\leqq t}\sup_{|z|\leqq\rho}||H(z,s)-H_n(z,s)||=0$$

implies

$$\lim_{n\to\infty}\sup_{s\leqq t}\sup_{|z|\leqq\rho}||KH(z,s)-K_n H_n(z,s)||=0.$$

The analogous lemma to be used in the proof of Theorem 1.5 is somewhat more complicated.

LEMMA 2.4. *Let H_n, $H\in P$. Suppose there exists a constant C such that for every $t\geqq 0$, $\rho<1$ and every sequence $\{t_n\}$ satisfying $t_n>t+C/n$ and $\lim_{n\to\infty}t_n=t$*

$$\lim_{n\to\infty}\sup_{|z|\leqq\rho}||H_n(z,t_n)-H(z,t)||=0. \tag{2.5}$$

If $G_n(t)$ satisfies the conditions of Theorem 1.5, then $K_n H_n$ and KH satisfy the above conditions with C replaced by $C+1$.

PROOF. Suppose $t_n>t+(C+1)/n$, and $\lim_{n\to\infty}t_n=t$. Let $k_n(s)$ be the smallest integer greater than or equal to ns. Then

$$t_n-k_n(s)/n>t-s+C/n.$$

Consequently, by (2.5)

$$\lim_{n\to\infty}\sup_{|z|\leqq\rho}||H_n(z,t_n-k_n(s)/n)-H(z,t-s)||=0. \tag{2.6}$$

We note that

$$K_n H_n(z, t_n) = z(I - G([nt_n]/n)) + \int_0^{[nt_n]/n} h_n(H_n(z, t_n - k_n(s)/n))\, dG(s),$$

and the lemma follows by (2.6) and the right continuity of G.

For each n, we consider the following approximation of $X_n(t)$: Let $X_n^{(m)}(t)$ be the vector giving the number of particles of each type from the generations $0, 1, 2, \cdots, m-1$ that are alive at time t. Define

$$F_n^i(z, t, m) = \sum_{l \in Z_0^k} P\{X_n^{(m)}(t) = l \,|\, X(0) = e(i)\} z^l.$$

Then

$$F_n(z, t, m) = K_n^m \mathbf{1}.$$

where $K_n^m \mathbf{1}$ is the mth power of the operator K_n operating on $H(z, t) \equiv \mathbf{1}$. (See Harris [1], page 132.)

It is clear that

$$\lim_{m \to \infty} X_n^{(m)}(t) = X_n(t),$$

and hence

$$\lim_{m \to \infty} F_n(z, t, m) = F_n(z, t).$$

However, the following lemma will be useful in establishing the rate of this convergence.

LEMMA 2.7. *Let $\Gamma_n(z)$ be the matrix with elements*

$$\gamma_{ij}^n(z) = \frac{\partial}{\partial z_i} h_n^j(z);$$

let

$$r_n(\rho, t, m) = F_n(0, t, m) + (\mathbf{1} - F_n(0, t, m)) \sup_{i \leqq k} \rho_i;$$

and let $\bar{G}_n(t)$ be the diagonal matrix with all diagonal elements equal to

$$G_n^0(t) = k^{-1} \textstyle\sum_{i=1}^k G_n^i(t).$$

These quantities, without the subscript n, are defined similarly for the limiting process. Then for every $t \geqq 0$ and $\rho \leqq \mathbf{1}$,

$$\begin{aligned}
\Delta_n(\rho, t, m+1) &\equiv \sup_{s \leqq t} \sup_{|z| \leqq \rho} |K_n^{m+1}\mathbf{1}(z, s) - F_n(z, s)| \\
&\leqq \int_0^t \Delta_n(\rho, t-s, m) \Gamma_n(r_n(\rho, t-s, m))\, dG_n(s) \qquad (2.8)\\
&\leqq k \int_0^t \Delta_n(\rho, t-s, m)\, d\bar{G}_n(s) \Gamma_n(r_n(\rho, t, m)),
\end{aligned}$$

and

$$\sup_{s \leqq t} \sup_{|z| \leqq \rho} |K_n^{m+l}\mathbf{1}(z, s) - F_n(z, s)| \leqq 2k^l P\{S_{n,l} \leqq t\} \mathbf{1} \Gamma_n^l(r_n(\rho, t, m)), \qquad (2.9)$$

where $S_{n,l}$ is the sum of l independent random variables with distribution $G_n^0(t)$ and Γ_n^l is the lth power of the matrix Γ_n.

PROOF. We first observe that $|z| \leqq \rho$ and $s \leqq t$ imply

$$|F_n(z,s)| \leqq F_n(\rho,s) \leqq F_n(\rho,s,m) \quad \text{and}$$

$$|F_n(z,s,m)| \leqq F_n(\rho,s,m) \leqq r_n(\rho,t,m),$$

and hence

$$|\Gamma_n(F_n(z,s))| \leqq \Gamma_n(r_n(\rho,t,m)) \quad \text{and}$$

$$|\Gamma_n(F_n(z,s,m))| \leqq \Gamma_n(r_n(\rho,t,m)).$$

Therefore

$$\begin{aligned}\Delta_n(\rho,t,m+1) &= \sup_{s\leqq t}\sup_{|z|\leqq\rho}\left|\int_0^s [h_n(F_n(z,s-u,m))-h_n(F_n(z,s-u))]\,dG_n(u)\right| \\ &\leqq \sup_{s\leqq t}\sup_{|z|\leqq\rho}\int_0^s |F_n(z,s-u,m)-F_n(z,s-u)|\,\Gamma_n(F_n(|z|,s-u,m))\,dG_n(u) \\ &\leqq \int_0^t \Delta_n(\rho,t-u,m)\Gamma_n(r_n(\rho,t,m))\,dG_n(u).\end{aligned}$$

The last inequality in (2.8) follows from

$$dG_n^{\,i}(s)/dG_n^{\,0}(s) \leqq k$$

and the fact that $\bar{G}_n(s)$ commutes with $\Gamma_n(z)$.

The inequality in (2.9) follows by iterating (2.8) l times and observing $\Delta_n \leqq 2\cdot\mathbf{1}$ and $r_n(\rho, t, m') \leqq r_n(\rho, t, m)$ for $m' > m$.

Let $t \geqq 0$, $\eta > 0$ and $\rho < \mathbf{1}$. Since

$$\lim_{m\to\infty} F(0,t+\eta,m) = F(0,t+\eta) < \mathbf{1},$$

there exists m such that $F(0, t+\eta, m) < \mathbf{1}$.

Lemma 2.1 implies

$$\lim_{n\to\infty}\int_0^t \sup_{|z|\leqq\rho}\|K^m\mathbf{1}(z,s)-K_n^{\,m}\mathbf{1}(z,s)\|\,ds = 0, \tag{2.10}$$

and since $r_n(\rho, t, m)$ is an increasing function of t

$$\limsup_{n\to\infty} r_n(\rho,t,m) \leqq r(\rho,t+\eta,m) < \mathbf{1}.$$

Letting S_l denote a sum of l independent random variables with distribution $G^0(t)$

$$\limsup_{n\to\infty} P\{S_{n,l} \leqq t\} \leqq P\{S_l \leqq t+\eta\}$$

and since

$$\lim_{n\to\infty}\gamma_{ij}^n(z) = \gamma_{ij}(z) \quad \text{for} \quad |z| < \mathbf{1},$$

(2.9) implies

$$\begin{aligned}\limsup_{n\to\infty}\sup_{s\leqq t}\sup_{|z|\leqq\rho}|K_n^{\,m+l}\mathbf{1}(z,s)-F_n(z,s)| \\ \leqq 2k^lP\{S_l \leqq t+\eta\}\mathbf{1}\Gamma^l(r(\rho,t+\eta,m)).\end{aligned} \tag{2.11}$$

Note that the right-hand side of (2.11) goes to zero as l goes to infinity faster than ε^l for any $0 < \varepsilon < 1$, since

$$\exp\{-\theta(t+\eta)\}\,P\{S_l \leqq t+\eta\} \leqq E(\exp\{-\theta S_l\}) \leqq \left[\int_0^\infty \exp\{-\theta t\}\,dG^0(t)\right]^l$$

and for θ sufficiently large

$$\int_0^\infty \exp\{-\theta t\}\,dG^0(t) < \varepsilon.$$

Finally we prove the theorems.

From (2.10) and (2.11) it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^t \sup_{|z|\leqq\rho} \|F(z,s)-F_n(z,s)\|\,ds \\
&\quad \leqq \lim_{n\to\infty} \int_0^t \sup_{|z|\leqq\rho} \|K_n{}^{m+l}\mathbf{1}(z,s)-K^{m+l}\mathbf{1}(z,s)\|\,ds \\
&\qquad + \limsup_{n\to\infty} \sup_{s\leqq t} \sup_{|z|\leqq\rho} t\,\|K_n{}^{m+l}\mathbf{1}(z,s)-F_n(z,s)\| \\
&\qquad + \sup_{s\leqq t} \sup_{|z|\leqq\rho} t\,\|K^{m+l}\mathbf{1}(z,s)-F(z,s)\| \\
&\quad \leqq 2t\,\|2k^l P\{S_l \leqq t+\eta\}\mathbf{1}\Gamma^l(r(\rho,t+\eta,m))\|,
\end{aligned}$$

and Theorem 1.3 follows.

Under the conditions of Theorem 1.4, Lemma 2.3 implies

$$\lim_{n\to\infty} \sup_{s\leqq t} \sup_{|z|\leqq\rho} \|K_n{}^{m+l}\mathbf{1}(z,s)-K^{m+l}\mathbf{1}(z,s)\| = 0,$$

and Theorem 1.4 follows similarly to Theorem 1.3.

Under the conditions of Theorem 1.5, Lemma 2.4 implies

$$\lim_{n\to\infty} \sup_{|z|\leqq\rho} \|K_n{}^{m+l}\mathbf{1}(z,t_n)-K^{m+l}\mathbf{1}(z,t)\| = 0 \tag{2.12}$$

for every sequence $t_n \to t$ with $t_n > t+(m+l)/n$.

Since $t+n^{-\frac{1}{2}} > t+(m+l)/n$ for n sufficiently large, (2.12) holds for all $l \geqq 0$ and $t_n = t+n^{-\frac{1}{2}}$, and Theorem 1.5 follows.

3. Example. Let $F(z, t)$ be the generating function for a continuous parameter Markov branching process with offspring generating function $f(z)$ and lifetime distribution $G(t) = 1-e^{-\alpha t}$.

Let $F_n(z, 1/n) = (1-\alpha/n)z+(\alpha/n)f(z)$ and let $F_n(z, m/n)$ denote the mth iterate of $F_n(z, 1/n)$. Then $F_n(z, m/n)$ is the mth generation generating function of a discrete parameter Markov branching process. We observe, however, that defining

$$F_n(z,t) = F_n(z,k/n), \qquad k/n \leqq t < (k+1)/n$$

we may interpret $F_n(z, t)$ as the generating function of an age dependent branching process with offspring generating function $f(z)$ and lifetime distribution

$$G_n(t) = \textstyle\sum_{l=0}^m \alpha/n(1-\alpha/n)^l = 1-(1-\alpha/n)^{m+1} \quad \text{for} \quad m/n \leqq t < (m+1)/n.$$

Since $\lim_{n\to\infty} G_n(t) = 1-e^{-\alpha t}$, Theorem 1.4 implies for every $t \geqq 1$, $\rho < 1$

$$\lim_{n\to\infty} \sup_{s\leqq t} \sup_{|z|\leqq\rho} \|F_n(z,t)-F(z,t)\| = 0.$$

Consequently, we have a natural way of approximating a continuous parameter Markov branching process by discrete parameter Markov branching processes.

REFERENCE

[1] HARRIS, T. E. (1963). *The Theory of Branching Processes.* Prentice-Hall, Inc., Englewood Cliffs, N.J.