The Annals of Mathematical Statistics
1971, Vol. 42, No. 1, 395–397

A SERIES OF TOURNAMENTAL DESIGNS

BY C. C. YALAVIGI

Government College, Mercara, India

1. Introduction. Let

$v =$ a given number of elements,

$m^* =$ number of varieties associated with each element,

$k =$ number of varieties in a block with k_1 varieties in one configuration and k_2 varieties in another,

$v_1 =$ number of times 2 varieties occur together in a configuration except the varieties associated with the same element,

$v_2 =$ number of times 2 varieties occur in the opposite configurations except the varieties associated with the same element,

$r_1 =$ number of times each variety occurs in configurations of k_1 elements,

$r_2 =$ number of times each variety occurs in configurations of k_2 elements,

$b =$ total number of blocks.

In a tournamental design (see C. C. Yalavigi [2]), these parameters satisfy the following system of equations viz.,

$$(1) \qquad (k_1-1)r_1+(k_2-1)r_2 = v_1(v-1), \qquad k_2r_1+k_1r_2 = v_2(v-1),$$
$$b(k_1+k_2) = v(r_1+r_2)$$

and designs of this type do not seem to have been considered for $k_1 \neq k_2$. Our aim is therefore to determine the solution of a general series given by

$$(2) \qquad \begin{array}{lll} v = 2(2t+1)+1, & b = v(2t+1), & k_1 = t+1, \\ k_2 = t, & r_1 = (2t+1)(t+1), & r_2 = (2t+1)t, \\ v_1 = t^2, & v_2 = t^2+t, & m^* = 1, \end{array}$$

where v denotes a prime integer or a power of a prime integer of the form p^n.

2. Designs for $k_1 = t+1$, $k_2 = t$ and $v = 2(2t+1)+1 = p^n$. This section will employ the method of generators due to R. C. Bose [1], for constructing balanced incomplete block designs. We identify the elements with the elements of the Galois field GF(p^n) and let

$$(3) \qquad B_i = (A_{0+i}, A_{1+i}, \cdots, A_{t+i}; A_{t+1+i}, \cdots, A_{2t+i}), \qquad i = 0, 1, \cdots, 2t$$

Received April 29, 1969; revised March 30, 1970.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.*
www.jstor.org

generate all the blocks of a design viz.,

$$B_{ia} = (A_{0+i}+a, A_{1+i}+a, \cdots, A_{t+i}+a; A_{t+1+i}+a, \cdots, A_{2t+i}+a), \tag{4}$$
$$i = 0, 1, \cdots, 2t,\ a = 0, 1, \cdots, 2(2t+1).$$

This system of blocks will not produce the desired design unless three sets of numbers called the differences of elements in configurations and the differences of elements in opposite configurations satisfy certain conditions. They are designated as:

Differences of elements in configurations:

$$A_{r+i}A_{r'+i} \text{ differences } = \pm(A_{r+i}-A_{r'+i}), \qquad r \neq r', r, r' = 0, 1, \cdots, t$$
$$A_{s+i}A_{s'+i} \text{ differences } = \pm(A_{s+i}-A_{s'+i}), \qquad s \neq s', s, s' = t+1, t+2, \cdots, 2t.$$

Differences of elements in the opposite configurations:

$$A_{r+i}A_{s+i} \text{ differences } = \pm(A_{r+i}-A_{s+i}).$$

These differences must satisfy the hypothesis of the following theorem.

THEOREM. *A set of* $2t+1$ *initial blocks* $B_0, B_1, \cdots, B_{2t}$ *generates the design of this section in the above described manner*, iff

(A) *the* $A_{r+i}A_{r'+i}$ *and* $A_{s+i}A_{s'+i}$ *differences are* $2(2t+1)$ *distinct sets of* v_1 *equal differences,*
(B) *the* $A_{r+i}A_{s+i}$ *differences are* $2(2t+1)$ *distinct sets of* v_2 *equal differences and*
(C) *none of the* $A_{r+i}A_{r'+i}$, $A_{s+i}A_{s'+i}$, $A_{r+i}A_{s+i}$ *differences is equal to zero.*

The proof is left to the reader. We proceed to construct a system of initial blocks satisfying the conditions of the Theorem. Let

$$B_i = (x^{0+i}, x^{1+i}, \cdots, x^{t+i}; x^{t+1+i}, x^{t+2+i}, \cdots, x^{2t+i}), \qquad i = 0, 1, \cdots, 2t, \tag{5}$$

where x denotes a primitive element in $GF(p^n)$. The differences of this set of initial blocks are

$$\begin{aligned} A_{r+i}A_{r'+i} &\text{ differences } = \pm(x^{r+i}-x^{r'+i}),\\ A_{s+i}A_{s'+i} &\text{ differences } = \pm(x^{s+i}-x^{s'+i}),\\ A_{r+i}A_{s+i} &\text{ differences } = \pm(x^{r+i}-x^{s+i}). \end{aligned} \tag{6}$$

These differences must satisfy

$$(x^r - x^{r'}) = \pm x^{C_{rr'rr''}}(x^r - x^{r''}), \qquad r \neq r' \neq r'', r'' = 0, 1, \cdots, t, \tag{7a}$$

$$(x^s - x^{s'}) = \pm x^{C_{ss'ss''}}(x^s - x^{s''}), \qquad s \neq s' \neq s'', s'' = t+1, t+2, \cdots, 2t, \tag{7b}$$

$$(x^r - x^s) = \pm x^{C_{rsrs_1}}(x^r - x^{s_1}), \qquad s \neq s_1, s_1 = t+1, t+2, \cdots, 2t, \tag{7c}$$

$$(x^{r+i}-x^{r'+i}) \neq 0, \qquad (x^{s+i}-x^{s'+i}) \neq 0, \qquad (x^{r+i}-x^{s+i}) \neq 0, \tag{7d}$$

where $C_{rr'rr''}$, $C_{ss'ss''}$, C_{rsrs_1} denote related elements. The proof of (7) is as follows. Consider (7a). Let

$$x^{c_{rr'}} = x^r - x^{r'}, \qquad x^{c_{rr''}} = x^r - x^{r''}. \tag{8}$$

The proof of (7a) requires that $x^{C_{rr'rr''}+c_{rr''}-c_{rr'}} = \pm 1$. This is true because a unique element $C_{rr'rr''}$ exists for $c_{rr'}$ and $c_{rr''}$ such that $C_{rr'rr''}+c_{rr''} \equiv c_{rr'}$ (mod $2(2t+1)$) or $C_{rr'rr''}+c_{rr''} \equiv c_{rr'}$ (mod $2t+1$). The proofs of (7b) and (7c) are similar. For proving (7d), note that $x, x^2, \cdots, x^{2(2t+1)} = 1$ are all different. Hence (5) satisfies the Theorem. Some illustrations will follow.

Set of initial blocks for $v = 2(2t+1)+1$, $k_1 = t+1$, $k_2 = t$.

v	Initial blocks
11	(1, 2, 4; 8, 5), (2, 4, 8; 5, 10), (4, 8, 5; 10, 9), (8, 5, 10; 9, 7), (5, 10, 9; 7, 3)
19	(1, 3, 9, 8, 5; 15, 7, 2, 6), (3, 9, 8, 5, 15; 7, 2, 6, 18), (9, 8, 5, 15, 7; 2, 6, 18, 16), (8, 5, 15, 7, 2; 6, 18, 16, 10), (5, 15, 7, 2, 6; 18, 16, 10, 11), (15, 7, 2, 6, 18; 16, 10, 11, 14), (7, 2, 6, 18, 16; 10, 11, 14, 4), (2, 6, 18, 16, 10; 11, 14, 4, 12), (6, 18, 16, 10, 11; 14, 4, 12, 11).

REFERENCES

[1] BOSE, R. C. (1939). On the construction of balanced incomplete block designs. *Ann. Eugenics* **9** 353–399.

[2] YALAVIGI, C. C. (1968). On the construction of tournamental designs, read at the 34th conference of Indian Math Society.