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A NOTE ON UNIFORM CONVERGENCE OF STOCHASTIC PROCESSES[1]

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0. Introduction. Our aim in this note is to extend Theorems 5.1 and 5.2 of [4]. Let $R(\cdot, \cdot)$ be the covariance kernel of a Gaussian process with index set S, here S will always mean a compact metric space. R is assumed throughout to be continous on $S \times S$. Let $H(R)$ be the reproducing kernel Hilbert space of R. It is a Hilbert space of continuous functions k on S with the following properties:

$$R(\cdot, t) \in H(R) \qquad \text{for each} \quad t \in S; \tag{0.1}$$

$$\langle k, R(\cdot, t)\rangle = k(t), \tag{0.2}$$

where $\langle , \rangle$ denotes the inner product in $H(R)$. For a discussion of reproducing kernel Hilbert spaces and their application to the study of Gaussian processes we refer to [1] and [5]. In what follows $C(S)$ will denote the Banach space of real-valued continuous functions on S with the sup norm, and $\mathscr{C}$ the σ-algebra of Borel sets of $C(S)$. x will denote a generic element of $C(S)$.

Before stating the main results we would like to record here for later reference the fact that if $\{X_t, t \in S\}$ is a Gaussian process on some probability space $(\Omega, \mathscr{F}, P)$, then there is an isometric isomorphism between $H(R)$ and the closure of the linear space spanned by $\{X_t, t \in S\}$ in $L_2(\Omega, \mathscr{F}, P)$. We shall denote this closure by $\mathscr{L}_2(X_t, t \in S)$ and this isometric isomorphism by θ, where for $t \in S$ we have $\theta(R(\cdot, t)) = X_t$. We now state the main results.

THEOREM 1. *Let $\{X_t, t \in S\}$ be a Gaussian process with covariance R and almost all paths continuous on a complete probability space $(\Omega, \mathscr{F}, P)$. Let $\{\psi_j\}_{j=1}^{\infty}$ be a complete orthonormal system* (CONS) *in $H(R)$ and let $\{\xi_j\}_{j=1}^{\infty}$ be the sequence of independent random variables on $(\Omega, \mathscr{F}, P)$ each distributed normally with mean 0 and variance 1, given by $\xi_j = \theta(\psi_j)$. Then the partial sums*

$$\textstyle\sum_{j=1}^{n} \xi_j(\omega)\psi_j(t) = S_n(t, \omega) \tag{0.3}$$

converge uniformly in $t \in S$ to $X_t(\omega)$ as $n \to \infty$ a.e. (P).

THEOREM 2. *Let $\{\eta_j\}_{j=1}^{\infty}$ be a sequence of independent random variables on a complete probability space $(\Omega, \mathscr{F}, P)$, each distributed normally with mean 0 and variance 1. Let R be a covariance such that there exists a Gaussian process with this covariance and with almost all sample paths continuous (on some probability space). Let $\{\psi_j\}_{j=1}^{\infty}$ be a* CONS *in $H(R)$. If $S = [0, 1]$, then the partial sums*

$$\textstyle\sum_{j=1}^{n} \eta_j(\omega)\psi_j(t) = S_n'(t, \omega) \tag{0.4}$$

converge uniformly in $t \in [0, 1]$ to a Gaussian process on $(\Omega, \mathscr{F}, P)$ *whose covariance is R and almost all of whose sample paths are continuous as $n \to \infty$* a.e. (P).

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We would like to remark here that in the case of standard Brownian motion $H(R)$ is the space of absolutely continuous functions $f(t) = \int_0^t f'(u)\,du$ with $\int_0^1 f'^2(u)\,du < \infty$. If $\{\varphi_n\}_{n=1}^\infty$ is a CONS in $L_2[0, 1]$, then $\int_0^t \varphi_n(u)\,du$, $n = 1, 2, \cdots$, is a CONS in $H(R)$ for Brownian motion and so Theorems 5.1 and 5.2 [4] become particular cases of our theorems 1 and 2 respectively. It should be observed, however, that in Theorem 5.2 [4] it is not presumed that standard Brownian motion on [0, 1] can be defined to have continuous paths; this fact is automatically proved there. However, in a general result such as Theorem 2, one would need some conditions on R to guarantee that a Gaussian process with covariance R and having continuous paths exists.

1. Proofs. The following lemma is given for later use.

LEMMA 1. *Let* $\{\psi_n\}_{n=1}^\infty$ *be a* CONS *in* $H(R)$, *then* $\sum_{n=1}^\infty \psi_n^2(t)$ *converges uniformly in* $t \in S$ *to* $R(t, t)$.

PROOF. Given $t \in S$, $R(\cdot, t) \in H(R)$, hence by the Parseval relation $\langle R(\cdot, t), R(\cdot, t)\rangle = \sum_{n=1}^\infty \langle R(\cdot, t), \psi_n\rangle^2 = \sum_{n=1}^\infty \psi_n^2(t)$. On the other hand $\langle R(\cdot, t), R(\cdot, t)\rangle = R(t, t)$. Hence $\sum_{n=1}^\infty \psi_n^2(t)$ converges to $R(t, t)$ for every $t \in S$. Dini's theorem applied to $f_n(t) = R(t, t) - \sum_{j=1}^n \psi_j^2(t)$ now shows that $f_n(t) \to 0$ uniformly in $t \in S$ as $n \to \infty$.

Before proceeding with the proofs of Theorems 1 and 2 we introduce some more notation. z will stand for a generic element of the topological dual of $C(S)$; thus z is a finite signed Borel measure on S. For $x \in C(S)$, (z, x) will denote the value z takes at x. We define by $S_n(S_n')$ the mapping of Ω into $C(S)$ (C[0, 1]) given for $\omega \in \Omega$ by $S_n(\omega) = S_n(t, \omega)$, $t \in S$ $(S_n'(\omega) = S_n(t, \omega), 0 \leqq t \leqq 1)$. Then $S_n(S_n')$ are strongly measurable $C(S)$ (C[0, 1])-valued random variables on $(\Omega, \mathscr{F}, P)$.

PROOF OF THEOREM 1. $R(\cdot, t)$ has an expansion in $H(R)$ given by

$$\sum_{j=1}^\infty \langle \psi_j, R(\cdot, t)\rangle \psi_j(\cdot) = \sum_{j=1}^\infty \psi_j(t)\psi_j(\cdot).$$

By the isometric ismorphism between $H(R)$ and $\mathscr{L}_2(X_t, t \in S)$ we conclude that $\sum_{j=1}^\infty \psi_j(t)\xi_j$ converges in $\mathscr{L}_2(\Omega, \mathscr{F}, P)$ to X_t for every $t \in S$. Since $\sum_{j=1}^\infty \psi_j(t)\xi_j$ is a series of independent random variables, it follows that it converges to X_t a.e. (P). Let X denote the mapping of Ω into $C(S)$ corresponding to $(X_t, t \in S)$ defined the same way as S_n and S_n'. We now show that $S_n \to X$ a.e. (P) as $C(S)$-valued random variables. The argument is essentially the same as in [4] page 45. It is enough by Theorem 4.1 [4] $((e) \Rightarrow (a))$ that the random variables (z, S_n) converge in probability to (z, X) for every continuous linear functional z on $C(S)$. We have

$$\begin{aligned} E[|(z, S_n) - (z, X)|] &= E[|\textstyle\int_S z(dt)(S_n(t) - X_t)|] \\ &\leqq \textstyle\int_S |z|(dt)E|S_n(t) - X_t|, \\ &\leqq \textstyle\int_S |z|(dt)E^{\frac{1}{2}}[S_n(t) - X_t]^2 \\ &= \textstyle\int_S |z|(dt)(\sum_{j=n+1}^\infty \psi_j^2(t))^{\frac{1}{2}}, \end{aligned}$$

(where $|z|$ = total variation of z).

But $\sum_{j=n+1}^\infty \psi_j^2(t) \to 0$ uniformly in $t \in [0, 1]$ as $n \to \infty$ by Lemma 1, hence the last expression above tends to 0 as $n \to \infty$. This completes the proof.

PROOF OF THEOREM 2. We will reduce the proof of this theorem to that of Theorem 1. We first prove that for a fixed $t \in [0, 1]$, $S_n'(t, \omega)$ converges a.e. (P) to a random variable $Y_t(\omega)$, where $\{Y_t, t \in [0, 1]\}$ is a Gaussian process on $(\Omega, \mathscr{F}, P)$ with covariance R. The proof of this is similar to that of Theorem 4 [6]. Note first that an application of Parseval's relation as in Lemma 1 shows that the series $\sum_{j=1}^{\infty} \psi_j(s)\psi_j(t)$ converges to $R(s, t)$. For $t \in [0, 1]$, the a.e. (P) convergence of $S_n'(t, \omega)$ follows from the fact that the random variable $\eta_j(\omega)\psi_j(t)$, $j = 1, 2, \cdots$, are independent with mean 0 and variance $\psi_j^2(t)$ and by Lemma 1 we have $\sum_{j=1}^{\infty} \psi_j^2(t) < \infty$. We denote this limit of $S_n'(t, \omega)$ by $Y_t(\omega)$, defining Y_t arbitrarily on the P-null set where the limit may not exist. It is now clear that the process $\{Y_t, t \in [0, 1]\}$ is a Gaussian process on $(\Omega, \mathscr{F}, P)$ with covariance R. Let $\{Z_t, t \in [0, 1]\}$ be a separable version of $\{Y_t, t \in [0, 1]\}$, which exists by Theorem 2.4 [3] on the same probability space $(\Omega, \mathscr{F}, P)$. Noting that there exists a Gaussian process on some probability space with covariance R and almost all paths continuous and the fact that the Z_t process is separable, we conclude from Theorem 9.2 [2] that there is a set $\Omega_0 \in \mathscr{F}$ with $P(\Omega_0) = 1$ such that for $\omega \in \Omega_0$ the Z_t process has continuous paths. Let Z be the corresponding $C[0, 1]$-valued strongly measurable random variable on $(\Omega, \mathscr{F}, P)$. For a discussion of such facts as Z defining a strongly measurable random variable we refer to page 57 [2]; what is needed is that Z map Ω into $C[0, 1]$ and Z_t be a random variable for each $t \in [0, 1]$. This remark applies also to the definition of S_n, S_n' and X. We thus have proved so far that for each $t \in [0, 1]$ the random variables $S_n'(t, \omega)$ converge a.e. (P) to $Z_t(\omega)$, where the Z_t process is a Gaussian process on $[0, 1]$ with almost all paths continuous and covariance R. We now show exactly as in the proof of Theorem 1 that for any continuous linear functional z on $C[0, 1]$ the random variables (Z, S_n') converge in probability to the random variable (z, Z); all we have to do is to replace S by $[0, 1]$ everywhere in the very last part of the proof of Theorem 1. This proves Theorem 2.

REMARK. Let $S = [0, 1]$. Let $\{\lambda_j\}_{j=1}^{\infty}$, $\lambda_j > 0$, be the eigenvalues of R and $\{\varphi_j\}_{j=1}^{\infty}$ the corresponding normalized (in $L_2[0, 1]$) eigenfunctions. Then it is known that $\{\lambda_j^{\frac{1}{2}}\varphi_j\}_{j=1}^{\infty}$ is a CONS in $H(R)$. Thus whenever Theorem 1 or Theorem 2 applies, the uniform convergence in t of the "Karhunen-Loève" expansion for the process follows as a special case.

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