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# AN INFINITESIMAL DECOMPOSITION FOR A CLASS OF MARKOV PROCESSES[1]

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**1. Introduction.** An unavoidable initial issue in treating many questions on Markov processes is that specifying the objects to be studied. There are various different possible means of specification, and the one chosen will depend, of course, both upon the purposes of the work at hand and upon the particular process, or class thereof, to be specified. If we assume the process to have stationary transition probabilities, one important means (which provides a good analytical access to the process) is to give an "infinitesimal generator" defined in any one of several ways. A standard example is based on the well-known formula of Lévy and Khintchine for the characteristic functions of infinitely divisible processes in $R^n$, which can be interpreted as giving an instantaneous decomposition of the process into independent Gaussian and Poissonian components ([6] page 550).

Quite a different method of specification is usually employed in studies of general potential-theoretic questions. Here the necessary requirements involve stopping times, and are therefore expressed qualitatively in terms of the behavior of the path functions of the process together with some general requirements on the regularity of the transition function.

This distinction in method naturally raises the problem of determining how the two approaches are connected, which is not an easy matter to settle. Much recent work on Markov processes, particularly that of A. V. Skorokhod [12]–[15], can be considered as being directed toward obtaining the explicit generators of a sufficiently wide abstract class of processes.[2] Partly for technical reasons, this work is often done in the reverse order. The generators are specified first and the corresponding processes are then constructed. Up to the present, however, the constructive method has not attained the goal of producing a complete class of processes free of unnatural restrictions, except perhaps in the very particular case of one-dimensional diffusion processes. On the other hand, general methods involving square integrable martingales have been developed recently for treating the same problem by starting with the processes themselves. These methods, which are due to several French, Russian, and Japanese probabilists, are fundamental to the present paper.[3]

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[2] Under the term "explicit generator" we do not include the general existence theories of the strong, weak or Dynkin generators, but only those generators in which the Gaussian and Poissonian components are distinguished.

[3] The related problem of establishing properties such as quasi-left continuity and the strong Markov property for processes whose generator or semigroup is known has also been extensively studied, but it seems to be quite separate from that of obtaining generators for a given class of

In more detail, we use some ideas and results of P. A. Meyer [7]–[9], A. V. Skorokhod [12], [14], [15], M. Motoo [11], S. Watanabe [16], H. Kunita [5], and E. Dynkin [3], [4] to obtain an instantaneous decomposition in explicit terms for a rather large class of processes which is specified intrinsically in terms of the qualitative behavior of the paths globally in time. This decomposition is similar to that of Skorokhod [15], although considerably more detailed, and the class of processes treated is only slightly different. To be explicit, we treat all Hunt processes on compact spaces with resolvants mapping $C$ into $C$ and satisfying Meyer's hypothesis of absolute continuity. The present method, which relies upon applying the more complete results of the other authors cited above at various points in Skorokhod's argument, is evidently susceptible to further extensions. Thanks, indeed, to these known results, our problem here reduces largely to organizing the material in such a way as to achieve the desired end. This does not apply to Corollary 4.1, however (which is new), or to an essential feature which appears at the outset, for which the name "excessive coordinates" is appropriate. This idea is simply that if there is a sequence $\{f_n\}$ of excessive functions which separates points in the state space, the Markov process $X(t)$ can be "coordinated" by the vector process $\{f_n(X(t)), 1 \leqq n\}$ and the components $f_n(X(t))$ become supermartingales. The known results on decomposition of supermartingales can then be utilized to study the local behavior of the original process.

This approach is perhaps slightly artificial, but it has the compensating merit of illustrating the meaning and use of several broadly related theorems which have appeared in different places and in such a way that their relevance to Markov processes was not always clear. Indeed, since our results are perhaps still not the final ones in this direction, the paper is best considered rather as a demonstration of what can be done in the direction indicated by using these methods than as a conclusive treatment.

The author wishes to thank Professor S. Orey for helpful discussion and suggestions on this work. Furthermore, the seminar lectures of Meyer [8], although largely avoided in the references below, have been of great assistance. They provide a single account covering most of the results needed from [5], [10], [11], [16].

**1. Representation of a process by martingales and additive functionals.** We shall be concerned only with Hunt processes, and therefore begin by recalling their definition ([9] page 97). For reasons which become clear momentarily, the definition is specialized to a compact state space $E$. (This can always be achieved via the usual one-point compactification by a trap state.)

DEFINITION 1.1. Let $E$ be a compact metric space with Borel field $\mathscr{E}$, and let $p(t, x, A)$, $t \geqq 0$, $x \in E$, $A \in \mathscr{E}$, be a Markovian transition function, i.e.

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processes. It will not concern us here, but it does serve to point up the fact that there is a wide gap to be covered in connecting the "instantaneous time" and "finite positive time" properties of Markov processes.

(i) $p(t, \cdot, A)$ is $\mathscr{E}$-measurable,
(ii) $p(t, x, \cdot)$ is a probability measure on $\mathscr{E}$,
(iii) $p(0, x, A) = I_A(x)$, where $I_A$ is the indicator function of $A$, and
(iv) $p(t_1+t_2, x, A) = \int p(t_1, x, dy)p(t_2, y, A)$.

The semigroup $P_t: P_t f(x) = \int p(t, x, dy)f(y), f \in b(\mathscr{E})$ (the bounded $\mathscr{E}$-measurable functions) is called a "Hunt semigroup" if

(a) For each probability measure $\mu$ on $\mathscr{E}$ there exists a Markov process with right continuous paths and initial distribution $\mu$ having $p(t, x, A)$ as transition function,

(b) Every process satisfying (a) is a strong Markov process, and

(c) Every process satisfying (a) is quasi-left continuous.

A Markov process $X(t)$ is called a "Hunt process" if with its initial distribution $\mu$ it satisfies (a), and hence (a)–(c), for some Hunt semigroup.

It follows from (b) that if $\mathscr{F}^\mu(t)$ denotes the "completion" with respect to *all* null sets generated by the process with initial measure $\mu$ of the $\sigma$-field generated by $X(s)$, $s \leqq t$, then $\mathscr{F}^\mu(t) = \bigcap_{\delta>0}\mathscr{F}^\mu(t+\delta)$, ([8] XIII, T13), and $X(t)$ is Markovian relative to these $\sigma$-fields. We assume henceforth, except where otherwise indicated, that all of the processes and stopping times considered are relative to these $\sigma$-fields.

Next we introduce our only hypothesis which is in any degree innovative, postponing until Section 3 a discussion of its range of validity.

HYPOTHESIS 1.1. *There exists a sequence $\{f_n\}$ of continuous excessive functions bounded by* 1 *and separating points in $E$.*

*Note.* For the reader unfamiliar with the terminology of probabilistic potential theory (for example "excessive") we refer once and for all to [1].

We now have a preliminary topological observation.

THEOREM 1.1. *Let $I_\infty$ denote the parallelotope $\times_{j=1}^{\infty}[0, 1]$ with the product topology. The mapping $\varphi: \varphi(x) = (f_1(x), \cdots, f_n(x), \cdots)$ is a homeomorphism of $E$ onto a compact subset of $I_\infty$.*

PROOF. Since $\{f_n\}$ separates points in $E$, it is clear that $\varphi$ is one-to-one. From the compactness of $E$ it follows that if $\varphi$ is continuous then its range is compact and it is a homeomorphism. Let $x_n \to x$. Each neighborhood of $\varphi(x)$ contains an open set in the form of a finite product of open intervals at certain coordinates $n_1, \cdots, n_k$ with the remaining coordinates unrestricted. Since each $f_{n_j}$ is continuous, $i \leqq j \leqq k$, we have $\lim_{n\to\infty} f_{n_j}(x_n) = f_{n_j}(x)$. Thus for large $n$, $\varphi(x_n)$ is in the neighborhood, and the proof is complete.

In view of this theorem, we may replace for most purposes $E$ by $\varphi(E)$, $p(t, x, A)$ by $p(t, \varphi(x), \varphi(A))$, and $X(t)$ by $(f_1(X(t)), \cdots, f_n(X(t)), \cdots)$. From now on we assume that this replacement has already been made, and thus write $E$ for $\varphi(E)$, etc., and $(X_1(t), \cdots, X_n(t), \cdots)$ for the process. In most cases of interest, a finite number of excessive functions suffices to separate points, and then the process has as state space a compact subset of $R^n$. We now have a first decomposition result.

THEOREM 1.2. *The coordinates* $X_n(t)$ *can be written uniquely (up to a set of* $P^\mu$-*measure* 0 *for all initial distributions* $\mu$) *in the form* $X_n(t) = M_n(t) - A_n(t)$, *where the decomposition (on the sample space) does not depend on* $\mu$, $A_n(t)$ *is a positive continuous additive functional* ($A_n(0) = 0$), *and, for each* $\mu$, $M_n(t)$ *is a right continuous, quasi-left continuous martingale with* $E^\mu M^2(t) < 7$.

REMARK. Regarding the sample space, we assume only that it is sufficient to define both the processes and the "shift" operators which appear in the definition of additive functionals ([1] page 20).

PROOF. For each $\mu$ it is obvious that $X_n(t)$ is a right continuous supermartingale, and therefore ([2] page 361) the limit $X_n(\infty) = \lim_{t\to\infty} X_n(t)$ exists a.s. and also, since $X_n(t) < 1$, in $L_1$. Thus if we write $X_n(t) = E(X_n(\infty) \mid \mathscr{F}(t)) + (X_n(t) - E(X_n(\infty) \mid \mathscr{F}(t)) = E^{X(t)}X_n(\infty) + (X_n(t) - E^{X(t)}(X_n(\infty)))$ we have a decomposition free of $\mu$ in which the first term is a martingale and the second term is a potential of class $D$ ([7] VI # 9, VI T 20). The second term then has a unique decomposition (for each $\mu$, $P^\mu$ a.s.) into the difference of a right continuous martingale and a natural increasing process $A_n(t)$ ([7] VII, T 31). Moreover, the $\sigma$-fields $\mathscr{F}^\mu(t)$ are free of times of discontinuity ([8] XIV, T 36), and hence the potential is quasi-left continuous, and thus regular ([1] IV, Definition 3.2). It follows from ([1] IV, (3.8)) that $A_n(t)$ is equivalent to a continuous additive functional of the process (and thus we may choose it free of $\mu$). Since, finally, $E^\mu A_n^2(\infty) < 2$ ([7] VII, T 24), one notes that $E^\mu(X_n(\infty) + A_n(\infty))^2 < 7$, and setting $M_n(t) = X_n(t) + A_n(t)$, the quasi-left continuity of $M_n(t)$ follows from that of $X(t)$. This completes the proof.

For our next step in the analysis of the component processes $X_n(t)$, we confine attention to the martingales $M_n(t)$.

THEOREM 1.3. *Each* $M_n(t)$ *decomposes (for each* $\mu$) *uniquely into* $M_n(t) = M_n^c(t) + M_n^d(t)$, *such that* $M_n^c(t)$ *is a boundedly square integrable martingale with continuous paths, and* $M_n^d(t)$ *is a boundedly square integrable martingale, orthogonal to every such martingale having no common discontinuities with it, with* $M_n^d(0) = 0$.

*Note.* In the terminology of ([8] I, Section 3) $M_n^d$ is a "compensated sum of jumps."

PROOF. This result is a direct application of ([7] VIII, T 32) (the "second decomposition of square integrable martingales.").

To relate these martingales further to the Markov property of the process, we need the well-known hypothesis of "absolute continuity" ([9] XV, Section 4):

HYPOTHESIS 1.2. *There exists a finite measure* $v$ *on* $(E, \mathscr{E})$ *such that an excessive function for* $P_t$ *which is* 0 a.e. *with respect to* $v$ *must be identically* 0.

THEOREM 1.4. *The martingales* $M_n^c(t) - M_n^c(0)$ *and* $M_n^d(t)$ *can be chosen (by modification for each* $\mu$ *on a fixed* $P^\mu$-*null set) to be additive functionals of* $X(t)$, *the former continuous.*

*Note.* Obviously the latter remains a compensated sum of jumps.

PROOF. It is clear that $X_n(t)-X_n(0)$ is an additive functional, and since $A_n(t)$ is also, this implies that $M_n(t)-M_n(0)$ is an additive functional. Moreover, from $E^x M_n^{\,2}(t) < 7$ it follows that $M_n(t)-M_n(0)$ is in the space $\mathscr{M}'$ of martingale additive functionals of Motoo and Watanabe [11], [16]. Let us redefine $M_n^{\,c}(t)-M_n^{\,c}(0)$ as the projection in $\mathscr{M}'$ of $M_n(t)-M_n(0)$ onto the stable subspace of continuous elements. Setting $M_n^{\,d}(t) = M_n(t)-M_n^{\,c}(t)$, it is known that $M_n^{\,d}(t)$ is in $\mathscr{M}'$ and orthogonal to this subspace. On the other hand, as defined in Theorem 1.4, the analogous martingales are the corresponding projections in the space $\mathscr{M}^\mu$ of $P^\mu$-square integrable martingales [5]. To show that these are equal to their analogues $P^\mu$-almost everywhere it suffices to show that (a) the continuous elements vanishing at $t = 0$ which are in $\mathscr{M}^\mu \cap \mathscr{M}'$ are dense in the corresponding subspace of $\mathscr{M}^\mu$, and (b) the elements of the space complementary to the continuous subspace of $\mathscr{M}'$ which are in $\mathscr{M}^\mu \cap \mathscr{M}'$ are also dense in the corresponding subspace of $\mathscr{M}^\mu$. Indeed, since the projections of $M_n(t)-M_n(0)$ in $\mathscr{M}'$ are clearly in $\mathscr{M}^\mu$, and since they are determined in each of $\mathscr{M}'$ and $\mathscr{M}^\mu$ by their orthogonality to the complementary subspaces, it suffices that the intersection of these subspaces contain dense sets in those of $\mathscr{M}^\mu$. To establish this point, we can use the "generating system" of elements of the form $u(X(t))-u(X(0))-\int_0^t \alpha u(X(s))-f(X(s))\,ds$, $\alpha > 0$, $f \in b(\mathscr{E})$, $u = R_\alpha f$, where $R_\alpha = \int_0^\infty e^{-\alpha t} P_t\,dt$ ([5]; [11] page 464). It follows exactly as in ([5] Theorem 4.2) that these elements are in $\mathscr{M}^\mu \cap \mathscr{M}'$, vanish at $t = 0$, and are dense in the subspace of $\mathscr{M}^\mu$ vanishing at $t = 0$. Let $\mathscr{M}^\mu_{c,0}$ and $\mathscr{M}^\mu_{d,0}$ denote the two orthogonal subspaces of $\mathscr{M}^\mu$ required for (a) and (b). For $M_1 \in \mathscr{M}^\mu_{d,0}$, we can choose for each $t$ elements of the generating system which approximate $M_1$ in the sense of the seminorm $(E^\mu M_1^{\,2}(t))^{\frac{1}{2}}$. Let these elements be written as sums of orthogonal projections onto the continuous and (purely) discontinuous subspaces of $\mathscr{M}'$. Clearly these projections remain in $\mathscr{M}^\mu$, and the continuous one is orthogonal to $M_1$. It follows that the discontinuous ones approximate $M_1$ even better than the original sums, which implies that if we now choose any $M \in \mathscr{M}^\mu$, $M(0) = 0$, and write $M = M_1 + M_2$, $M_1 \in \mathscr{M}^\mu_{d,0}$, $M_2 \in \mathscr{M}^\mu_{c,0}$, then in approximating $M$ by elements of the generating system, the continuous projections in $\mathscr{M}'$ of those elements also approximate $M_2$. This proves (a), and to complete the proof of (b) it suffices to observe that the discontinuous projections in $\mathscr{M}'$, being orthogonal to the continuous subspace of $\mathscr{M}'$ are by (a) also orthogonal to that of $\mathscr{M}^\mu$. Hence they are in the discontinuous subspace of $\mathscr{M}^\mu$, and, as shown above, approximate any $M_1 \in \mathscr{M}^\mu_{d,0}$. This completes the proof.

**2. An instantaneous decomposition.** In this section, the two martingale additive functionals $M_n^{\,c}(t)-M_n^{\,c}(0)$ and $M_n^{\,d}(t)$ will be analyzed separately. It will emerge that they correspond to the Gaussian and Poissonian components in a rough Lévy–Khintchine type of decomposition of the process. We consider $M_n^{\,d}(t)$ first. From the earlier remarks it is immediate that this is quasi-left continuous, "purely discontinuous," and boundedly square integrable.

Let $H(t)$ denote a continuous increasing additive functional such that $E^x H(t)$ is finite for all $(t, x)$,[4] and with respect to which, for every square integrable martingale additive functional $M(t)$, $\langle M \rangle_t$ is absolutely continuous with respect to $H(t)$ in the sense ([11] or [16] page 56) that there exists an $\mathscr{E}$-measurable function $f \geqq 0$ such that for each $t$ and $x$, $E^x \int_0^t f(X(s))\, dH(s) < \infty$ and $\langle M \rangle_t = \int_0^t f(X(s))\, dH(s)$. We recall that $\langle M \rangle_t$ is the unique continuous increasing additive functional $F(t)$ such that $M^2(t) - F(t)$ is a martingale, and that two additive functionals are considered to be equal if they differ only on a set having $P^\mu$-measure 0 for all $\mu$. Replacing $H(t)$ by $H(t)+t$ if necessary, we can and do assume that $H(t)$ is strictly increasing and exceeds $t$. We now introduce the Lévy system $(n(x, dy), H(t))$ of $X(t)$ relative to $H(t)$, as constructed, for example in [16]. This system, in which the kernel $n(x, A)$ is defined for each $A \in \mathscr{E}$ uniquely up to a set of $H$-potential 0 in $x$ and $n(x, \{x\}) = 0$, has the property that for each purely discontinuous, quasi-left continuous martingale additive functional $M(t) \in \mathscr{M}'$, there is an $\mathscr{E} \times \mathscr{E}$-measurable $f(x, y) \geqq 0$ with $f(x, x) = 0$ for all $x$ and

$$E^x \int_0^t \int_E f^2(X(s), y) n(X(s), dy)\, dH(s) = E^x \langle M \rangle_t < \infty,$$

such that $M(t) = \sum_{s \leqq t} f(X(s-), X(s)) - \int_0^t \int_E f(X(s), y) n(X(s), dy)\, dH(s)$. Here $\int_E f(x, y) n(x, dy)$ is well defined except perhaps on a set of potential 0 (i.e. of $v$-measure 0), and the summation is defined over the jump times $s \leqq t$. Conversely, every $f$ with the stated properties defines such an $M(t)$, by ([16] Theorem 3.1). Now since $X_n(x)$ is a bounded excessive function it is known ([8] IV, Theorem 4) that $X_n(y) - X_n(x)$ is such an $f$. Moreover, an element of $\mathscr{M}'$ which is orthogonal to the continuous subspace of $\mathscr{M}'$ is uniquely determined by its jump, as we see by considering the difference of two such elements. Consequently, the above $f$ determines $M_n{}^d$ and we have

THEOREM 2.1. *For each $n$,*

$$M_n{}^d(t) = \sum_{s \leqq t} (X_n(s) - X_n(s-)) - \int_0^t \int_E (y_n - X_n(s)) n(X(s), dy)\, dH(s)$$

*where $y_n$ is the* nth *coordinate of $y \in E$.*

REMARK. It is not to be expected that this representation for all $n$ in any sense determines $n(x, dy)$. On the other hand, the analogue with functions $f(x_1, y_1 \cdots, x_n, y_n)$ in place of just $y_n - x_n$ would, if assumed for all $n$, determine $n(x, \cdot)$ $v$—a.e. in $x$. This follows easily from the fact that since $\infty > E^x \sum_{s \leqq t} \sum_{j=1}^n (X_j(s) - X_j(s-))^2$ any function $f(x_1, y_1, \cdots, x_n, y_n)$ with $|f| \leqq \sum_{j=1}^n |y_n - x_n|$ would be admissible.

We consider next the continuous components $M_n{}^c(t) - M_n{}^c(0)$. Here it is possible to treat the joint distributions of $n$-tuples directly. The first step is to introduce certain matrix-valued functions which can be thought of as transformations to orthogonal local coordinates.

THEOREM 2.2. *For each $n > 0$ there exists a nonnegative, Borel measurable, $n \times n$ matrix-valued function $W_n(x) = (W_n(x; i, j))$ with $W_n(x; i, i) = 1$ and $W_n(x; i, j) = 0$*

[4] Meyer (9) III shows that $E^x H(t)$ may be assumed bounded in $x$, hence $E^\mu H(t) < \infty$ for all $\mu$.

*for $i > j$, such that the equation*

$$(M_1{}^c(t)-M_1{}^c(0), \cdots, M_n{}^c(t)-M_n{}^c(0)) = \int_0^t d(B_1(s), \cdots, B_n(s)) \cdot W_n(X(s))$$

*defines the left side as a stochastic integral (in the sense of M. Motoo and S. Watanabe;* [11] *and* [5]) *of $n$ orthogonal and continuous square integrable martingale additive functionals $B_1, \cdots, B_n$ with $E^x B_k{}^2(t) \leqq E^x(M_k(t)-M_k(0))^2$, $x \in E$. Here each $W_n$ is triangular with unit diagonal, and, for $m < n$, $W_m$ is the first $m \times m$ submatrix of $W_n$.*

PROOF. One begins by setting $B_1(t) = M_1{}^c(t)-M_1{}^c(0)$, and proceeds by induction. Suppose that $B_1, \cdots, B_k$ have been defined, and set $B_{k+1}(t) = M_{k+1}^c(t)-M_{k+1}^c(0)$— Projection $(M_{k+1}(t)-M_{k+1}(0)$ onto the subspace of $\mathscr{M}'$ generated by $B_1, \cdots, B_k)$ where $\mathscr{M}'$ is the space of all martingale additive functionals in the sense of [11]. It follows that $B_{k+1}(t) = M_{k+1}(t)-M_{k+1}(0)-\sum_{j=1}^k \int_0^t W_n(X(s);j, k+1)\, dB_j(s)$, where $\int_0^t W_n(X(s);j, k+1)\, dB_j(s)$ defines the projection onto the subspace generated by $B_j$. The inequality in the theorem is now clear. Setting

$$\begin{aligned} W_n(x;j, k+1) &= 1; j = k+1 \\ &= 0; k+1 < j \leqq n \end{aligned}$$

the theorem follows.

By using another known result, we can partially understand the structure of the processes $(B_1(t), \cdots, B_n(t))$. The statement of the result is made awkward by the fact that $\langle B_k \rangle_t$ does not increase to $\infty$. Accordingly, we let $(\beta_1(t), \cdots, \beta_n(t))$ be an $n$-dimensional Brownian motion entirely independent of $X(t)$ for all initial distributions. For any $c \geqq 0$, we define the stopping times $T_k(c) = \inf\{t: \langle B_k \rangle_t > c\}$, or $T_k(c) = \infty$ if no such $t$ exists. Now set

$$\begin{aligned} \gamma_k(c) &= B_k(T_k(c)), & T_k(c) < \infty; \\ &= B_k(\infty)+\beta_k(c-\lim_{t\to\infty}\langle \beta_k \rangle_t), & \text{otherwise.} \end{aligned}$$

THEOREM 2.3. *For each initial distribution $\mu$, $P^\mu\{B_k(t) = 0, 0 \leqq t \leqq T_k(0)\} = 1$ and, for each $c > 0$, $(\gamma_1(c), \cdots, \gamma_n(c))$ is a vector of orthogonal normal components, each having mean* 0 *and variance $c$.*

PROOF. The first statement is clear if we recall ([5] page 211) that the martingales $B_k(t \wedge T_k(0))$ are associated with the continuous increasing processes $\langle B_k \rangle_{t \wedge T_k}(0)$. The same reasoning shows that if $r_1$ and $r_2$ vary over the nonnegative rationals, $P^\mu\{\bigcup_{r_1 < r_2}\{\langle B_k \rangle_t = \langle B_k \rangle_{r_1}, r_1 \leqq t \leqq r_2; B_k(t_0) \neq B_k(r_1)$ for some $t_0 \in [r_1, r_2]\}\} = 0$. Consequently, for a.e. path, $B_k$ remains constant in every interval in which $\langle B_k \rangle$ is constant. It now follows as in the case $N = 1$ of ([5] Theorem 3.1) that $\gamma_k(c)$ is in fact an ordinary Brownian motion. Thus it needs only be shown that the components are uncorrelated. Let $T_{j,k}(c) = T_j(c) \wedge T_k(c) \leqq \infty$. Then we have by orthogonality and uniform integrability $E^\mu(B_j(T_{j,k}(c))B_k(T_{j,k}(c))) = 0$. But, for example, we clearly have also

$$E^\mu[B_j(T_j(c))(B_k(T_k(c)-T_j(c))); T_{j,k}(c) = T_j(c) < \infty] = 0.$$

The conclusion is now obvious.

REMARK. In Section 4 we prove a comparatively deep theorem to the effect that $(\gamma_1(c), \cdots, \gamma_n(c))$ is an $n$-dimensional Brownian motion. The most we could show by the present method is that for $c_1 < c_2$ and $c_3 < c_4$, $\gamma_j(c_2)-\gamma_j(c_1)$ and $\gamma_k(c_4)-\gamma_k(c_3)$ are orthogonal $(j \neq k)$. The following example serves to illustrate both the theorem and the stronger assertion.

Let $X(t) = (X_1(t), X_2(t))$ be plane Brownian motion in the square $0 < x, y < 1$ with paths remaining fixed upon reaching the boundary, and consider the two excessive functions $f_1(x, y) = x \wedge y$, $f_2(x, y) = 1-(x \vee y)$ of this process, which can be used as the coordinates of a representation of $X(t)$ as in Section 1. Obviously both $f_1(X(t))$ and $f_2(X(t))$ become martingales if $X(t)$ is stopped upon reaching the diagonal $x = y$, and the continuous increasing processes which must be subtracted to obtain a representation as in Section 1 therefore increase only when $X_1(t) = X_2(t)$. By reason of symmetry and local homogeneity for translations along this diagonal, the increasing process is the same in both cases. Indeed, the only additive functional (up to a constant factor) satisfying all of the requirements is the "local time" of $X(t)$ on the line $x = y$. The simplest definition of this functional is as the local time at 0 of the one-dimensional "Brownian motion" $X_1(t)-X_2(t)$, in the original sense of P. Lévy (we omit a factor $2^{-\frac{1}{2}}$, which is irrelevant here). Denoting this local time by $L(t)$, letting $D$ and $D'$ denote the regions $0 \leqq y < x \leqq 1$ and $0 \leqq x \leqq y \leqq 1$ respectively, and using the fact that $|X_1(t)-X_2(t)|-L(t)$ is a martingale (by a well-known result of Lévy), it follows that the representation of Section 1 for these two coordinates is

$$f_1(X(t)) = f_1(X(0)) + \int_0^t I_{D'}(X(s))\, dX_1(s) + \int_0^t I_D(X(s))\, dX_2(s) - 2^{-1}L(t)$$

$$f_2(X(t)) = f_2(X(0)) - \int_0^t I_D(X(s))\, dX_1(x) - \int_0^t I_{D'}(X(s))\, dX_2(s) - 2^{-1}L(t).$$

Indeed, these are clearly valid up to the first passage time to $\{x = y\}$, and their sum reduces to $1-|X_1(t)-X_2(t)|$, for which $L(t)$ provides the necessary difference from a martingale. It is easily checked that the martingales $\int_0^t I_{D'}\, dX_1 + \int_0^t I_D\, dX_2$ and $\int_0^t I_D\, dX_1 + \int_0^t I_{D'}\, dX_2$ are orthogonal, and that they are Brownian motions up to the time when $X(t)$ is stopped at the boundary of $\{0 \leqq x, y \leqq 1\}$. Since one has $\int_0^t I_{D'}\, dX_1 + \int_0^t I_D\, dX_2 = \frac{1}{2}L(t) + f_1(X(t)) - f_1(X(0))$, and $-[\int_0^t I_D\, dX_1 + \int_0^t I_{D'}\, dX_2] = \frac{1}{2}L(t) + f_2(X(t)) - f_2(X(0))$, in which the common term $\frac{1}{2}L(t)$ is not bounded while the other terms are less than 1 in absolute value, it would seem that the Brownian motions cannot be independent. However, after adjoining the Brownian motions $\beta_1(t)$ and $\beta_2(t)$ as in Theorem 2.3, it follows from ([5] Theorem 2.3) that they do become independent. This fact will be shown in Section 4 to hold in general, although [5] does not apply unless $T_j(c)$ does not depend on $j$.

**3. An alternative hypothesis.** We continue to assume that $X(t)$ is a Hunt process on the compact metric space $E$, and recall the definition of the resolvant operators $R_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)\, dt$, $\lambda > 0$, $f \in b(\mathscr{E})$. It is to be shown that the decomposition of Section 1 and Section 2 can still be carried out, in all essentials and with some simplification, if in place of Hypothesis 1.1 we assume

HYPOTHESIS 3.1. *For $f \in C$ we have $R_\lambda f \in C$, where $C$ denotes the continuous functions on $E$.*

REMARK. We require this explicitly for only a single $\lambda > 0$, however it then holds for all $\lambda > 0$.

Our idea is to apply the earlier results to the process with semigroup $e^{-\lambda t} P_t$, and then dispense with the $e^{-\lambda t}$. Accordingly, we introduce the new sample space $\Omega \times R^+$ (where $\Omega$ is the sample space of $X(t)$) and construct the canonical subprocess of $X(t)$ corresponding to the multiplicative functional $e^{-\lambda t}$ ([1] III, Section 3). In short, we adjoint to $E$ an abstract point $\Delta$ as an isolated point, and let $e$ be an exponential random variable ($P\{e > t\} = e^{-\lambda t}$) independent of $X(t)$ for all $\mu$, with $e(w, x) = x$ on the new sample space. Then we define

$$X_\lambda(t) = X(t); \qquad t < e$$
$$= \Delta; \qquad t \geqq e.$$

We also adjoint another sample point $w_\Delta$, and define $X_\lambda(t) = \Delta$ at $w_\Delta$ for all $t$. The semigroup of $X_\lambda$ is given by $e^{-\lambda t} P_t$ on $E$, and by the identity at $\Delta$, where $P_t$ applies to $f$ restricted to $E$. It is known that $X_\lambda$ is a Hunt process, and if $C_0$ denotes $C$ extended by $f(\Delta) = 0$ to functions on $E \cup \Delta$, then the resolvent of $X_\lambda$ maps $C_0$ into $C_0$. We now have the

LEMMA 3.1. *Hypothesis* 1.1 *holds for $X_\lambda$ with*

$$f_n(x) = E^x \int_0^\infty g_n(X_\lambda(t))\, dt, \qquad 0 \leqq g_n \in C_0,\ 1 \leqq n.$$

PROOF. It is clear that any such an $f_n$ is excessive and is in $C_0$. Let $g_n$ be a sequence which is uniformly dense in the subset of those $C_0$ elements nonnegative and bounded by $\lambda$. The existence of such a sequence follows from that of a dense sequence in $C_0$ by considering $(g \vee 0) \wedge \lambda$. Then if there exist $x$ and $y$ in $E$ with $f_n(x) = f_n(y)$ for all $n$ it follows that $R_\lambda g(x) = R_\lambda g(y)$ for all $g \in C$, and also for all $\lambda > 0$ since $C$ is preserved by $R_\lambda$. Since $\lim_{\lambda \to \infty} \lambda R_\lambda g(x) = g(x)$, this is a contradiction. Finally, if $g_n(x) > 0$ then $f_n(x) > 0$, and hence $\Delta$ is also separated from $E$. The sequence $f_n$ thus satisfies the lemma.

Since Hypothesis 1.2 is known to carry over to $X_\lambda$ ([8] XV, page 160), the earlier results apply to $X_\lambda$ without change.[5] We assume as before that $E$ is replaced by a subset of $\mathsf{X}_{j=1}^{\infty}[0, 1]$. However, it is convenient to use, instead of the $\sigma$-fields generated by $X_\lambda$, the joint $\sigma$-fields generated by $\mathscr{F}^\mu(t)$ and the sets $\{e \leqq s\}$, $s \leqq t$. Since $X_\lambda$ is Markovian relative to these $\sigma$-fields the decomposition of $X_\lambda$ then yields martingales with respect to these larger $\sigma$-fields. More generally, we may enlarge the sample space to the form $\Omega \times \mathsf{X}_{j=1}^{\infty} R_j^+$, and define a sequence $e_1, e_2, \cdots$ of independent exponential random variables like $e$, entirely independent of $X(t)$, where

[5] In the same direction as Hypothesis 3.1, it is worth noting that Hypothesis 1.2 follows from the stronger assumption that $R_\lambda$ maps $b(\mathscr{E})$ into $C$. Indeed, this implies that $\lambda$-potentials of bounded nonnegative functions are continuous, and thus (by [1] II, (2.6)) that $\lambda$-excessive functions are lower semicontinuous. This implies Hypothesis 1.2 ([1] page 197).

$e_j(w; x_1, \cdots, x_j, \cdots) = x_j$. If we then introduce the joint $\sigma$-fields generated by $\mathscr{F}^\mu(t)$ and $\{\sum_{j=1}^n e_j \leqq s\}$, $s \leqq t$, $1 \leqq n$, the decomposition of $X_\lambda$ remains valid relative to these $\sigma$-fields. We are now in a position to obtain a decomposition of $X(t)$ from that of $X_\lambda$.

THEOREM 3.1. *Under Hypothesis* 3.1 *instead of Hypothesis* 1.1, *X may be decomposed in the form* $X_n(t) = X_n(0) + F_n{}^c(t) + F_n{}^d(t) - C_n(t)$, *where the respective terms on the right are processes like those in Section* 1, *except that the quadratic means are not necessarily bounded over t.*

PROOF. We begin by defining $F_n{}^c(t \wedge e_1) = M_{n,1}^c(t) - M_{n,1}^c(0)$, where $M_{n,1}^c(t)$ is the continuous martingale in the decomposition of $X_{\lambda,1}$ (with $e = e_1$). Let also

$$\begin{aligned} \hat{F}_n{}^d(t \wedge e_1) &= M_{n,1}^d(t); && 0 \leqq t < e_1 \\ &= M_{n,1}^d(e_1 -); && e_1 \leqq t, \end{aligned}$$

where "$e_1 -$" denotes the left limit at time $e_1$, and set finally $\hat{C}_n(t \wedge e_1) = A_{n,1}(t \wedge e_1)$. Consider now, on the one hand $X(t)$, $e_1 \leqq t < e_1 + e_2$ and on the other a process which is stochastically equivalent to $X_{\lambda,1}$ given the initial distribution of $X(e_1)$. It is clear that $X(t)$, $e_1 \leqq t \leqq e_1 + e_2$ will become such a process if we set $X(e_1 + e_2) = \Delta$, and then replace $t$ by $t - e_1$. Furthermore, by the strong Markov property of $X(t)$ at the stopping time $e_1$, this process is conditionally independent of the past if $X(e_1)$ is given. Accordingly, we can decompose this process as before, and define

$$F_n{}^c(t \wedge e_1) + (M_{n,2}^c(t \wedge (e_1 + e_2) - (t \wedge e_1)) - M_{n,2}^c(0)) = F_n{}^c(t \wedge (e_1 + e_2))$$

the subscript "2" denoting this decomposition. We also define

$$\begin{aligned} \hat{F}_n{}^d(t \wedge (e_1 + e_2)) &= \hat{F}_n{}^d(t \wedge e_1); && t < e_1 \\ &= \hat{F}_n{}^d(e_1) + M_{n,2}^d(t \wedge (e_1 + e_2)); && e_1 \leqq t < e_1 + e_2 \\ &= \hat{F}_n{}^d(e_1) + M_{n,2}^d(e_2 -); && e_1 + e_2 \leqq t \end{aligned}$$

and $\hat{C}_n(t \wedge (e_1 + e_2))$ analogously. Let us show that $F_n{}^c(t \wedge (e_1 + e_2))$ is a continuous martingale and an additive functional of $X_{\lambda,2}(t)$, where

$$\begin{aligned} X_{\lambda,2}(t) &= X(t); && t < e_1 + e_2 \\ &= \Delta; && t \geqq e_1 + e_2 \end{aligned}$$

(of course, $e_1 + e_2$ is an exponential random variable independent of $X(t)$, and $X_{\lambda,2}(t)$ is a Hunt process). First, we must be more explicit as to the interpretation of "additive functional," since the shift transformation has not been fully defined on the enlarged sample space. We consider here the operation $\theta_t$ which shifts the original paths $w(s)$ to $w(s+t)$ and at the same time replaces, for each $n$, $\sum_{j=1}^n e_j$ by $\sum_{j=1}^{n(t)} e_j - t$, where $n(t) = n - 1 + \inf\{k : \sum_{j=1}^k e_j > t\}$. Now choose $t_1 < t_2$, and denoting the (enlarged) past up to time $t$ by $\mathscr{F}(t)$ (for fixed $\mu$), consider the cases

$t_1 < e_1$ and $t_1 \geqq e_1$ separately. For $t_1 < e_1$, we have

$$\begin{aligned} E^\mu(F_n{}^c(t_2 \wedge (e_1+e_2)) \mid \mathscr{F}(t_1)) &= E^\mu(F_n{}^c(t_2 \wedge e_1) + (F_n{}^c(t_2 \wedge (e_1+e_2)) - F_n{}^c(t_2 \wedge e_1)) \mid \mathscr{F}(t_1) \\ &= F_n{}^c(t_1) + E^\mu(F_n{}^c(t_2 \wedge (e_1+e_2)) - F_n{}^c(t_2 \wedge e_1) \mid \mathscr{F}(t_1)). \end{aligned}$$

To show that the second term is 0, we distinguish the subcases $t_2 \wedge e_1 = t_2 < e_1$ and $t_2 \wedge e_1 = e_1$. The first subcase is trivial since $(t_2 \wedge (e_1+e_2)) = t_2 \wedge e_1$. In the second subcase, we write the term as

$$E^\mu(E^\mu(F_n{}^c(t_2 \wedge (e_1+e_2)) - F_n{}^c(e_1) \mid \mathscr{F}(e_1)) \mid \mathscr{F}(t_1)),$$

which is 0 by the martingale property of $M_{n,2}^c$ and the strong Markov property of $X(t)$ at $e_1$. The other case follows similarly. Since $F_n{}^c(t \wedge (e_1+e_2))$ is clearly continuous, it remains only to show that it is an additive functional. On $\{t_1 < t_2 \leqq e_1\}$ and $\{e_1 \leqq t_1 < t_2\}$ this follows immediately. Now on $\{t_1 < e_1 < t_2\}$ the shift $\theta_{t_1}$ yields a path with $e_1 - t_1$ in place of $e_1$, $e_2 - t_1$ in place of $e_2$, and initial point $X(t_1)$. The desired property follows since $M_{n,1}^c(t) - M_{n,1}^c(0)$ is an additive functional and the increment after time $e_1$ is not changed by the shift. In the same way it is clear that $\hat{F}_n{}^d(t \wedge (e_1+e_2))$ and $\hat{C}_n{}^d(t \wedge (e_1+e_2))$ are additive functionals.

We next proceed inductively on $k$ to define these expressions for $t \wedge (e_1 + \cdots + e_k)$, and letting $k \to \infty$ we obtain their definition for all $t$. Obviously the additive functional property is preserved in the limit. Let us show that $F_n{}^c(t)$ is a square integrable martingale. We know from Theorem 1.2 that $E^\mu(F_n{}^c(e_1))^2 < 7$, and it follows that $E^\mu(F_n{}^c(e_1 + \cdots + e_k))^2 < 7k$. Accordingly, setting $e_0 = 0$ we have the rough estimate

$$\begin{aligned} E^\mu(F_n{}^c(t))^2 &\leqq \textstyle\sum_{k=0}^\infty E^\mu(F_n{}^c(e_0 + \cdots + e_{k+1}))^2 P^\mu\{e_0 + \cdots + e_k \leqq t\} \\ &\leqq 7 \sum_{k=0}^\infty (k+1) \left( \sum_{j=k}^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} \right) \leqq 7 \sum_{j=0}^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} \frac{(j+2)^2}{2} < K(t), \end{aligned}$$

where we can introduce the

DEFINITION 3.1. Let $K(t)$ denote $28((\lambda t)^2 + 1)$.
Then one has

$$E^\mu(\max_{s \leqq t} |F_n{}^c(s)|)^2 = \lim_{k \to \infty} E^\mu(\max_{s \leqq t} |F_n{}^c(s \wedge (e_1 + \cdots + e_k))|)^2 \leqq 4K(t),$$

by a theorem of Doob ([2] VII, page 317). From this it follows by the dominated convergence theorem for conditional expectations that

$$\begin{aligned} E^\mu(F_n{}^c(t_2) \mid \mathscr{F}(t_1)) &= \lim_{k \to \infty} E^\mu(F_n{}^c(t_2 \wedge (e_1 + \cdots + e_k)) \mid \mathscr{F}(t_1)) \\ &= \lim_{k \to \infty} F_n{}^c(t_1 \wedge (e_1 + \cdots + e_k)) \\ &= F_n{}^c(t_1) \text{ a.s. } P^\mu. \end{aligned}$$

We have thus established all of the asserted properties of $F_n^c$. Turning to $\hat{F}_n^d(t)$, we can express it in terms of the Lévy system (Theorem 2.1) of $X_\lambda$. Let $H(t)$ denote the additive functional obtained from its analogue for $X_\lambda$ by the same piecing together as was used above for $F_n^c$ (since $H(t)$ is continuous, the values at $e_1$, $e_1+e_2$, etc. cause no difficulty, and $H(t)$ is integrable in view of footnote 4)). It is clear upon replacing $t$ by $(t \wedge e_1)$– that we have the formula

$$\hat{F}_n^d(t) = \sum_{s \leqq t}(X_n(s)-X_n(s-)) - \int_0^t \int_{E\cup\{\Delta\}} (y_n - X_n(s))n(X(s), dy)\, dH(s).$$

This is not a martingale, but it is evident from the properties of $(n(x, A), H(t))$ (see for instance [16]) that for $x \in E$, $A \in \mathscr{E}$, this is also a Lévy system for $X(t)$, and hence that the expression

$$F_n^d(t) = \sum_{s \leqq t}(X_n(s)-X_n(s-)) - \int_0^t \int_E (y_n - X_n(s))n(X(s), dy)\, dH(s)$$

is a martingale, orthogonal to $F_n^c(t)$. Indeed, it follows from ([16] (3.11) page 63) that

$$\begin{aligned} E^\mu(F_n^d(t))^2 &= E^\mu \int_0^t \int_E (y_n - X_n(s))^2 n(X(s), dy)\, dH(s) \\ &\leqq E^\mu \int_0^t \int_{E\cup\{\Delta\}} (y_n - X_n(s))^2 n(X(s), dy)\, dH(s) \\ &= E^\mu(\hat{F}_n^d(t) + (M_{n,1}(e_1) - M_{n,1}(e_1-))I_{\{e_1 \leqq t\}} \\ &\qquad + (M_{n,2}(e_2) - M_{n,2}(e_2-))I_{\{e_1+e_2 \leqq t\}} + \cdots)^2 \\ &\leqq K(t), \end{aligned}$$

where the last inequality is proved by the same estimates as for $F_n^c$ above, using $E^\mu(M_{n,1}^d(e_1))^2 < 7$.

To complete the proof of Theorem 3.1, it now remains only to define $C_n(t) = \hat{C}_n(t) - \int_0^t X_n(s)n(X(s), \{\Delta\})\, dH(s)$, in view of the fact that $y_n = 0$ when $y = \Delta$.

Let us return now to the martingales $F_n^c$ and show that they can be orthogonalized and reduced to normal variates as in Theorem 2.2 and Theorem 2.3. Let $W_n(x)$ denote the matrix which orthogonalizes the $n$-tuple $(F_1^c(t \wedge e_1), \cdots, F_n^c(t \wedge e_1))$, and let $(B_1(t \wedge e_1), \cdots, B_n(t \wedge e_1))$ denote the resulting orthogonal martingales, the notation being justified since they are certainly constant for $t \geqq e_1$. It is obvious from the proof of Theorem 2.2 that, for all $n$, $E^\mu(B_n(t \wedge e_1))^2 \leqq E^\mu(F_n^c(t \wedge e_1))^2$, and therefore by the same estimates as used for $F_n^c$ we can piece the $B_n$ together to define continuous martingale additive functionals $B_n(t)$ of $X(t)$, with $E^\mu B_n^2(t) \leqq K(t)$. It also follows by continuing both sides in $t$ beyond $t \wedge e_1$ that the identity of Theorem 2.2 remain valid. Finally, since $\langle B_i, B_j \rangle_t$ is also the continuation of $\langle B_i, B_j \rangle_{t \wedge e_1}$, the martingales $B_n(t)$ are orthogonal.

Before summarizing these observations in a theorem, let us remark on the structure of $\hat{C}_n(t)$. We have the

LEMMA 3.2.

$$\hat{C}_n(t) = \int_0^t g_n(X(s))\, ds.$$

PROOF. Since both sides are additive functionals, it suffices to prove the result with $t \wedge e_1$ in place of $t$. By definition, $\hat{C}_n(t \wedge e_1) + f_n(X(t \wedge e_1))$ is to be a martingale, where $X(t \wedge e_1) = X_\lambda(t)$ and $f_n(x) = E^x \int_0^\infty g_n(X_\lambda(t))\,dt$ (Lemma 3.1). But for $t_1 < t_2$,

$$\begin{aligned} E^\mu(f_n(X_\lambda(t_2)) \mid \mathscr{F}(t_1)) &= E^{X_\lambda(t_1)} \int_0^\infty g_n(X_\lambda(t_2 - t_1 + t))\,dt \\ &= f_n(X_\lambda(t_1)) - E^{X_\lambda(t_1)} \int_0^{t_2 - t_1} g_n(X_\lambda(t))\,dt \\ &= f_n(X_\lambda(t_1)) - (E^\mu(\hat{C}_n(t_2 \wedge e_1) \mid \mathscr{F}(t_1)) - \hat{C}_n(t_1 \wedge e_1)), \end{aligned}$$

which completes the proof.

Collecting these assertions, we have finally

COROLLARY 3.1. *In the representation of Theorem* 3.1, $(F_1{}^c(t), \cdots, F_n{}^c(t)) = \int_0^t d(B_1(s), \cdots, B_n(s)) \cdot W_n(X(s))$ *where* $(B_1, \cdots, B_n)$ *is an n-tuple of orthogonal continuous martingales satisfying Theorem* 2.3;

$$F_n{}^d(t) = \sum_{s \leqq t} (X_n(s) - X_n(s-)) - \int_0^t \int_E (y_n - X_n(s)) n(X(s), dy)\,dH(s)$$

*where* $(n(x, dy), H(s))$ *is a Lévy system for* $X(t)$; *and* $C_n(t) = \int_0^t g_n(X(s))\,ds - \int_0^t X_n(s) n(X(s), \{\Delta\})\,dH(s)$.

**4. The quasi-infinitesimal operator and uniqueness.** In the preceding section we obtained from a given process $X(t)$ a collection of data in the form $((n(x, dy), H(t)), W_n(x), B_n(t), g_n(x)(n \geqq 1))$, that is, of functions on $E$ and of additive functionals. It is natural to suppose that these data then determine $X(t)$ uniquely up to stochastic equivalence. Unfortunately, we are not able to prove this in the present generality. What we are able to do (using known methods) is somewhat less general, but does give additional insight into the process. In brief, we introduce an invertible time change, the appearance of additive functionals in the representation of the time-changed process is largely eliminated, and the uniqueness problem is reduced to that for this new process. Even for it, however, the solution seems to require a further hypothesis. Let us remark that a time-changed process is simply one with the same hitting probabilities as the given process, by a theorem of Blumenthal, Getoor, and McKean ([1] V, Section 5). The present use of a time change is due to Skorokhod [12].

We base the time change on $H(t)$. Recall that $H(t)$ is continuous, strictly increasing, and $H(t) \geqq t$. Accordingly, let $h_n(x)$, $1 \leqq n$, denote $\mathscr{E}$-measurable functions such that

$$\langle B_n \rangle_t = \int_0^t h_n(X(s))\,dH(s),$$

and let $0 \leqq h(x)$ be chosen such that

$$t = \int_0^t h(X(s))\,dH(s)$$

(clearly $t$ is absolutely continuous with respect to $H(t)$, so that the necessary theorem of Motoo [11] applies). We now introduce the time change $\tau(t) = \inf\{s : H(s) > t\}$, and the resulting process $Y(t) = X(\tau(t))$, which is again a Hunt process satisfying

Hypothesis 1.2 with the same measure $v$. Let us consider the representation of $Y(t)$, obtained by substituting $\tau(t)$ for $t$ in that of $X(t)$. We have

$$(F_1{}^c(\tau(t)), \cdots F_n{}^c(\tau(t))) = \int_0^{\tau(t)} d(B_1(s), \cdots, B_n(s)) \cdot W_n(X(s))$$
$$= \int_0^t d(B_1(\tau(s)), \cdots, B_n(\tau(s))) \cdot W_n(Y(s)),$$

where

$$\langle B_i, B_j \rangle_{\tau(t)} = 0; \qquad i \neq j$$
$$= \int_0^t h_j(Y(s))\, ds; \qquad i = j,$$

the last steps being relevant since, in view of the inequality $\tau(t) \leqq t$, the $B_j(\tau(t))$ are square integrable martingale additive functionals. Next, we have easily

$$F_n{}^d(\tau(t)) = \sum_{s \leqq t} (Y_n(s) - Y_n(s-)) - \int_0^t \int_E (y_n - Y_n(s)) n(Y(s), dy)\, ds,$$

and finally,

$$C_n(\tau(t)) = \int_0^t g_n(Y(s))\, d\tau(s) - \int_0^t Y_n(s) n(Y(s), \{\Delta\})\, ds$$
$$= \int_0^t g_n(Y(s)) h(Y(s)) - Y_n(s) n(Y(s), \{\Delta\})\, ds,$$

where, since $dt \leqq dH(t)$, we may assume that $0 \leqq h(y) \leqq 1$ and $0 \leqq n(y, \{\Delta\}) \leqq 1$, and thus the integrand is between 0 and $\lambda + 1$. We have thus obtained for $Y(t)$ a representation entirely analogous to that of $X(t)$ but relatively free of additive functionals dependent upon the sample space. It is also clear that if $Y(t)$ is determined uniquely by the sample-space-free data $(n(x, dy), W_n(x), h_n(x), h(x), g_n(x))$ of this representation, then $X(t)$, which is obtained from $Y(t)$ by the time change $H(t)$, is uniquely determined by its data listed before.

Using $Y(t)$, we shall now prove the stronger form of Theorem 2.3.

THEOREM 4.1. *The process* $(\gamma_1(c), \cdots, \gamma_n(c))$ *of Theorem* 2.3 *or Corollary* 3.1 *is an n-dimensional Brownian motion.*

PROOF. It is clear that this process is the same whether one begins with $X(t)$ or $Y(t)$ in defining it, and in view of the properties of $Y(t)$ this permits us to assume from the start that $\langle B_k \rangle_t = \int_0^t h_k(X(s))\, ds$ for suitable $h_k \geqq 0$. The first step in the proof is to justify the assumption that $B_k(t) = \int_0^t h_k{}^{\frac{1}{2}}(X(s))\, dD_k(s)$, where $(D_1, \cdots, D_n)$ is an $n$-dimensional Brownian motion. Formally, the process $D_k$ is given by $D_k(t) = \int_0^t h_k{}^{-\frac{1}{2}}(X(s))\, dB_k(s)$, but this expression is meaningless if $h_k(X(s)) = 0$. Accordingly, let $(\delta_1(t), \cdots, \delta_n(t))$ be an $n$-dimensional Brownian motion entirely independent of $X(t)$, and consider the product sample space on which both are defined. On this space the process $(Y(t), \delta_1(t), \cdots, \delta_n(t))$ is a Hunt process, in an evident way, but it seems difficult to verify Hypothesis 1.2. Nevertheless, if we set $S_k = \{x \in E : h_k(x) \neq 0\}$, and $S_k' = E - S_k$, the expressions

$$D_k(t) = \int_0^t h_k{}^{-\frac{1}{2}}(X(s)) I_{S_k}(X(s))\, dB_k(s) + \int_0^t I_{S_k'}(X(s))\, d\delta_k(s)$$

have a meaning for all $t$ a.s. The last term can be simply interpreted as a conditional integral when $X(s)$, $0 \leqq s < \infty$, is given and thus $I_{S_k'}(X(s))$ is a fixed function independent of $\delta_k$. It follows without difficulty that relative to the $\sigma$-fields $\mathscr{F}^\mu(t) \times \mathscr{F}_t(\delta_1, \cdots, \delta_n)$ (where the last factor has the obvious meaning) $D_1, \cdots, D_n$ is a martingale and the equation $\langle D_k \rangle_t = \int_0^t h_k I_{S_k} d_s\langle B_k \rangle + \int_0^t I_{S_k'} ds = t$ implies by a well-known criterion that $D_k$ is a Brownian motion. Furthermore, one has $\langle D_j, D_k \rangle_t = 0$, and hence by ([5] Theorem 2.3) $(D_1, \cdots, D_n)$ is an $n$-dimensional Brownian motion. Now we have $B_k(t) = \int_0^t h_k^{\frac{1}{2}}(X(s))\, dD_k(s)$, where the integral can be interpreted in the space $\mathscr{M}$ of [5], and therefore our theorem will be an immediate consequence of the general

Lemma 4.1. *Let $(D_1, \cdots, D_n)$ be an n-dimensional Brownian motion relative to a right continuous family $\mathscr{F}^*(t)$ of $\sigma$-fields (containing all null sets), and let $\phi_1, \cdots, \phi_n$ be processes for which the expressions $K_j(t) = \int_0^t \phi_j(s)\, dD_j(s)$ are in $\mathscr{M}$ in the sense of* [5]. *Then $(K_1, \cdots, K_n)$ reduces to Brownian motion via time changes $T_1, \cdots, T_n$ as in Theorem* 2.3.

Proof. We present first the main steps of the proof, and afterwards discuss the technicalities needed to make it rigorous. Since the result is known for $n = 1$ we shall assume it for $n$ and proceed by induction. Suppose now that $D_{n+1}(t)$ is given for all $t \geqq 0$. Granting that $\phi_1, \cdots, \phi_n$ can be conditionally defined in such a way that $K_1, \cdots, K_n$ still make sense, it is clear that they are still an $n$-tuple of orthogonal continuous martingales to which the induction hypothesis applies. Their conditional reduced process is thus $n$-dimensional Brownian motion. Since the reduced path is also a fixed function of the $d_j$ and $D_j$ paths almost surely, this is also a conditional version of the original reduced process. But this conclusion is free of the given process $D_{n+1}$, and it immediately follows that if the reduced process of $K_1, \cdots, K_n$ is given then $D_{n+1}$ remains conditionally a Brownian motion. This implies by the same reasoning as before that $K_{n+1}$ is conditionally a martingale, and therefore the reduced process of $K_{n+1}$ is conditionally a Brownian motion. This is obviously equivalent to the assertion of the lemma.

There is little difficulty in making this argument rigorous. We suppose at first that $\phi_1, \cdots, \phi_{n+1}$ are bounded, right continuous processes having left limits (of course, $\varphi_j(t)$ is $\mathscr{F}^*(t)$-measurable by definition). It is required to define on the sample space a conditional measure for the process $\phi_1, D_1, \cdots, \phi_n, D_n$ given $D_{n+1}$. For this it suffices to define a conditional distribution over the $\sigma$-field generated by $\{\varphi_j(r), D_j(r), 1 \leqq j \leqq n, 0 \leqq r \text{ rational}\}$ given $\{D_{n+1}(r), 0 \leqq r \text{ rational}\}$. Indeed, the resulting processes $\varphi_j(r)$ will have right and left limits along the rationals at all $t$ a.s., and it is easy to see that if $\varphi_j(t)$ is defined by right continuity it will be a conditional process with respect to its generated $\sigma$-field. Moreover, $D_j(t)$ if defined in the same way will remain independent of $D_{n+1}$. According to a known theorem ([6] 27.2, page 361) such a conditional distribution exists provided that the range of $\{\varphi_j(r), D_j(r), 1 \leqq j \leqq n\}$ is a Borel set in the corresponding infinite product space. This can easily be arranged by choosing the original sample space sufficiently large (for example, by adjoining as a null set the entire product space of all $2n+2$ components, and relaxing the continuity assumptions on this null set), and since the conclusion of the lemma does not depend on the sample space (as is clear) no generality is thereby lost. It is now easy to see that with probability 1 the conditional processes $\phi_1, \cdots, \phi_n$ will be suitable integrands for defining $\int_0^t \phi_j(s)\, dD_j(s)$. The only property which remains to be verified is that $E(\int_0^t \varphi_j^2(s)\, ds \mid D_{n+1}) < \infty$, which is immediate. The same method also applies to define $\phi_{n+1}$, $D_{n+1}$, and $K_{n+1}$ conditionally given the reduced process of $(K_1, \cdots, K_n)$. As we have noted, $D_{n+1}$ remains a Brownian motion, and thus the proof is rigorous for $\varphi_1, \cdots, \varphi_{n+1}$ bounded, right continuous, and having left limits.

Proceeding to the general case, the definitions of ([5] Section 2) indicate[6] that every permissible process $\varphi_j(t)$ is in the closure of this class with respect to the seminorms $E^{\frac{1}{2}}\int_0^t \varphi_j^2(s)\, ds$, $0 \leqq t$. Thus to complete the proof it suffices to show that if $(\varphi_{j,r}, 1 \leqq j \leqq n+1)$ is a sequence of integrands for which the conclusion holds, and which converges with respect to these seminorms to a permissible set $(\varphi_1, \cdots, \varphi_{n+1})$, then it also holds for these. But since $E(\int_0^t \varphi_{j,r}\, dD_j - \int_0^t \varphi_j\, dD_j)^2 = E\int_0^t (\varphi_{j,r} - \varphi_j)^2\, ds \to 0$, by passing if necessary to a subsequence, we may assume that the martingales converge a.s. uniformly in finite time intervals to the corresponding limits. Moreover, the corresponding increasing processes $\int_0^t \varphi_{j,r}^2(s)\, ds$ converge in the same sense. It is then clear by the continuity of the paths of all these processes that over $\{T_j(c) < \infty\}$ the reduced process $\gamma_j(c)$ is the a.s. limit of the approximating Brownian motions $\gamma_{j,r}(c)$. Finally, since $\sup\{c: T_j(c) < \infty\}$ is also the limit of the corresponding expression for $T_{j,r}$, the adjoined independent Brownian motions operate for approximately the same time periods, and if we choose them the same for all $r$ it is evident that the Brownian reduced processes converge a.s. for each $c$ to the limit process. The latter is thus an $n$-dimensional Brownian motion, and the proof is complete.

The following corollary, although not directly related to Markov processes, is of general interest in that it extends Theorem 2.3 of [5] and subsumes Theorem 4.1 as well.

COROLLARY 4.1. *Let $B_1, \cdots, B_n$ be orthogonal continuous square integrable martingales adapted to a right continuous family $\mathscr{F}(t)$ of $\sigma$-fields containing all null sets. Define $T_j(c) = \inf\{t: \langle B_j \rangle_t > c\}$ and let $\beta_1, \cdots, \beta_n$ be an $n$-dimensional Brownian motion independent of $B_1, \cdots, B_n$. Then the process $(\gamma_1, \cdots, \gamma_n)$, where*

$$\gamma_j(c) = B_j(T_j(c)); \qquad T_j(c) < \infty$$
$$= B_j(\infty) + \beta_j(c - \sup s: T_j(s) < \infty); \qquad \text{otherwise,}$$

*is an $n$-dimensional Brownian motion.*

PROOF. Since one has only to imitate the proof of Theorem 4.1 we will be content with presenting a sketch. Set $H(t) = \langle B_1 \rangle_t + \cdots + \langle B_n \rangle_t + t$, and introduce the time change $\tau(t) = H^{-1}(t)$. Evidently $\tau(t)$ is a stopping time, and the process $(B_1(\tau(t)), \cdots, B_n(\tau(t)))$ is an orthogonal $n$-tuple of continuous square integrable

[6] A different class of integrands is used in [8]. However, it is not hard to check that in the case of continuous martingales they lead to the same class of stochastic integrals.

martingales relative to the family of $\sigma$-fields $\mathscr{F}^*(t) = \{S \in \mathscr{F}(\infty): S \cap \{\tau \leqq t\} \in \mathscr{F}(t)$ for all $t\}$. In fact, $\langle B_i(\tau(t)), B_j(\tau(t)) \rangle_t = \langle B_i, B_j \rangle_{\tau(t)}$, and it follows that the "reduced process" of this $n$-tuple is the same as that of $(B_1, \cdots, B_n)$. Because $\langle B_i \rangle_{\tau(t+\Delta)} - \langle B_i \rangle_{\tau(t)} < \Delta$, it follows for example by the argument of ([8] I, Proposition 1) that there are well-measurable processes $H_i(t) \geqq 0$ such that $\langle B_i \rangle_{\tau(t)} = \int_0^t H_i(s)\,ds$. It is easy to see that the processes

$$\theta_i(t) = 1, \qquad H_i(t) = 0;$$
$$= 0, \qquad \text{otherwise}.$$

are well-measurable, and adjoining a Brownian motion $(\delta_1, \cdots, \delta_n)$ to the sample space as before, the processes

$$D_j(t) = \int_0^t H_j^{-\frac{1}{2}}(s) I_{\{s: H_j(s) \neq 0\}}(s)\, d_s B_j(\tau(s)) + \int_0^t \theta_j(s)\, d\delta_j(s)$$

define $n$-dimensional Brownian motion (by [5] Theorem 2.3). Here the interpretation of the stochastic integrals can be based on ([8] I, Section 2, #3) where the change from well-measurable to very-well-measurable integrands utilizes ([7] VIII, T 20) and the continuity of $B_j$ and $\delta_j$, as in the remarks of the former reference (this continuity can replace the absence of discontinuities of $\mathscr{F}(t)$ assumed there). By considering the corresponding increasing processes it follows that $B_j(\tau(t)) = \int_0^t H_j^{\frac{1}{2}}(s)\, dD_j(s)$, and the proof is now completed by applying Lemma 4.1.

To investigate the uniqueness for $Y(t)$, and obtain at the same time a more formal kind of infinitesimal operator, we shall next apply the extension of Itô's formula due to Kunita and Watanabe ([5] Theorem 5.1, page 229). Let $F(x_1, \cdots, x_n)$ be a function with bounded continuous 2nd order partial derivatives on $\mathsf{X}_{j=1}^n[0, 1]$, written $F \in C_n^{\,2}$. Then if we introduce (for $y \in E$), $\mathbf{y}_n = (y_1, \cdots, y_n)$, and $\mathbf{Y}_n(t) = (Y_1(t), \cdots, Y_n(t))$, we obtain

$$\begin{aligned} F(\mathbf{Y}_n(t)) - F(\mathbf{Y}_n(0)) \\ = \sum_{j=1}^n \int_0^t \frac{\partial F}{\partial x_j}(\mathbf{Y}_n(s))\, d(B_1(\tau(s)), \cdots, B_n(\tau(s))) \cdot W_n(Y(s)) \\ + \int_0^t \int_E F(\mathbf{y}_n) - F(\mathbf{Y}_n(s)) q(ds, dy) \\ + \int_0^t \int_E \left[ (F(\mathbf{y}_n) - F(\mathbf{Y}_n(s))) - \sum_{j=1}^n (y_j - Y_j(s)) \frac{\partial F}{\partial x_j}(\mathbf{Y}_n(s)) \right] n(Y(s), dy)\, ds \\ + \sum_{j=1}^n \int_0^t \frac{\partial F}{\partial x_j}(\mathbf{Y}_n(s))(g_j(Y(s)) h(Y(s)) - Y_j(s) n(Y(s), \{\Delta\}))\, ds \\ + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(\mathbf{Y}_n(s)) \sum_{k=1}^n h_k(Y(s)) W_n(Y(s); k, i) W_n(Y(s); k, j)\, ds, \end{aligned}$$

where $q(t, A) + \int_0^t n(Y(s), A)\, ds$, $A \in \mathscr{E}$, is by definition the number of jumps of

$Y(s)$, $s \leqq t$, which terminate in $A$ (one notes that $q(ds, dy)$ is a kind of "random Poisson measure").

It is necessary to obtain some bounds on the 5 terms of this sum. Let $K$ be an upper bound on $|F|$ and the absolute values of its first and second order derivatives. The first term is clearly a square integrable martingale for each initial value $x$ (or distribution $\mu$) and will cause no difficulty. It follows as in the proof of Theorem 3.1 that

$$E^{\mu} \int_0^t \int_E \sum_{j=1}^n (y_j - Y_j(s))^2 n(Y(s), dy)\, ds < K(t),$$

where $K(t)$ is given by Definition 3.1, whence the third term is absolutely integrable with expectation bounded by $K(t)K$ in view of Taylor's Theorem. The integrand of the fourth term is bounded by $(\lambda+1)K$, while the last term may be bounded by $\frac{1}{2}K \sum\sum_{i,j=1}^n \int_0^t |d_s\langle F_i^c, F_j^c\rangle_{\tau(s)}|$, which is no greater in expectation than $\frac{1}{2}n^2 K\, K(t)$. Finally, as for the second term we shall only require that it have expectation 0 for each $t$ and $x \in E$, which follows immediately from its definition in ([5] page 231) (it is even square integrable).

As a consequence of these estimates and Fubini's Theorem we have

$$E^{\mu}(F(\mathbf{Y}_n(t)) - F(\mathbf{Y}_n(0))) = \int_0^t E^{\mu}\Omega F(\mathbf{Y}_n(s))\, ds,$$

where the operator $\Omega$ is well defined except, perhaps, on a set of potential 0 by the following expression

$$\text{(4.1)} \qquad \Omega F(z) = \int_E \left( F(\mathbf{y}_n) - F(\mathbf{z}_n) - \sum_{j=1}^n (y_j - z_j) \frac{\partial F}{\partial x_j}(\mathbf{z}_n) \right) n(z, dy)$$
$$+ \sum_{j=1}^n (g_j(z)h(z) - z_j\, n(z, \{\Delta\})) \frac{\partial F}{\partial x_j}(\mathbf{z}_n)$$
$$+ \sum\sum_{i,j=1}^n \left( \sum_{k=1}^n f_k(z) W_n(z; k, i) W_n(z; k, j) \right) \frac{\partial^2 F}{\partial x_i \partial x_j}(\mathbf{z}_n),$$

the integral being in the sense of $L^1(n(z, dy))$. We note that in this definition $F$ may be regarded as a function over $\mathsf{X}_{j=1}^{\infty}[0, 1]$, depending on only a finite (but arbitrary) set of coordinates. The operator $\Omega$ is an example of what is called in [3] a "quasi-infinitesimal operator" of $Y(t)$.

Letting $T_t$ denote the semigroup of $Y(t)$, it may appear that $\Omega$ automatically determines $T_t$ from the equation $T_t F(x) - F(x) = \int_0^t T_s \Omega F(s)\, ds$. This depends, however, upon the collection of such $F$'s being "sufficiently large" in some precise sense, and thus requires a further hypothesis. To arrive at a statement of it, we apply the resolvent to the above equation and integrate by parts on the right side. Thus we obtain

$$\int_0^{\infty} e^{-\lambda t}(T_t F(x) - F(x))\, dt$$
$$= \lim_{N \to \infty} \left[ \frac{1 - e^{-\lambda N}}{\lambda} \int_0^N T_s \Omega F(x)\, ds + \frac{1}{\lambda} \int_0^N (e^{-\lambda t} - 1) T_t \Omega F(x)\, dt \right]$$
$$= \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda t} T_t \Omega F(x)\, dt,$$

where we have used the estimates obtained above in the form $\int_0^t T_s|\Omega F(x)|\,ds \leqq CK(t)$, in which $C$ depends only on $F$. This identity can be written in a more familiar way as follows:

$$\lambda G_\lambda F - G_\lambda \Omega F = F; \qquad F \in C^2 \tag{4.2}$$

where $G_\lambda$ is the resolvent of $Y(t)$, and $C^2$ denotes the twice continuously differentiable functions on $\mathsf{X}_{j=1}^{\infty}[0, 1]$ depending on only finite sets of coordinates.

To see under what conditions (4.2) determines the semigroup, we can introduce a comparison with the weak Dynkin generator $\mathscr{A}$. We recall from ([4] **1** I, Section 6) that $\mathscr{A}F = G$ holds for $F \in b(\mathscr{E})$ whenever $\lim_{t\to 0+} t^{-1}(T_tF - F) = G$ in the sense of bounded pointwise convergence on $E$ and $G \in b(\mathscr{E})$. Furthermore, $\mathscr{A}$ does determine $T_t$ since $G_\lambda F$ is defined at least for $F \in C$ and satisfies

$$\lambda G_\lambda F - \mathscr{A}\, G_\lambda F = F. \tag{4.3}$$

Comparing (4.2) and (4.3) we note that $\mathscr{A}G_\lambda F = G_\lambda \Omega F$ on $C^2$. Let $W$ denote the weak continuity set of $T_t$. Then $G_\lambda(W)$ is the domain of $\mathscr{A}$ (of course $C \subset W$) and for $F \in G_\lambda(W)$ one has $\mathscr{A}G_\lambda F = G_\lambda \mathscr{A}F$. Thus for $F \in G_\lambda(W) \cap C^2$ one has $G_\lambda \mathscr{A}F = G_\lambda \Omega F$, and the problem can be formulated as that of given the condition under which $\{G_\lambda \mathscr{A}F, F \in G_\lambda(W) \cap C^2, \lambda > 0\}$ determines $T_t$. This is a simple matter to provide.

HYPOTHESIS 4.1. *The weak closure of* $G_\lambda^{-1}(G_\lambda(W) \cap C^2)$ *is* $b(\mathscr{E})$.

REMARK. Since $G_\lambda$ is one-to-one on $W$, the hypothesis is well stated. Moreover, as is noted below, it does not depend on $\lambda > 0$. Finally, it obviously suffices that the same closure contain $C$.

THEOREM 4.2. *Under Hypothesis* 4.1, *the operator* $\Omega$ *of* (4.1) *determines the semigroup of* $Y(t)$ *uniquely.*

PROOF. For $F \in G_\lambda(W) \cap C^2$ we have by (4.2) the equation $G_\lambda(\lambda F - \Omega F) = F$, and also $G_\lambda(\lambda F - \mathscr{A}F) = F$, from which it follows that $G_\lambda(\Omega F - \mathscr{A}F) = 0$. Now let $L^+ = (\Omega F - \mathscr{A}F) \vee 0$ and $L^- = -((\Omega F - \mathscr{A}F) \wedge 0)$. Then $G_\lambda L^+ = G_\lambda L^-$, where both sides are finite because $G_\lambda(|\Omega F|) < \infty$. We consider the continuous additive functionals of $Y(t)$ given by $\int_0^t L^+(Y(s))\,ds$ and $\int_0^t L^-(Y(s))\,ds$. Since they have the same finite $\lambda$-potentials, it is known ([1] IV, Theorem 2.13) that they must be equivalent, and therefore $L^+ = L^-$ except on a set of potential 0. Consequently, we have $\Omega F = \mathscr{A}F$ except on a set of potential 0. This implies that $\Omega F$ determines $\mathscr{A}F$ uniquely, for if $\mathscr{A}F_1 = \mathscr{A}F_2$ except on a set of potential 0 then $\lambda G_\lambda(\mathscr{A}F_1) = \lambda G_\lambda(\mathscr{A}F_2)$, and as $\lambda \to \infty$ we obtain $\mathscr{A}F_1 = \mathscr{A}F_2$ everywhere. Thus for $F \in G_\lambda(W) \cap C^2$ we have determined $\mathscr{A}F$ such that $\lambda F - \mathscr{A}F = G_\lambda^{-1}F$, i.e. $G_\lambda^{-1}F$ is determined. But if $f_\iota = G_\lambda^{-1}F_\iota$ is a weakly convergent net with limit $f$, then since $G_\lambda$ is weakly continuous it follows that $F_\iota$ converges weakly to a limit $F$. It is known that $\lambda - \mathscr{A}$ is a closed operator, and hence $\lambda F - \mathscr{A}F = f$. By hypothesis, any $f \in W$ may be obtained in this manner, and this determines $\mathscr{A}$. We have already remarked that $\mathscr{A}$ determines $T_t$, and the proof is complete.

In conclusion, we show that Hypothesis 4.1 does not depend on $\lambda$. Choose $0 < \mu \neq \lambda$, and let $f_\iota$ be a net in $G_\lambda^{-1}(G_\lambda(W) \cap C^2)$ such that $f_\iota$ converges weakly to $f$ (and $G_\lambda f_\iota$ converges weakly to $G_\lambda f$). Since $G_\lambda f_\iota \in G_\lambda(W) \cap C^2 = G_\mu(W) \cap C^2$, there is a net $g_\iota$ in $G_\mu^{-1}(G_\mu(W) \cap C^2)$ such that $G_\lambda f_\iota = G_\mu g_\iota$. Furthermore, since we have

$$g_\iota = \mu G_\mu g_\iota - \mathscr{A} G_\mu g_\iota = \mu G_\lambda f_\iota - \mathscr{A} G_\lambda f_\iota$$
$$= (\mu - \lambda) G_\lambda f_\iota + f_\iota,$$

the net $g_\iota$ converges to $(\mu - \lambda) G_\lambda f + f$. If $f \in W$ then $(\mu - \lambda) G_\lambda f + f \in W$, and since $(\mu - \lambda) G_\lambda f + f = G_\mu^{-1}(G_\lambda f)$ and $G_\lambda f$ is an arbitrary element in $G_\lambda(W) = G_\mu(W)$, this implies that the weak closure of $G_\mu^{-1}(G_\mu(W) \cap C^2)$ contains $W$, and therefore is $b(\mathscr{E})$. The proof is complete.

REMARK. It would be worthwhile to obtain a reformulation of Hypothesis 4.1 in terms of the original process $X(t)$, but this problem has presented no easy solution.

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