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ABSORPTION PROBABILITIES FOR CERTAIN TWO-DIMENSIONAL RANDOM WALKS[1]

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1. Introduction. In Section 2 of this paper, we consider absorption of certain finite random walks on three boundaries amenable to a method of images for the plane. Each of the three boundaries determines a class of walks to which the method is applicable; specifically, in the case of a rectangle or a right isosceles triangle with sides oriented along the axes, walks involving unit steps in directions 0, $\pi/2$, π and $3\pi/2$ and walks involving steps of length $2^{\frac{1}{2}}$ in directions $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$; in addition, for the rectangle, walks involving steps of all eight types; in the case of an equilateral triangle, walks involving unit steps in directions 0, $2\pi/3$ and $4\pi/3$, walks involving unit steps in directions $\pi/3$, π and $5\pi/3$, and walks involving steps of all six types.

For each of these eight boundary-walk combinations, it is possible to compute certain "untied" and "tied" probabilities. The first of these is the probability $P_{\Gamma,n}$ of absorption under a uniform distribution over all paths consisting of n steps of the specified types. The second is the probability $P^e_{\Gamma,\alpha,n}$, under a uniform distribution over all paths consisting of n steps of the specified types and ending at a specified interior point e, that absorption occurs at or before the $[\alpha n]$th step, $0 < \alpha \leqq 1$. We note that the tied absorption probability for one of the eight above boundary-walk combinations has been derived in [3], by an argument less direct than that presented here, for the case $\alpha = 1$.

Limits of expressions derived in Section 2 provide asymptotic absorption probabilities not only for the few cases examined there, but also, through the invariance principle, for rather large classes of walks, both "untied" and "tied". These are detailed in Section 3 and Section 4. The invariance principle simultaneously provides probabilities of absorption of two-dimensional[2] untied and tied Wiener processes on cylinders with triangular base, and hence the corresponding distributions of the time to absorption.

2. A method of images for the plane. We first illustrate the derivation of a "tied" absorption probability in terms of the boundary-walk combination treated in [3], i.e., the (open) equilateral triangle Γ together with a walk involving unit steps in directions 0, $2\pi/3$ and $4\pi/3$ (in the terminology of [3], steps of type A, B and C). We are interested in first computing the probability $P^e_{\Gamma,\alpha,n}$ that an n-step walk of this type, starting at the origin and terminating at the interior point e of Γ, exits Γ

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[2] The qualifier "two-dimensional" refers here to the range of the process (i.e. space) rather than, as for example in [6], to the domain (i.e. time).

at or before the $[\alpha n]$th step. In view of uniformity,

$$P^e_{\Gamma,\alpha,n} = |B_{\alpha,e}|/N_e, \tag{2.1}$$

where N_e is the number of n-step paths from the origin to e, and $|B_{\alpha,e}|$ is the number of these paths in the set $B_{\alpha,e}$ of paths exiting Γ at or before the $[\alpha n]$th step.

It is useful to write

$$|B_{\alpha,e}| = |D_{\alpha,e}| + |F_{\alpha,e}|, \tag{2.2}$$

where $D_{\alpha,e}$ is the subset of paths in $B_{\alpha,e}$ for which the $[\alpha n]$th step is outside Γ, and $F_{\alpha,e} = B_{\alpha,e} - D_{\alpha,e}$.

Analogously to the procedure in [3], we compute $|F_{\alpha,e}|$ by resorting to certain sets $A^+_{\alpha,e}$ and $A^-_{\alpha,e}$ of auxiliary paths. These sets are defined in terms of a grid of auxiliary points and triangles in the plane generated as follows; given the equilateral triangle Γ, a system of contiguous triangles is generated by successive "unfolding" of Γ. The triangles so generated are called auxiliary triangles. Those obtained as a result of an odd number of unfoldings are called $\oplus$ triangles, and those obtained as a result of an even number of unfoldings are called $\ominus$ triangles. If one imagines the point e as participating in the successive unfoldings, one obtains analogously a certain grid of points. Clearly there will be precisely one such grid point within each of the auxiliary triangles. Grid points located within $\oplus$ triangles are called $\oplus$ points, and grid points located within $\ominus$ triangles are called $\ominus$ points. $A^+_{\alpha,e}$ is the set of n-step paths, consisting of steps of type A, B or C, starting at the origin, which, at the $[\alpha n]$th step, is inside (and not on the boundary of) some $\oplus$ triangle and ends at the $\oplus$ point in that same triangle. $A^-_{\alpha,e}$ is defined analogously. The reader may wish to refer to [3] for a further discussion and diagram for the case when e is the origin. Note that, in the terminology of [3], $|A^+_{1,0}| = N_\oplus$ and $|A^-_{1,0}| = N_\ominus$.

We now proceed to verify that

$$|F_{\alpha,e}| = |A^+_{\alpha,e}| - |A^-_{\alpha,e}|. \tag{2.3}$$

Since $F_{\alpha,e} \cap A^-_{\alpha,e} = \emptyset$, we need only construct a 1–1 map of $F_{\alpha,e} \cup A^-_{\alpha,e}$ onto $A^+_{\alpha,e}$: given an auxiliary path in $A^+_{\alpha,e}$, the $[\alpha n]$th step of such a path ends inside some $\oplus$ triangle. Hence there is a last point of entry into that triangle, prior to the $[\alpha n]$th step. "Reflection" of the path segment following that entry point provides the desired map.

(2.1), (2.2) and (2.3) now yield

$$P^e_{\Gamma,\alpha,n} = (|D_{\alpha,e}| + |A^+_{\alpha,e}| - |A^-_{\alpha,e}|)/N_e. \tag{2.4}$$

Note that $|D_{1,e}| = 0$, so that $|B_{1,e}| = |A^+_{1,e}| - |A^-_{1,e}|$, which is essentially Equation (18) of [3]. Relation (2.4), written in a form analogous to Equation (19) of [3], is, for $3l$ the length of a side of Γ,

$$P^e_{\Gamma,\alpha,n} = \frac{|D_{\alpha,e}|}{N_e} + 3\sum_{i=1}^{[n/3l]} \sum_{j \in J(i)} r(i,j) n(\alpha, e(i,j))/N_e, \tag{2.5}$$

where $e(i,j)$ is the auxiliary point (i.e., $\oplus$ point or $\ominus$ point) corresponding to the auxiliary triangle Γ_{ij} centered at U_{ij}: $(3lj/2, 3^{\frac{1}{2}}l(2i-j)/2)$, $n(\alpha, e(i,j))$ is the number of n-step paths from the origin to $e(i,j)$ whose $[\alpha n]$th step terminates in Γ_{ij}, $J(i)$ is the set $[2-i, 3-i, 5-i, 6-i, \cdots 2i]$ and $r(i,j) = -1$ for $i+j = 3, 6, 9, 12, \cdots$ and 1 for $i+j = 2, 5, 8, 11, \cdots$.

Similar considerations apply as well when the walk is assumed to begin at a location other than the center of Γ, or when considering the other seven boundary-walk combinations, with the walk beginning at an arbitrary interior point that can be reached from the origin by steps of the relevant type.

For example, in the case that Γ is a right isosceles triangle centered at the origin with equal sides of length $3l$ oriented along the axes, and the walk begins at the origin and involves unit steps in the directions 0, $\pi/2$, π and $3\pi/2$, the auxiliary points and triangles are obtained as in the first case, by "unfolding" Γ and e. We then have

$$P^e_{\Gamma,\alpha,n} = \frac{|D_{\alpha,e}|}{N_e} + \sum_{k=1}^{[n/2l]} \sum_{i \in S;\, j \in S;\, |i|+|j|=2k} r(i,j)n(\alpha, e(i,j))/N_e, \tag{2.6}$$

where $e(i,j)$ is the auxiliary point corresponding to the auxiliary triangle Γ_{ij} centered at U_{ij}: (li, lj), S is the set $[\cdots, -5, -3, -2, 0, 1, 3, 4, 6, \cdots]$ and $r(i,j) = 1$ if $i = \cdots, -9, -5, -3, 1, 3, 7, 9, \cdots$ and $j-i = \cdots, -12, -6, 0, 6, 12, \cdots$ or if $i = \cdots, -8, -6, -2, 0, 4, 6, 10, \cdots$ and $j-i = \cdots, -8, -2, 4, 10, \cdots$, and $r(i,j) = -1$ otherwise.

Again if Γ is a rectangle centered at the origin, with sides of length $2l_1$ and $2l_2$ oriented along the axes, and the walk begins at the origin and involves steps of length $2^{\frac{1}{2}}$ in the directions $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$,

$$P^e_{\Gamma\,\alpha,n} = \frac{|D_{\alpha,e}|}{N_e} + 2\sum_{i=-[n/2l_1]}^{[n/2l_1]} \sum_{\substack{j=1 \text{ if } i \leq 0 \\ =0 \text{ if } i>0}}^{[(n-2l_1|i|)/2l_2]} r(i,j)n(\alpha, e(i,j))/N_e, \tag{2.7}$$

where $e(i,j)$ is the auxiliary point corresponding to the auxiliary rectangle centered at U_{ij}: $(2l_1 i, 2l_2 j)$ and $r(i,j) = 1$ if $|i-j|$ is odd and $r(i,j) = -1$ if $|i-j|$ is even.

We consider next the derivation of "untied" absorption probabilities for the same eight boundary-walk combinations. Define A^+ to be the set of all paths whose nth step terminates within a $\oplus$ region (triangle or rectangle), and similarly for A^-. Define as well A to be the set of all paths whose nth step terminates on the boundary of some auxiliary region. Finally, define B to be the set of all paths absorbed by the nth step, and C to be the subset of B consisting of paths whose nth step terminates within Γ. Then

$$|B| = |A^+| + |A^-| + |A| + |C|. \tag{2.8}$$

But

$$|A^+| = \sum_e |A^+_{1,e}|, \qquad |A^-| = \sum_e |A^-_{1,e}|, \qquad |C| = \sum_e |F_{1,e}|$$

so that summing (2.3) over e implies

$$|C| = |A^+| - |A^-|, \tag{2.9}$$

which, together with (2.8), yields

$$|B| = 2|A^+|+|A|. \tag{2.10}$$

Relations somewhat more explicit than (2.10), analogous to (2.5), (2.6) and (2.7), can of course be written down. For example, for the equilateral triangle with steps of type A, B and C,

$$P_{\Gamma,n} = \frac{|A|}{3^n}+\frac{2}{3^{n-1}}\sum_{i=1}^{[n/3]]}\sum_{j\in J(i)} S(i,j)n_{ij}. \tag{2.11}$$

where $S(i,j) = (r(i,j)+1)/2$(cf. (2.5)), and n_{ij} is the number of paths whose nth step terminates in the interior of the auxiliary triangle centered at U_{ij}.

3. Asymptotic untied absorption probabilities. Theorem 3.1 gives the asymptotic implications of the untied absorption computations of Section 2.

THEOREM 3.1. *Let Γ be either an equilateral triangle containing the origin or an isosceles right triangle containing the origin. Let Γ^+ be that "half" of the plane that consists of the auxiliary $\oplus$ triangles corresponding to Γ. Let the walk Π_n consist of n* i.i.d. *two-dimensional vector steps X_1^n, X_2^n, $\cdots$, X_n^n, where $E[X_k^n] = \emptyset$ and $V[X_k^n] = n^{-1}I$. Then*

$$\lim_{n\to\infty} P_{\Gamma,n} = P_\Gamma = 2P_\Gamma{}^+, \tag{3.1}$$

where P_Γ, $P_{\Gamma,n}$ and $P_\Gamma{}^+$ are the probability that the two-dimensional independent Wiener process is absorbed by Γ, the probability that Π_n is absorbed by Γ, and the probability that an independent bivariate normal with standard deviation 1 *is in Γ^+, respectively.*

PROOF. The proof is detailed for Γ an equilateral triangle.

Consider the specialization of the assumptions regarding X_k^n to the case where, in terms of axes suitably oriented with respect to Γ, $X_k^n = Z_1^n:((2/n)^{\frac{1}{2}}, 0)$, $Z_2^n:(-(1/2n)^{\frac{1}{2}}, (3/2n)^{\frac{1}{2}})$ or $Z_3^n:(-(1/2n)^{\frac{1}{2}}, -(3/2n)^{\frac{1}{2}})$, each with probability $\frac{1}{3}$. Let $P_{\Gamma,n}^*$ be the probability that this special walk is absorbed by Γ. The invariance principle gives

$$\lim_{n\to\infty} P_{\Gamma,n} = P_\Gamma = \lim_{n\to\infty} P_{\Gamma,n}^*, \tag{3.2}$$

of which the first equality is the first equality of (3.1). In view of the second equality of (3.2) the second equality of (3.1) will have been established if it can be shown that

$$\lim_{n\to\infty} P_{\Gamma,n}^* = 2P_\Gamma{}^+. \tag{3.3}$$

To this end assume that the dimension $l(\Gamma)$ of Γ is such that, for some n, any point of intersection of a path composed of steps Z_i^n with the boundary must equal $k_1 Z_1^n+k_2 Z_2^n+k_3 Z_3^n$. Then in view of (2.10),

$$P_{\Gamma,\nu}^* = 2P\{\textstyle\sum_{k=1}^{\nu} X_k^\nu \in \Gamma^+\}+P\{\sum_{k=1}^{\nu} X_k^\nu \in G\}, \tag{3.4}$$

where $\{\nu\}$ is a suitable subsequence of the positive integers and G is the set of points on the boundary of the $\oplus$ triangles. Now (3.3) follows from the convergence in law of $\{\sum_{k=1}^{\nu} X_k^{\nu}\}$ to the independent bivariate normal with covariance matrix I, and the fact that the left-hand side of (3.3) exists in view of (3.2). Finally, the restriction on $l(\Gamma)$ can be removed since the set of so restricted $l(\Gamma)$ is dense in the reals and since, by the countable additivity of Wiener measure, P_Γ is continuous in $l(\Gamma)$.

The same argument applies for the right isosceles triangle, with the steps Z_i^n replaced by steps $((2/n)^{\frac{1}{2}}, 0)$, $(0, (2/n)^{\frac{1}{2}})$, $(-(2/n)^{\frac{1}{2}}, 0)$ and $(0, -(2/n)^{\frac{1}{2}})$.

The sort of assertion made in Theorem 3.1 applies as well to the rectangular Γ of Section 2, but follows directly from one-dimensional results.

It follows from the argument in Theorem 3.1 that the probability $P_{\Gamma,T}$ of absorption of the standard two-dimensional Wiener process by Γ by time T is $2P_{\Gamma,T}^{+}$, where $P_{\Gamma,T}^{+}$ is the probability assigned to Γ^{+} by an independent bivariate normal distribution with covariance matrix TI.

4. Asymptotic tied absorption probabilities. Theorem 4.1 below gives the asymptotic implications of the tied absorption computations of Section 2.

We begin by verifying an invariance principle for a bivariate tied random walk. Let S_n be a $(q-1)$-dimensional simplex centered at the origin with edge length proportional to $n^{-\frac{1}{2}}$. Let $V_1^n, \cdots, V_q^n$ be the set of rays from the origin to the vertices of S_n. Define Z_j^n to be the projection of V_j^n into a two-dimensional subspace. Let $W_1^n, \cdots, W_n^n$, n a multiple of q, be an ordered sequence of n/q vectors Z_1^n, n/q vectors $Z_2^n, \cdots$, and n/q vectors Z_q^n.

NOTE. Zero correlation of the two components of W (i.e., $\sum_{i=1}^{n} W_{i,1} W_{i,2} = 0$) presumably is basic to the convergence to the independent Brownian bridge discussed in Lemma 4.1 below. Construction of the W sequence by means of the simplex S_n is one manageable way of achieving zero correlation. Had it been possible to deal with, simply, zero correlation, a bivariate result at the level of generality of Theorem 4 of [4] or Theorem 24.1 of [1] would have been achieved.

Let $\xi_1^n, \cdots, \xi_n^n$ be a random permutation of the W_j^n's. Define the random walk Π_n^0 by

$$\Pi_n^0(t) = \sum_{j=1}^{[nt]} \xi_j^n$$

with $\Pi_n^0(t) = 0$ for $0 \leqq t < 1/n$.

LEMMA 4.1. *Let Γ be a bounded open set in the plane, with $\emptyset \in \Gamma$, and let Π_n^0 be as defined above. Also, let $W^0(t)$ be the two-dimensional independent Brownian bridge. Then, for* $0 < \alpha < 1$,

$$\lim_{n\to\infty} P_{\Gamma,\alpha,n}^0 = P_{\Gamma,\alpha}^0,$$

where $P_{\Gamma,\alpha}^0$ and $P_{\Gamma,\alpha,n}^0$ are, respectively, the probability that $W^0(t) \notin \Gamma$ for some $t \in [0, \alpha]$ and the probability that $\Pi_n^0(t) \notin \Gamma$ for some $t \in [0, \alpha]$.

PROOF. The proof proceeds by verifying the analogs of conditions (i) and (ii) in Theorem 3.1 of [2]. First

$$(\Pi_n^0(\alpha/k), \cdots, \Pi_n^0(\alpha)) \to_{\mathscr{D}} (W^0(\alpha/k), \cdots, W^0(\alpha)) \tag{4.2}$$

where $\Pi_n{}^0(i\alpha/k) = \sum_{j=1}^{n_i} \xi_j{}^n$, $n_i = [i\alpha n/k]$, and the right-hand side is a $2k$-dimensional random variable distributed according to the appropriate finite dimensional distribution of W^0. To verify (4.2), note that ([5])

$$(4.3) \qquad (\Pi_n{}^*(\alpha/k), \cdots, \Pi_n{}^*(\alpha)) \to_{\mathscr{D}} (W^*(\alpha/k), \cdots, W^*(\alpha)).$$

Here the $\Pi_n{}^*(i\alpha/k)$ are $k(q-1)$-dimensional partial sums of a random permutation of an ordered sequence of n/q vectors $V_1{}^n$, n/q vectors $V_2{}^n, \cdots$, and n/q vectors $V_q{}^n$; also the $W^*(i\alpha/k)$ have the $k(q-1)$-dimensional normal distribution appropriate to the $(q-1)$-dimensional independent Brownian bridge. In addition ([2], Theorem 2.1)

$$(4.4) \qquad \mathscr{P}(\Pi_n{}^*(\alpha/k), \cdots, \Pi_n{}^*(\alpha)) \to_{\mathscr{D}} \mathscr{P}(W^*(\alpha/k), \cdots, W^*(\alpha))$$

where $\mathscr{P}$ is the (continuous) orthogonal projection from $R^{(q-1)k}$ to R^{2k}. (4.2) then follows from (4.3) and (4.4) and the fact that

$$(4.5) \qquad \begin{aligned} (\Pi_n{}^0(\alpha/k), \cdots, \Pi_n{}^0(\alpha)) &= \mathscr{P}(\Pi_n{}^*(\alpha/k), \cdots, \Pi_n{}^*(\alpha)) \\ (W^0(\alpha/k), \cdots, W^0(\alpha)) &= \mathscr{P}(W^*(\alpha/k), \cdots, W^*(\alpha)). \end{aligned}$$

(4.5) states that the projection of a vector of partial sums of permuted V's is in fact a vector of partial sums of permuted Z's.

Next, for any $\varepsilon > 0$,

$$(4.6) \quad \begin{aligned} P^0_{\Gamma,\alpha,n} &= \textstyle\sum_{i=1}^k \sum_{r \in (n_{i-1}, n_i]} \sum_{c \in C} P[E_{\Gamma,\alpha,n,r,c} \cap \|\Pi_n{}^0(r/n) - \Pi_n{}^0(r'/n)\| \geqq \varepsilon] \\ &\quad + \textstyle\sum_i \sum_r \sum_c P[E_{\Gamma,\alpha,n,r,c} \cap \|\Pi_n{}^0(r/n) - \Pi_n{}^0(r'/n)\| < \varepsilon] \end{aligned}$$

where $E_{\Gamma,\alpha,n,r,c}$ is the event $\Pi_n{}^0(t) \notin \Gamma$ for the first time for $t = r/n \leqq \alpha$, with path composition "c" up to time r/n, and r' is the smallest of the n_i greater than or equal to r.

The event whose probability is given by the second term of the right-hand side of (4.6) implies that, for some n_i, $\Pi_n{}^0(n_i/n) \in \Gamma^\varepsilon$ where Γ^ε is the union of all (open) ε-neighborhoods centered at points of $\bar{\Gamma} = R^2 - \Gamma$.

As for the first term,

$$(4.7) \quad \begin{aligned} P[E_{\Gamma,\alpha,n,r,c} \cap \|\Pi_n{}^0(r/n) - \Pi_n{}^0(r'/n)\| \geqq \varepsilon] &= P[\|\Pi_n{}^0(r/n) - \Pi_n{}^0(r'/n)\| \\ &\qquad \geqq \varepsilon \mid E_{\Gamma,\alpha,n,r,c}] P[E_{\Gamma,\alpha,n,r,c}] \end{aligned}$$

where

$$(4.8) \qquad P[\|\Pi_n{}^0(r/n) - \Pi_n{}^0(r'/n)\| \geqq \varepsilon \mid E_{\Gamma,\alpha,n,r,c}] = O_n(1)/k,$$

uniformly in r and c. The last equality is verified as follows:

$$(4.9) \quad \begin{aligned} P[\|\Pi_n{}^0(r/n) - \Pi_n{}^0(r'/n)\| \geqq \varepsilon \mid E_{\Gamma,\alpha,n,r,c}] &\leqq P[|X_r - X_{r'}| \geqq \varepsilon/2^{\frac{1}{2}} \mid E_{\Gamma,\alpha,n,r,c}] \\ &\quad + P[|Y_r - Y_{r'}| \geqq \varepsilon/2^{\frac{1}{2}} \mid E_{\Gamma,\alpha,n,r,c}] \end{aligned}$$

where X and Y are respectively the horizontal and vertical projections of Π. Given $E_{\Gamma,\alpha,n,r,c}$,

$$\Pi_n{}^0(r/n) - \Pi_n{}^0(r'/n) = \textstyle\sum_{i=1}^{r'-r} \zeta_i{}^n$$

where $(\zeta_1{}^n, \cdots, \zeta_{r'-r}^n)$ is distributed as a random sample of size $r'-r$ from a finite population $(W_1{}^{*n}, \cdots, W_{n-r}^{*n})$ of $n-r$ vectors, composed of the set $(W_1{}^n, \cdots, W_n{}^n)$, with r vectors specified by "c" deleted. Hence, if $X_i{}^{*n}$ is the horizontal projection of $W_i{}^{*n}$,

$$X_{r'} - X_r = \sum_{i=1}^{r'-r} \gamma_i{}^n$$

where $(\gamma_1{}^n, \cdots, \gamma_{r'-r}^n)$ is distributed as a random sample $\mathscr{S}$ of size $r'-r$ from a finite population $(X_1{}^{*n}, \cdots, X_{n-r}^{*n})$ of size $n-r$. Hence

$$E(X_{r'} - X_r) = \sum_{i=1}^{n-r} E(\delta_i{}^n) X_i{}^n = (r'-r)/(n-r) \sum_{i=1}^{n-r} X_i{}^n, \tag{4.10}$$

where $\delta_i{}^n = 1$ if $X_i{}^{*n}$ is in $\mathscr{S}$ and is 0 otherwise. But

$$E(\delta_i{}^n) = (r'-r)/(n-r) \leqq (\alpha n/k)/n(1-\alpha) = \alpha/k(1-\alpha) \tag{4.11}$$

and

$$\left|\sum_{i=r+1}^n X_i{}^n\right| = |X_r| = O_n(1) \tag{4.12}$$

uniformly in r and c, which follows from the fact that Γ is bounded. In view of (4.11) and (4.12)

$$|E(X_r - X_{r'})| \leqq O_n(1)/k \tag{4.13}$$

so that, for n, k large, $|E(X_r - X_{r'})| \leqq \varepsilon/2^{3/2}$, uniformly in r and c. Also given $E_{\Gamma,\alpha,n,r,c}$,

$$\begin{aligned} &V(X_r - X_{r'}) \\ &= \frac{(r'-r)(n-r')}{(n-r)(n-r-1)} \sum_{i=r+1}^n X_i{}^n + \frac{r'-r}{n-r}\left[\frac{r'-r-1}{n-r-1} - \frac{r'-r}{n-r}\right]\left(\sum_{i=r+1}^n X_i{}^n\right)^2 \\ &= O_n(1)/k \end{aligned} \tag{4.14}$$

uniformly in r and c. Thus, for n, k large,

$$\begin{aligned} &P[|X_r - X_{r'}| \geqq \varepsilon/2^{\frac{1}{2}} \mid E_{\Gamma,\alpha,n,r,c}] \\ &\leqq P[|(X_r - X_{r'}) - E(X_r - X_{r'})| \geqq \varepsilon/2\,(2^{\frac{1}{2}}) \mid E_{\Gamma,\alpha,n,r,c}] \\ &\leqq V(X_r - X_{r'})/\varepsilon^2 \\ &= O_n(1)\ k \end{aligned} \tag{4.15}$$

uniformly in r and c.

A similar argument applies as well to $Y_r - Y_{r'}$, yielding (4.8). Thus, in view of (4.7), the first term of the right-hand side of (4.6) is $O_n(1)/k$.

Hence, (4.6) leads to

$$\begin{aligned} &P[\Pi_n{}^0(n_i/n) \notin \Gamma \text{ for some } i = 1, 2, \cdots, k] \\ &\leqq P_{\Gamma,\alpha,n}^0 \leqq O_n(1)/k + P[\Pi_n{}^0(n_i/n) \in \Gamma^\varepsilon \text{ for some } i = 1, 2, \cdots, k] \end{aligned} \tag{4.16}$$

or

$$\begin{aligned} 1 - P[\Pi_n{}^0(n_i/n) \in \Gamma^\varepsilon, \text{ some } i] - O_n(1)/k &\leqq 1 - P_{\Gamma,\alpha,n}^0 \\ &\leqq P[\Pi_n{}^0(n_i/n) \in \Gamma, \forall\, i] \end{aligned} \tag{4.17}$$

and, using (4.2) and letting $n \to \infty$,

$$(4.18)\quad 1 - P[W^0(i\alpha/k) \in \Gamma^\varepsilon, \text{ some } i] - o_k(1) \leqq \liminf (1 - P^0_{\Gamma,\alpha,n}) \leqq \limsup (1 - P^0_{\Gamma,\alpha,n}) \leqq P[W^0(i\alpha/k) \in \Gamma, \forall\, i].$$

Since Γ^ε and Γ are open and W^0 is continuous on [0, 1], letting $k \to \infty$,

$$(4.19)\quad P[\mathscr{F}] \equiv P[W^0(t) \in \bar{\Gamma}, \text{ some } t \in [0, \alpha]] \leqq \liminf P^0_{\Gamma,\alpha,n} \leqq \limsup P^0_{\Gamma,\alpha,n} \leqq P[W^0(t) \in \Gamma^\varepsilon, \text{ some } t \in [0, \alpha]] \equiv P[\mathscr{E}^{(\varepsilon)}].$$

Now, letting ε_i be a sequence tending to zero, the event $\bigcap_i \mathscr{E}^{(\varepsilon_i)}$ implies that $\inf_{t \in [0,\alpha],\, x \in \bar{\Gamma}} \|x - W^0(t)\| = 0$. But the continuity of $W^0(t)$ on $[0, t]$ implies that of $\inf_{x \in \bar{\Gamma}} \|x - W^0(t)\|$, so that, since $\bar{\Gamma}$ is closed, $\inf_{t \in [0,\alpha]} [\inf_{x \in \bar{\Gamma}} \|x - W^0(t)\|] = 0$ implies the existence of $(\hat{t}, \hat{x})$ with $\|\hat{x} - W^0(\hat{t})\| = 0$. In other words, $\bigcap_i \mathscr{E}^{(\varepsilon_i)} \subset \mathscr{F}$. But clearly $\mathscr{F} \subset \bigcap_i \mathscr{E}^{(\varepsilon_i)}$, and the $\mathscr{E}^{(\varepsilon_i)}$ decrease monotonically, so that

$$(4.20)\quad \lim_{i \to \infty} P[\mathscr{E}^{(\varepsilon_i)}] = P[\mathscr{F}] \equiv P^0_{\Gamma,\alpha},$$

and (4.19) and (4.20) imply

$$\lim_{n \to \infty} P^0_{\Gamma,\alpha,n} = P^0_{\Gamma,\alpha}.$$

LEMMA 4.2. *For* Γ, $\Pi_n{}^0$, $P^0_{\Gamma,\alpha,n}$ *and* $P^0_{\Gamma,\alpha}$ *as defined in Lemma* 4.1 *and* $\alpha \leqq 1$,

$$(4.21)\quad \lim_{n \to \infty} P^0_{\Gamma,\alpha,n} = P^0_{\Gamma,\alpha}.$$

PROOF. All that remains to be shown is that

$$(4.22)\quad \lim_{n \to \infty} P^0_{\Gamma,1,n} = P^0_{\Gamma,1}.$$

The fact that

$$(\Pi_n{}^0(1/k), \cdots, \Pi_n{}^0(1)) \to_{\mathscr{D}} (W^0(1/k), \cdots, W^0(1))$$

follows in the same way as in the previous lemma.

Further, for any $\varepsilon > 0$,

$$(4.23)\quad P^0_{\Gamma,1,n} = \sum_{i=1}^{[k-k^{1/2}]} \sum_{r \in (n_{i-1}, n_i]} \sum_c P[E_{\Gamma,n,r,c} \cap \|\Pi_n{}^0(r/n) - \Pi_n{}^0(r'/n)\| \geqq \varepsilon] + \sum_{r \in (n_{[k-k^{1/2}]}, n_k]} \sum_c P[E_{\Gamma,n,r,c} \cap \|\Pi_n{}^0(r/n) - \Pi_n{}^0(r'/n)\| \geqq \varepsilon] + \sum_{i=1}^{k} \sum_r \sum_c P[E_{\Gamma,n,r,c} \cap \|\Pi_n{}^0(r/n) - \Pi_n{}^0(r'/n)\| < \varepsilon].$$

The event whose probability is given by the last term of the right-hand side of (4.23) implies that $\Pi_n{}^0(n_i/n) \in \Gamma^\varepsilon$ for some n_i.

As for the second term,

$$(4.24)\quad \begin{aligned} &\sum_{r \in (n_{k-k^{1/2}}, n_k]} \sum_c P[E_{\Gamma,n,r,c} \cap \|\Pi_n{}^0(r/n) - \Pi_n{}^0(r'/n)\| \geqq \varepsilon] \\ &\leqq \sum_{r \in (n_{k-k^{1/2}}, n_k]} \sum_c P[E_{\Gamma,n,r,c}] \\ &= P[\Pi_n{}^0(r/n) \in \bar{\Gamma} \text{ for the first time for } r \geqq [[k-k^{\frac{1}{2}}]n/k]] \\ &\leqq P[\Pi_n{}^0(r/n) \in \bar{\Gamma} \text{ for some } r \geqq [[k-k^{\frac{1}{2}}]n/k]] \\ &= P[\Pi_n{}^0(r/n) \in \bar{\Gamma} \text{ for some } r \leqq n - [[k-k^{\frac{1}{2}}]n/k]] \\ &\to_{n \to \infty} P[W^0(t) \in \bar{\Gamma} \text{ for some } t \leqq 1/k^{\frac{1}{2}}] = P^0_{\Gamma,1/k^{1/2}} \end{aligned}$$

where the last conclusion comes from Lemma 4.1 with $\alpha = 1/k^{\frac{1}{2}}$. Since the sequence of events, E_k: $W^0(t) \in \bar{\Gamma}$, $t \leqq 1/k^{\frac{1}{2}}$, is monotone decreasing and $\bigcap E_k = \emptyset$, $P^0_{\Gamma,1/k^{1/2}}$ is $o_k(1)$.

The first term of the right-hand side of (4.23) is $O_n(1)/k^{\frac{1}{2}}$. The portion of the argument in the proof of Lemma 4.1 corresponding to the verification here is that between (4.9) and (4.15), with α essentially set equal to $1-1/k^{\frac{1}{2}}$.

The rest of the argument proceeds as before.

Analogously to the arguments in Theorem 3.1, we now give an explicit form for the left-hand side of (4.21) in two special cases—Γ an open equilateral triangle and Γ an open right-isosceles triangle, both centered at the origin of $\Pi_n{}^0$:

$$\lim_{n\to\infty} P^0_{\Gamma,\alpha,n} = P^0_{\Gamma,\alpha} = P^{0*}_{\Gamma,\alpha}, \qquad 0 < \alpha \leqq 1 \tag{4.25}$$

where the right-hand side is the limit of (2.5) and (2.6) respectively for the two cases. Expression (4.25) gives, analogously to (3.1), absorption probabilities, indeed absorption waiting time distributions, for the tied Wiener process in absorbing triangular cylinders.

THEOREM 4.1. *Let Γ be either of the two triangles described in Section* 2. *Let the walk $\Pi_n{}^0$ be as described in Lemma* 4.1. *Then relation* (4.25) *holds, with $P^{0*}_{\Gamma,\alpha}$ given by* (4.27) *and* (4.34) *respectively for the two triangles.*

PROOF. The proof is detailed only for Γ an equilateral triangle.

Consider the specialization of the assumptions regarding $\Pi_n{}^0$ to the case where the sequence $Z_1{}^n, \cdots, Z_n{}^n$, $n = 3m$, consists of m each of the three vectors $((2/n)^{\frac{1}{2}}, 0)$, $(-(1/2n)^{\frac{1}{2}}, (3/2n)^{\frac{1}{2}})$, and $(-(1/2n)^{\frac{1}{2}}, -(3/2n)^{\frac{1}{2}})$. Let $P^{0*}_{\Gamma,\alpha,n}$, $0 < \alpha \leqq 1$, be the probability that this special walk is absorbed on Γ at or before the $[\alpha n]$th step. The invariance principle of Lemma 4.2 gives, for $0 < \alpha \leqq 1$

$$\lim_{n\to\infty} P^0_{\Gamma,\alpha,n} = P^0_{\Gamma,\alpha} = \lim_{n\to\infty} P^{0*}_{\Gamma,\alpha,n} = P^{0*}_{\Gamma,\alpha}, \tag{4.26}$$

and it remains only to derive the form of the right-hand side. For the equilateral triangle case we show that

$$\begin{aligned} P^{0*}_{\Gamma,\alpha} = {} & 1 - P[N \in \Gamma(0,0; 3l/(\alpha(1-\alpha))^{\frac{1}{2}})] + 3\textstyle\sum_{i=1}^{\infty} \sum_{j\in J(i)} r(i,j) \\ & \cdot P[N \in \Gamma_{(+)}(3lj[(1-\alpha)/\alpha]^{\frac{1}{2}}/2, 3l(2i-j)[(1-\alpha)/\alpha]^{\frac{1}{2}}/2; 3l/(\alpha(1-\alpha))^{\frac{1}{2}})] \\ & \cdot \exp[-3l^2(i^2+j^2-ij)/2], \end{aligned} \tag{4.27}$$

where $N \in \Gamma(a, b; d)$ denotes the event that a bivariate normal vector with covariance matrix I is in the region bounded by an equilateral triangle centered at (a, b) with sides of length d. Γ_+ refers to regions ∇, corresponding to $i+j = 2, 5, 8, 11, \cdots$ while Γ_- refers to regions Δ, corresponding to $i+j = 3, 6, 9, \cdots$.

For $l(\Gamma)$ restricted as in the proof of Theorem 3.1 and the corresponding subsequence $\{\nu\}$, (2.5) may be rewritten

$$\begin{aligned} P^{0*}_{\Gamma,\alpha,n} = {} & P[\textstyle\sum_{k=1}^{[\alpha\nu]} Z_k{}^\nu \notin \Gamma \mid \sum_{k=1}^{\nu} Z_k{}^\nu = U_{00}] \\ & + 3\textstyle\sum_{i=1}^{[n/3l]} \sum_{j\in J(i)} r(i,j) P[\sum_{k=1}^{[\alpha\nu]} Z_k{}^\nu \in \Gamma_{ij} \mid \sum_{k=1}^{\nu} Z_k{}^\nu = U_{ij}] N_{ij}/N_0. \end{aligned} \tag{4.28}$$

It is readily shown that, conditional on $\sum_{k=1}^{\nu} Z_k^{\nu} = U_{ij}$,

$$(4.29)\qquad \sum_{k=1}^{[\alpha\nu]} Z_k^{\nu} \to_{\mathscr{D}} N(3\alpha lj/2, 3^{\frac{1}{2}}\alpha l(2i-j)/2; \alpha(1-\alpha)I).$$

Also,

$$(4.30)\qquad N_{ij}/N_0 = ([n/3]!)^3/\{[(n/3)+jl(n/2)^{\frac{1}{2}}]!\,[(n/3)+(i-j)l(n/2)^{\frac{1}{2}}]!\cdot[(n/3)-il(n/2)^{\frac{1}{2}}]!\}$$
$$\doteq \exp[-3l^2(i^2+j^2-ij)/2] \text{ for } n \text{ large}.$$

In view of (4.28)–(4.30) it suffices to show, for $0 < \alpha \leqq 1$ and for k large, that

$$(4.31)\qquad R(n,k,l,\alpha) = \sum_{i=k}^{[n/3l]}\sum_{j\in J(i)} r(i,j)P[\sum_{k=1}^{[\alpha\nu]} Z_k^{\nu} \in \Gamma_{ij} \mid \sum_{k=1}^{\nu} Z_k^{\nu} = U_{ij}]N_{ij}/N_0$$

is arbitrarily small uniformly in n for n large and also, for k large,

$$(4.32)\qquad |\sum_{i=k}^{\infty}\sum_{j\in J(i)} r(i,j)P[N\in\Gamma_{ij}]\exp[-3l^2(i^2+j^2-ij)/2]|$$

is arbitrarily small. Looking first at (4.32), this is bounded by

$$|2\sum_{i=k}^{\infty} i\exp[-3l^2(i^2-[i/2]^2-i[i/2])/2]|$$

which is small for k large.

Looking at (4.31) and using an argument similar to that in [3],

$$(4.33)\qquad R(n,k,l,\alpha) \leqq 3\sum_{i=k}^{[n/3l]}\{[(n/3)!]^3/[(n/3)-iln^{\frac{1}{2}}]!\,[(2n/3)+iln^{\frac{1}{2}}]!\}$$
$$\cdot(|\sum_{j\in J(i)} r(i,j)P[\sum_{k=1}^{[\alpha\nu]} Z_k^{\nu}\in\Gamma_{ij} \mid \sum_{k=1}^{\nu} Z_k^{\nu} = U_{ij}]\cdot\binom{(2n/3)+iln^{1/2}}{(n/3)+jln^{1/2}}|).$$

For fixed i, since $|r(i,j)P(\cdot)| \leqq 1$,

$$|\sum_{j\in J(i)} r(i,j)P[\sum_{k=1}^{[\alpha\nu]} Z_k^{\nu}\in\Gamma_{ij} \mid \sum_{k=1}^{\nu} Z_k^{\nu} = U_{ij}]\binom{(2n/3)+iln^{1/2}}{(n/3)+jln^{1/2}}|$$
$$\leqq \sum_{j\in J(i)}\binom{(2n/3)+iln^{1/2}}{(n/3)+jln^{1/2}}$$
$$\leqq 2i\max_{j\in J(i)}\binom{(2n/3)+iln^{1/2}}{(n/3)+jln^{1/2}}$$
$$= 2i\binom{(2n/3)+iln^{1/2}}{(n/3)+[i/2]ln^{1/2}}.$$

Therefore, following the argument in [3] and using the fact that

$$(n/3!)^3/[(n/3)-iln^{\frac{1}{2}}]!\,[(n/3)+[i/2]ln^{\frac{1}{2}}]!\,[(n/3)+(i-[i/2])ln^{\frac{1}{2}}]!$$
$$\doteq \exp[-l^2(k^2+[k/2]^2-k[k/2])],$$

$R(n, k, l, \alpha)$ is essentially bounded by

$$2k\exp[-l^2(k^2+[k/2]^2-k[k/2])]/(1-\exp(-l^2[k/2]))$$

for n large, and is arbitrarily small uniformly in n.

Applying the results of (4.29)–(4.32) to (4.28) we have

$$\lim_{n\to\infty} P_{\Gamma,\alpha,n}^{0*} = \text{right-hand side of (4.27)}$$

for $0 < \alpha \leqq 1$.

Finally, as in Theorem 3.1, we can remove the restriction on $l(\Gamma)$.

A similar argument applies as well to the right isosceles triangle with steps $Z_i^n, i = 1, \cdots, n$, consisting of an equal number of steps $((2/n)^{\frac{1}{2}}, 0)$, $(0, (2/n)^{\frac{1}{2}})$, $(-(2/n)^{\frac{1}{2}}, 0)$, and $(0, -(2/n)^{\frac{1}{2}})$. For this case

$$(4.34) \quad P_{\Gamma,\alpha}^{0*} = 1 - P[N \in \Gamma(0, 0; 3l/(\alpha(1-\alpha))^{\frac{1}{2}}] \\ + \sum_{k=1}^{\infty} \sum_{i \in S;\, j \in S;\, |i|+|j|=2k} r(i, j) P[N \in \Gamma(li[(1-\alpha)/\alpha]^{\frac{1}{2}}, \\ lj\,[(1-\alpha)/\alpha]^{\frac{1}{2}}; 3l/(\alpha(1-\alpha))^{\frac{1}{2}}] \exp[-2l^2(2k^2 - 2k|i| + i^2)].$$

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