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BERNOULLI FLOWS WITH INFINITE ENTROPY[1]

BY J. FELDMAN[2] AND M. SMORODINSKY[3]

University of California, Berkeley

As an easy consequence of the recent deep work of D. Ornstein on Bernoulli systems, together with an entropy calculation for continuous time and finite state space stationary Markov processes, we shall show that the flow obtained from an ergodic and irreducible such process is, at each time, isomorphic to a Bernoulli shift with infinite entropy. Ornstein [5] already has given an example of a flow S_t which is isomorphic to a Bernoulli shift with finite entropy for each t.

Let X_t, $-\infty < t < \infty$, be a continuous Markov process with state space $\{1, \cdots, N\}$, transition probabilities $P_t(i,j) = \Pr\{X_t = j \mid X_0 = i\}$, and stationary starting probabilities $\Pr\{X_0 = i\} = \pi_i$. We may as well reduce the state space so that no π_i is zero. Let $\lambda_i = \lim_{t\to 0}(1-P_t(i,i))/t$; this limit exists (see [2]). Let the process live on $\Omega = \{\omega:(-\infty, \infty) \to \{1, \cdots, N\},\ \omega \text{ right-continuous}\}$, with $X_t(\omega) = \omega(t)$, and let $\mathscr{F}$ be the σ-field generated by the coordinates. Such a version of the process always exists.

Let $\mathscr{F}_t$ be the σ-field generated by $\{X_{nt}: n = 0 \pm 1, \cdots\}$, $\{X_{nt}: n = 0 \pm 1, \cdots\}$. Let $(S_t\omega)(u) = \omega(t+u)$. Then S_t is a measure preserving transformation on $(\Omega, \mathscr{F}, \Pr)$, and leaves $\mathscr{F}_t$ invariant. The process $X_{nt, n=0\pm 1}, \ldots$, is a stationary Markov chain on $(\Omega, \mathscr{F}_t, \Pr)$, with starting probabilities π_i and transition probabilities $P_t(i,j)$; consequently the corresponding shift $S_t \mid \mathscr{F}_t$[4] has entropy $H(S_t \mid \mathscr{F}_t) = \sum_i \pi_i \sum_j \rho(P_t(i,j))$, where

$$\begin{aligned} \rho(r) &= -r\log r, & 0 < r \leqq 1 \\ &= \quad 0, & r = 0. \end{aligned}$$

This is shown for example in Billingsley [1].

THEOREM 1. *$\lim_{t\to 0} H(S_t \mid \mathscr{F}_t)/\rho(|t|)$ exists, and equals $\sum_{i=1}^N \pi_j \lambda_j$.*

PROOF. $P_t(i,i) = 1 - \lambda_i t + o(t)$, while $\sum_{j\neq i} P_t(i,j) = \lambda_i t + o(t)$. Consequently, for $j \neq i$, $P_t(i,j) = q_{ij}\lambda_i t + o(t)$, where $q_{ij} \geqq 0$ and $\sum_{j\neq i} q_{ij} = 1$. For any i, if $\lambda_i = 0$, then it is known (see [2]) that $P_t(i,i) = 1$ for all t, and $P_t(i,j) = 0$ if $i \neq j$. So $\sum_j \rho(P_t(i,j)) = 0$ for any such i. On the other hand, if $\lambda_i > 0$, then

$$\begin{aligned} \frac{\rho(P_t(i,i))}{\rho(t)} &= \frac{(1-\lambda_i t + o(t))\log(1-\lambda_i t + o(t))}{\rho(t)} \\ &= \frac{(1-\lambda_i t + o(t))}{\log|t|} \cdot \frac{(-\lambda_i t + o(t))}{|t|} \to 0 \qquad \text{as } t \to 0, \end{aligned}$$

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[4] $S_t \mid \mathscr{F}_t$ is the transformation S_t on $(\Omega, \mathscr{F}_t, \Pr)$.

while for $j \neq i$ we have

$$\frac{\rho(P_t(i,j))}{\rho(|t|)} = \frac{(q_{ij}\lambda_i t + o(t))}{|t|} \cdot \frac{\log(q_{ij}\lambda_i t + o(t))}{\log|t|} \to q_{ij}\lambda_i \qquad \text{as } t \to 0.$$

Thus

$$\sum_{j=1}^{N} \frac{\rho(P_t(i,j))}{\rho(|t|)} \to \lambda_i \qquad \text{as } t \to 0, \text{ and}$$

$$\frac{H(S_t \mid \mathscr{F}_t)}{\rho(t)} \to \textstyle\sum_i \pi_i \lambda_i. \qquad \square$$

Assume also that $N \geqq 2$ and $P_t(i,j)$, $t > 0$ is an irreducible Markov matrix. Then

THEOREM 2. *S_t is for each $t \neq 0$ a Bernoulli shift of infinite entropy.*

PROOF. The process X_{nt}, $n = 0, \pm 1, \cdots$ is mixing Markov; see for example [2].

Thus $S_t \,|\, \mathscr{F}_t$ is isomorphic to a Bernoulli shift, by Friedman–Ornstein [3]. Now, $S_t \,|\, \mathscr{F}_{t/n} = (S_{t/n} \,|\, \mathscr{F}_{t/n})^n$, so $S_t \,|\, \mathscr{F}_{t/n}$ is likewise a Bernoulli shift, since any power of a Bernoulli shift is again a Bernoulli shift. Finally, since $\mathscr{F}_{t/2^k} \uparrow$ and $\bigcup_{k=1}^{\infty} \mathscr{F}_{t/2^k}$ generates $\mathscr{F}$ up to sets of probability zero, it follows, by Ornstein [4], that S_t is isomorphic to a Bernoulli shift.

As for the entropy of S_t:

$$\begin{aligned} H(S_t) &\geqq H(S_t \mid \mathscr{F}_{t/n}) = H((S_{t/n} \mid \mathscr{F}_{t/n})^n) \\ &= nH(S_{t/n} \mid \mathscr{F}_{t/n}) \\ &= n\rho(|t|/n)(\textstyle\sum_i \pi_i \lambda_i + o(t/n)) \\ &= |t|\,(\log|t| + \log n)(\textstyle\sum_i \pi_i \lambda_i + o(t/n)). \end{aligned}$$

Now: since $N \geqq 2$, and the process is ergodic, each λ_i must be > 0; for otherwise $\{\omega : \omega(t) = i \text{ for all } t\}$ would be a set of positive probability < 1. So $H(S_t) = +\infty$. $\square$

ADDED IN PROOF. The fact that $H(S_t)$ is infinite was also noticed by U. Krengel.

REFERENCES

[1] BILLINGSLEY, P. (1965). *Ergodic Theory and Information.* Wiley, New York.

[2] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[3] FRIEDMAN, N. and ORNSTEIN, D. S. (1971). On isomorphism of weak Bernoulli transformations. To appear in *Advances in Math.*

[4] ORNSTEIN, D. S. (1971). Two Bernoulli shifts with infinite entropy are isomorphic. To appear in *Advances in Math.*

[5] ORNSTEIN, D. S. (1970). Embedding a Bernoulli shift in a flow. To appear in *Lecture Notes in Mathematics.* Springer-Verlag, Berlin 178–218.