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RANDOM TIME TRANSFORMATIONS OF SEMI-MARKOV PROCESSES

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1. Introduction. Let $\{X_t: t \geqq 0\}$ be a right continuous stochastic process taking values in some space $(E, \mathscr{E})$ and adapted to a right continuous family of σ-algebras $\{\mathscr{F}_t: t \geqq 0\}$, and let $\{\tau_t: t \geqq 0\}$ be a nonnegative real-valued right continuous stochastic process such that each τ_t is an $\mathscr{F}_t$-stopping time. Then the process $X_t' = X(\tau_t)\ t \geqq 0$, which is adapted to $\{\mathscr{F}_{\tau_t}\}$, is called the random time transformation (RTT) of $\{X_t\}$ determined by the change of time $\{\tau_t\}$.

Two classical RTT's that transform Markov processes into Markov processes are found in Dynkin [3] (or in Blumenthal and Getoor [2]) and in Feller [4]. For the RTT in Dynkin, $\{\tau_t\}$ is the inverse mapping of a strictly increasing continuous additive functional of $\{X_t\}$. This RTT is used in constructing generalized Brownian motion from a Wiener process. Feller discusses RTT's under the heading of subordination of processes. He shows that if $\{X_t\}$ is a Markov process with continuous transition probabilities and $\{\tau_t\}$ has stationary, independent, nonnegative increments and is independent of $\{X_t\}$, then $\{X_t'\}$ is a Markov process. This type of RTT was used by Bochner to construct symmetric, stable processes from a Wiener process. In addition to their theoretical significance, RTT's are useful in applications where $\{X_u: u \geqq 0\}$ describes some phenomena as a function of some parameter u, which increases in time according to a process $\{\tau_t\}$, and one is interested in $\{X_t'\}$ which depicts the phenomena as a function of time.

In this paper we present several RTT's of semi-Markov step processes (SMP's) (Definition 2). In Theorem 1, our major result, we identify some general conditions on a time process $\{\tau_t\}$ under which $\{X_t'\}$ is an SMP whenever $\{X_t\}$ is an SMP. We use this result in Section 4 to identify four types of RTT's. Two of these RTT's are analogous to those discussed by Dynkin and Feller for Markov processes, and a special case of another is similar to the RTT presented by Yackel [15]. Special cases of the RTT's presented transform Markov chains or regular step Markov processes or SMP's into any one of these three classes of processes. We conclude in Section 5 by presenting two other RTT's for special SMP's $\{X_t\}$ when $\{\tau_t\}$ is a step process independent of $\{X_t\}$.

2. Preliminaries. Let $(\Omega, \mathscr{F}, P)$ be a probability space: all stochastic processes introduced herein will be defined on this space. Let E be a locally compact Hausdorff space with a countable base, and set $\mathscr{E} = \mathscr{B}(E)$, the smallest σ-algebra containing the Borel sets of E. Let $R_+ = [0, \infty)$ and set $\mathscr{B}_+ = \mathscr{B}(R_+)$.

Let $Q(x, G)$ be a real-valued function defined for each $x \in E$ and $G \in \mathscr{E} \times \mathscr{B}_+$ with the properties:

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(i) $G \to Q(x, G)$ is a probability measure on $\mathscr{E} \times \mathscr{B}_+$ for each $x \in E$, and
(ii) $x \to Q(x, G)$ is $\mathscr{E}$-measurable for each $G \in \mathscr{E} \times \mathscr{B}_+$.

For each $n \geqq 0$, $x \in E$ and $G \in \mathscr{E} \times \mathscr{B}_+$ we define $Q^n(x, G)$ by setting $Q^0(x, G) = 1$ or 0 accordingly as $\{x\} \times \varnothing$ is or is not in G, and for $n \geqq 0$ we set

$$Q^{n+1}(x, G) = \int_0^\infty Q(x, dy)Q^n(y, G).$$

Note that Q is uniquely determined by the values it takes on the cylinder sets of $\mathscr{E} \times \mathscr{B}_+$. That is, if (i) and (ii) hold for cylinder sets of $\mathscr{E} \times \mathscr{B}_+$, then for each $x \in E$ we let $G \to \hat{Q}(x, G)$ be the unique extension, of the measure $G \to Q(x, G)$, to the σ-algebra $\mathscr{E} \times \mathscr{B}_+$. The mapping $x \to \hat{Q}(x, G)$ turns out to be $\mathscr{E}$-measurable for each $G \in \mathscr{E} \times \mathscr{B}_+$. This follows since $(E \times R_+, \mathscr{E} \times \mathscr{B}_+)$ being a locally compact Hausdorff space with a countable base implies that $G \to \hat{Q}(x, G)$ is a regular probability measure (see [8]), and so for each $G \in \mathscr{E} \times \mathscr{B}_+$ we can write

$$\hat{Q}(x, G) = \lim_{n \to \infty} \sum_{k=1}^{\infty} Q(x, A_{nk} \times B_{nk})$$

for an appropriate sequence of cylinder sets $\{A_{nk} \times B_{nk}\}$ of $\mathscr{E} \times \mathscr{B}_+$.

Definition 1. Let $\{\mathscr{G}_n : n \geqq 0\}$ be a non-decreasing family of sub-σ-algebras of $\mathscr{F}$ (which are complete with respect to P) and let Q be as above. Let $\{Y_n, W_n : n \geqq 0\}$ be a Markov chain (on $(\Omega, \mathscr{F}, P)$) with respect to $\{\mathscr{G}_n\}$ taking values in $(E \times R_+, \mathscr{E} \times \mathscr{B}_+)$ and with kernel Q. That is:

(i) Each (Y_n, W_n) is $\mathscr{G}_n$-measurable.
(ii) For each n and $G \in \mathscr{E} \times \mathscr{B}_+$

$$P[(Y_{n+1}, W_{n+1}) \in G \mid \mathscr{G}_n] = Q(Y_n, G) \quad \text{a.s.} \tag{2.1}$$

We call $\{Y_n, W_n\}$ a Markov renewal process (MRP) with respect to $\{\mathscr{G}_n\}$ with kernel Q.

The name MRP is motivated by properties (a) and (b) below. Several authors also use this name for the processes $\{N_t\}$ and $\{X_t\}$ in Definition 2. Note that without loss of generality we can assume that Q is of the form

$$Q(x, A \times [0, t]) = \int_A K(x, dy)F_{xy}(t), \tag{2.2}$$

where K is a Markov kernel on $(E, \mathscr{E})$ and $\{F_{xy}(\cdot) : x, y \in E\}$ is a family of distribution functions such that $(x, y) \to F_{xy}(t)$ is $\mathscr{E} \times \mathscr{E}$-measurable for each t. This follows since

$$\begin{aligned} Q(x, A \times [0, t]) &= P[Y_1 \in A, W_1 \leqq t \mid Y_0 = x] \\ &= \int_A P[Y_1 \in dy \mid Y_0 = x]P[W_1 \leqq t \mid Y_0 = x, Y_1 = y], \end{aligned}$$

and we can take $K(x, A)$ and $F_{xy}(t)$ to be the regular conditional distribution functions $P[Y_1 \in A \mid Y_0 = x]$ and $P[W_1 \leqq t \mid Y_0 = x, Y_1 = y]$ (see [8]) respectively.

An MRP $\{Y_n, W_n\}$ with respect to $\{\mathscr{G}_n\}$ with kernel Q has the following properties.

(a) $\{Y_n\}$ is a Markov chain with respect to $\{\mathscr{G}_n\}$ with kernel K.

(b) For any $n_1, \cdots, n_k$ and $t_1, \cdots, t_k$

$$P[W_{n_1} \leqq t_1, \cdots, W_{n_k} \leqq t_k \mid Y_n : n \geqq 0] = \prod_{i=1}^{k} F_{Y_{n_i-1} Y_{n_i}}(t_i).$$

That is, the $\{W_n\}$ are conditionally independent given the σ-algebra $\sigma(Y_n : n \geqq 0)$.

(c) Set $T_0 = 0$ a.s. and $T_n = \sum_{k=1}^{n} W_k$. Then $\{Y_n, T_n\}$ is a Markov chain with respect to $\{\mathscr{G}_n\}$ and for any $n \geqq 1$, and $G \in \mathscr{E} \times \mathscr{B}_+$,

$$P[(Y_n, T_n) \in G \mid \mathscr{G}_0] = Q^n(Y_0, G) \quad \text{a.s.}$$

(d) Set $\zeta = \sup_n T_n$. Then a necessary and sufficient condition for

$$P[\zeta = +\infty \mid X_0 = x] = 1 \qquad \text{for each } x \in E,$$

is that for each $\lambda > 0$ and $x \in E$

$$\int_0^\infty e^{-\lambda t} Q^n(x, E \times dt) \to 0 \qquad \text{as } n \to \infty. \tag{2.3}$$

Properties (a)–(c) follow directly from (2.1) and (2.2). Property (d) follows since, by (c), the integral in (2.3) represents $E[\exp(-\lambda T_n) \mid X_0 = x]$. Another useful property of MRP's is as follows.

Proposition 1. *Let* $\{Y_n, W_n\}$ *be an* MRP *with respect to* $\{\mathscr{G}_n\}$ *with kernel* Q *which satisfies* (2.3). *Set* $\nu_0 = 0$ *and for each* $n \geqq 0$ *let* $\nu_{n+1} = \inf\{m > \nu_n : W_m > 0\}$. *Set* $\tilde{Y}_n = Y_{\nu_n}$, $\tilde{W}_n = W_{\nu_n}$ *and* $\tilde{\mathscr{G}}_n = \mathscr{G}_{\nu_n}$. *Then* $\{\tilde{Y}_n, \tilde{W}_n\}$ *is an* MRP *with respect to* $\{\tilde{\mathscr{G}}_n\}$ *with kernel* $\tilde{Q}$ *satisfying*

$$\tilde{Q}(x, A \times [0, t]) = \sum_{k=0}^{\infty} \int_0^\infty Q^k(x, dy \times \{0\}) Q(y, A \times (0, t]),$$

for each $x \in E$, $A \in \mathscr{E}$ *and* $t \geqq 0$.

Proof. The variables $\{\nu_n\}$ are well defined and finite valued due to condition (2.3). We see by induction that each ν_n is a $\mathscr{G}_n$-stopping time. Obviously $\nu_0 = 0$ is a $\mathscr{G}_n$-stopping time and assuming that $\nu_0, \cdots, \nu_n$ also are, we have

$$\{\nu_{n+1} > m\} = \bigcup_{k=1}^{m} \{\nu_n = k, T_m = T_k\} \cup \{\nu_n > m\} \in \mathscr{G}_m,$$

and so ν_{n+1} is a $\mathscr{G}_n$-stopping time.

By the strong Markov property for $\{Y_n, W_n\}$, for each n, t and $A \in \mathscr{E}$

$$\begin{aligned} P[\tilde{Y}_{n+1} \in A, \tilde{W}_{n+1} \leqq t \mid \tilde{\mathscr{G}}_n] &= \sum_{k=0}^{\infty} \int_0^\infty P[Y_{\nu_n+k+1} \in A, W_{\nu_n+k+1} \in (0, t], Y_{\nu_n+k} \in dy, T_{\nu_n+k} = T_{\nu_n} \mid Y_{\nu_n}] \\ &= \sum_{k=0}^{\infty} \int_0^\infty Q^k(\tilde{Y}_n, dy \times \{0\}) Q(y, A \times (0, t]) \\ &= \tilde{Q}(\tilde{Y}_n, A \times [0, t]) \quad \text{a.s.} \end{aligned}$$

This proves the assertion. □

Definition 2. Let $\{Y_n, W_n : n \geqq 0\}$ be a stochastic process (on $(\Omega, \mathscr{F}, P)$), taking values in $(E \times R_+, \mathscr{E} \times \mathscr{B}_+)$ such that

$$W_0 = 0 \quad \text{and} \quad W_n > 0 \qquad \text{for } n \geqq 1. \tag{2.4}$$

For each $n \geqq 0$ set $T_n = \sum_{k=0}^{n} W_k$, and let $\zeta = \sup_n T_n$. For each $0 \leqq t < \zeta$ let $N_t = n$ if $T_n \leqq t < T_{n+1}$ and $X_t = Y_n$ if $T_n \leqq t < T_{n+1}$. Let $\{\mathscr{F}_t : t \in R_+\}$ be a non-decreasing right continuous family of sub-σ-algebras of $\mathscr{F}$ (which are complete with respect to P). Assume that $\{X_t, N_t\}$ is adapted to $\{\mathscr{F}_t\}$, i.e., for each n, t and $A \in \mathscr{E}$

$$\{X_t \in A, N_t = n\} \in \mathscr{F}_t.$$

If $\{Y_n, W_n\}$ is an MRP with respect to $\mathscr{G}_n = \mathscr{F}_{T_n}$, $n \geqq 0$, then we call $\{X_t, \zeta\}$ a semi-Markov step process (SMP) with respect to $\{\mathscr{F}_t\}$. We say that $\{X_t, \zeta\}$ is induced by the MRP $\{Y_n, W_n\}$. The process $\{N_t, \zeta\}$ is called the counting process associated with $\{Y_n, W_n\}$, or with $\{X_t, \zeta\}$.

Calling $\{N_t, \zeta\}$ and $\{X_t, \zeta\}$ stochastic processes (on $(\Omega, \mathscr{F}, P)$) is justified for

$$\{N_t = n, t < \zeta\} = \{T_n \leqq t < T_{n+1}\} \in \mathscr{F}$$

and

$$\{X_t \in A, t < \zeta\} = \bigcup_n \{T_n \leqq t < T_{n+1}, Y_n \in A\} \in \mathscr{F}.$$

(Herein set equality is taken to mean equality up to a set of P-measure zero.) The assumption that $\{N_t\}$ is adapted to $\{\mathscr{F}_t\}$ guarantees that each T_n is an $\mathscr{F}_t$-stopping time, since $\{T_n \leqq t\} = \{N_t > n, t < \zeta\} \in \mathscr{F}_t$ for each $t \geqq 0$. This in turn justifies the definition of the $\mathscr{G}_n$. Without this assumption, the T_n may not be $\mathscr{F}_t$-stopping times.

The SMP $\{X_t, \zeta\}$ is a continuous time step process, which proceeds through its state space $(E, \mathscr{E})$ according to the Markov chain $\{Y_n\}$, at a speed which is given by the $\{T_n\}$. It makes N_t jumps in the interval $(0, t]$, the nth jump occurs at time T_n and takes the process to state Y_n where it remains for a time $W_{n+1} > 0$ a.s. Note that some of these jumps may not be actual jumps in that we may have $P[Y_n = Y_{n+1}] > 0$. (For this reason an SMP may be induced by more than one MRP.) The lifetime of the SMP is given by the random variable ζ: note that $\zeta = +\infty$ a.s. if (2.3) holds. Our results can be formulated in an obvious manner (see [3]) for SMP's with random lifetimes, but for clarity of exposition we will assume $\zeta = +\infty$ a.s. The basic properties of SMP's on countable state spaces appear in Lévy [6], Smith [14], Pyke [9], [10], Pyke and Schaufele [11], [12] and Feller [5].

REMARK 1. An SMP induced by an MRP with kernel $Q(x, A \times [0, t]) = K(x, A)H_1(t)$, where H_1 is the degenerate distribution with unit mass at 1, is a Markov chain with kernel K. An SMP induced by an MRP with kernel

$$Q(x, A \times [0, t]) = K(x, A)[1 - e^{-\lambda(x)t}],$$

where K satisfies $K(x, \{x\}) = 0$ for each $x \in E$, and λ is a positive real valued $\mathscr{E}$-measurable function, is a regular step Markov process [1]).

REMARK 2. For each $t \geqq 0$ let $X_t^+ = Y_{N_t+1}$, $U_t = T_{N_t}$ and $U_t^+ = T_{N_t+1}$. Then it can be shown that the processes $\{X_t, U_t\}$ and $\{X_t, X_t^+, U_t, U_t^+\}$ are regular step

Markov processes. They in general will not be Markov processes with respect to $\{\mathscr{F}_t\}$.

For our major result we require the following.

LEMMA 1. *Let f be a function from R_+ into R_+ which is non-decreasing, right continuous and $f(o) = 0$ and $f(t)\uparrow\infty$ as $t \to \infty$. For each $t \in R_+$ let $g(t) = \inf\{s : f(s) > t\}$. Then the following are satisfied:*

(a) *The function g is non-decreasing, right continuous and for each $s \in R_+$ $f(s) = \inf\{t : g(t) > s\}$.*

(b) *If f is continuous at s, then $f(s) > t$ is equivalent to $g(t) < s$ for any $t \in R_+$.*

PROOF. Part (a) is proved on page 108 of Meyer [7]. For part (b) first assume $f(s) > t$. Under the hypothesis, there exists a $\delta > 0$ such that $f(s) > f(s-\delta) > t$, and so

$$g(t) = \inf\{u : f(u) > t\} \leqq s-\delta < s.$$

Conversely, if $g(t) < s$, then since g is right continuous (by part (a)) there exists a $\delta > 0$ such that $g(t) \leqq g(t+\delta) < s$, and so from part (a),

$$f(s) = \inf\{u : g(u) > s\} \geqq t+\delta > t.$$

This completes the proof of (b).

3. The main result. For the remainder of this paper we take $\{X_t\}$ to be an SMP with respect to $\{\mathscr{F}_t\}$ (as in Definition 2), which is induced by the MRP $\{Y_n, W_n\}$ having kernel Q satisfying (2.3).

Let $\{\varphi_t : t \in R_+\}$ be a stochastic process on $(\Omega, \mathscr{F}, P)$, taking values in $(R_+, \mathscr{B}_+)$, adapted to $\{\mathscr{F}_t\}$ and with right continuous, non-decreasing paths, which for convenience, satisfy $\varphi_0 = 0$ a.s. and $\varphi_t \uparrow \infty$ a.s. as $t \uparrow \infty$. Assume the following:

(A1) For a.a. $\omega \in \Omega$, the mapping $s \to \varphi_s(\omega)$ is continuous at the points $T_n(\omega)$, $n \geqq 0$ (this is obviously satisfied if $\{\varphi_t\}$ has continuous paths).

(A2) For any n let $\tilde{\varphi}_t = \varphi(T_n+t) - \varphi(T_n)$. Then $\sigma(\tilde{\varphi}_t : t \geqq 0)$ is conditionally independent of $\mathscr{G}_n = \mathscr{F}_{T_n}$ given $\sigma(Y_n, Y_{n+1}, W_{n+1})$; and for any t, u, w and $x, y \in E$

$$P[\tilde{\varphi}_t \leqq u \mid Y_n = x, Y_{n+1} = y, W_{n+1} = w] = P[\varphi_t \leqq u \mid Y_0 = x, Y_1 = y, W_1 = w].$$

For each $t \geqq 0$ let $\tau_t = \inf\{s : \varphi_s > t\}$. We call $\{\tau_t\}$ the *inverse* of $\{\varphi_t\}$. For each $t \geqq 0$ let $X_t' = X(\tau_t)$ and $\mathscr{F}_t' = \mathscr{F}_{\tau_t}$. Our main result is as follows.

THEOREM 1. *The process $\{X_t'\}$ is an* SMP *with respect to $\{\mathscr{F}_t'\}$.*

PROOF. Set $\nu_0 = 0$ and for $n \geqq 0$ let

$$\nu_{n+1} = \inf\{m > \nu_n : \varphi(T_m) > \varphi(T_{\nu_n})\}.$$

We see by induction that each ν_n is a $\mathscr{G}_n$-stopping time. Obviously $\nu_0 = 0$ is a $\mathscr{G}_n$-stopping time. Assume that $\nu_0, \cdots, \nu_n$ are $\mathscr{G}_n$-stopping times. Since $\{\varphi_t\}$ is adapted to $\{\mathscr{F}_t\}$ and T_n is a $\mathscr{F}_t$-stopping time $\{\varphi(T_n) \leqq t\} \in \mathscr{G}_n$ for each t and n, and so for each m

$$\{\nu_{n+1} > m\} = \bigcup_{k=1}^m \{\nu_n = k, \varphi(T_m) = \varphi(T_k)\} \cup \{\nu_n > m\} \in \mathscr{G}_m,$$

that is, ν_{n+1} is a $\mathscr{G}_n$-stopping time. This completes the induction argument.

For each $n \geqq 0$ let

(3.1) $$Y_n' = Y_{\nu_n}, \qquad T_n' = T_{\nu_n} \quad \text{and} \quad \mathscr{G}_n' = \mathscr{G}_{\nu_n},$$

and set

(3.2) $$W_0' = 0, \quad \text{and} \quad W_n' = T_n' - T_{n-1}' \qquad \text{for } n \geqq 1.$$

We will show that $\{Y_n', W_n'\}$ is an MRP which induces $\{Y_t'\}$.

Since $\{\mathscr{F}_t\}$ is right continuous, for any s, t

$$\{\tau_t < s\} = \bigcup_n \{\varphi_{s-n^{-1}} > t\} \in \mathscr{F}_s,$$

and so each τ_t is a $\mathscr{F}_t$-stopping time. Moreover by Lemma 1(a), $\{\tau_t\}$ is right continuous. It follows that $\{X_t'\}$ is right continuous; and it is adapted to $\{\mathscr{F}_t'\}$, since for any t, u and $A \in \mathscr{E}$,

$$\{X_t' \in A\} \cap \{\tau_t \leqq u\} = \bigcup_{r \leqq u;\, r \text{ rational}} \{\tau_t = r, X_r \in A\} \in \mathscr{F}_u.$$

For each t we can write

(3.3) $$X_t' = Y_n \quad \text{if} \quad T_n \leqq \tau_t < T_{n+1}.$$

Under assumption (A1) it follows from Lemma 1(b) that for any n and t

(3.4) $$\{T_n \leqq \tau_t < T_{n+1}\} = \{\varphi(T_n) \leqq t < \varphi(T_{n+1})\},$$

and so (3.3) is equivalent to

(3.5) $$X_t' = Y_n \quad \text{if} \quad \varphi(T_n) \leqq t < \varphi(T_{n+1}).$$

Then by the definition of the $\{\nu_n\}$, (3.5) is equivalent to $X_t' = Y_n'$ if $T_n' \leqq t < T_{n+1}'$.

The counting process, $N_t' = n$ if $T_n' \leqq t < T_{n+1}'$, associated with $\{X_t'\}$ is obviously adapted to $\{\mathscr{F}_t'\}$ if each T_n' is a $\mathscr{F}_t'$-stopping time. We prove the latter by induction. First note that by an argument similar to the above it can be shown that $\{N(\tau_t)\}$ is a right continuous process adapted to $\{\mathscr{F}_t'\}$, and $N(\tau_t) = \nu_n$ if $T_n' \leqq t < T_{n+1}'$. (Note that each T_n' is a jump point of $N(\tau_t)$ since $\nu_n > \nu_{n-1}$.) Now obviously $T_0' = 0$ is a $\mathscr{F}_t'$-stopping time, and assuming that $T_0', \cdots, T_n'$ are $\mathscr{F}_t'$-stopping times, for each $t \geqq 0$

$$\{T_{n+1}' > t\} = \bigcup_{r \leqq t;\, r \text{ rational}} \{T_n' = r, N(\tau_t) = N(\tau_r)\} \cup \{T_n' > t\} \in \mathscr{F}_t'$$

which completes the induction argument.

It remains to show that $\{Y_n', W_n'\}$ is an MRP with respect to $\{\mathscr{F}_{T_n'}'\}$. One can show by their definitions that

(3.6) $$\mathscr{F}_{T_n'}' = \mathscr{F}_{\tau(T_n')} \quad \text{and} \quad \mathscr{G}_{\nu_n} = \mathscr{F}_{T_{\nu_n}}.$$

By assumption (A1), for any n

$$\begin{aligned}\tau(T_n') &= \inf\{u : \varphi_u > T_n'\}\\ &= \max\{u : \varphi_u = \varphi(T_{\nu_n})\}\\ &= T_{\nu_n},\end{aligned}$$

and so from (3.6) we have

$$\mathscr{F}'_{T_n'} = \mathscr{G}_{\nu_n} = \mathscr{G}_n'.$$

To show that $\{Y_n', W_n'\}$ is an MRP with respect to $\{\mathscr{G}_n'\}$ it suffices by Proposition 1 to show that $\{Y_n, \varphi(T_n) - \varphi(T_{n-1})\}$ is an MRP with respect to $\mathscr{G}_n$. The latter follows, since by the strong Markov property of $\{Y_n, W_n\}$, and (A2) we have for any $x \in E$, $A \in \mathscr{E}$ and any n, t

$$\begin{aligned}&P[Y_{n+1} \in A, \varphi(T_{n+1}) - \varphi(T_n) \leqq t \mid \mathscr{G}_n, Y_n = x]\\ \text{(3.7)}\qquad &= \textstyle\int_{A \times R_+} Q(x, dy \times dw) P[\varphi(T_n + W_{n+1}) - \varphi(T_n) \leqq t \mid Y_n = x, Y_{n+1} = y,\\ &\qquad\qquad\qquad\qquad W_{n+1} = w]\\ &= \textstyle\int_{A \times R_+} Q(x, dy \times dw) P[\varphi(W_1) \leqq t \mid Y_0 = x, Y_1 = y, W_1 = w]\\ &= P[Y_1 \in A, \varphi(T_1) \leqq t \mid Y_0 = x].\end{aligned}$$

Thus the proof is complete. □

REMARK 3. The SMP $\{X_t'\}$ with respect to $\{\mathscr{F}_t'\}$ is induced by the MRP $\{Y_n', W_n'\}$ (defined in (3.1) and (3.2)) which by Proposition 1 and the above proof, has a kernel Q' satisfying

$$Q'(x, A \times [0, t]) = \textstyle\sum_{k=0}^{\infty} \int_0^{\infty} \tilde{Q}^k(x, dy \times \{0\}) \tilde{Q}(y, A \times (0, t])$$

where

$$\tilde{Q}(x, A \times [0, t]) = \textstyle\int_A K(x, dy) P[\varphi(W_1) \leqq t \mid Y_0 = x, Y_1 = y].$$

REMARK 4. The above theorem also holds if we replace (A2) by:

(A2′) There exists a sequence $\{\alpha_n\}$ of finite $\mathscr{G}_n$-stopping times that satisfy

$$\alpha_{n+1} = \alpha_n + \alpha_1 \circ \mathcal{O}_{\alpha_n}, \tag{3.8}$$

where $\{\mathcal{O}_n\}$ is the family of translation operators associated with $\{Y_n, W_n\}$ (see [2]), and in addition

$$\varphi(T_{\alpha_{n+1}}) - \varphi(T_{\alpha_n+1}) = 0 \qquad \text{for any } n \geqq 0. \tag{3.9}$$

Set

$$\tilde{Y}_n = Y_{\alpha_n}, \qquad \tilde{T}_n = T_{\alpha_n}, \qquad \tilde{\mathscr{G}}_n = \mathscr{G}_{\alpha_n}, \qquad \tilde{V}_0 = 0 \quad\text{and}\quad \tilde{V}_n = \tilde{T}_n - \tilde{T}_{n-1}; \tag{3.10}$$

and assume that (A2) holds with Y_n, T_n and $\mathscr{G}_n$ replaced by $\tilde{Y}_n$, $\tilde{T}_n$ and $\tilde{\mathscr{G}}_n$ respectively.

Note that (A2) was only invoked in proving that $\{Y_n', W_n'\}$ is an MRP with respect to $\{\mathscr{G}_n'\}$. Under (A2′) it can be shown that the $\{v_n\}$ defined in the above proof satisfy

$$v_{n+1} = \inf\{m > v_n : \varphi(\tilde{T}_m) > \varphi(\tilde{T}_{v_n})\}$$

and so

$$Y_n' = \tilde{Y}_{v_n}, \qquad W_n' = \varphi(\tilde{T}_{v_n}) - \varphi(\tilde{T}_{v_{n-1}}) \quad \text{and} \quad \mathscr{G}_n' = \tilde{\mathscr{G}}_{v_n}.$$

Then to show that $\{Y_n', W_n'\}$ is an MRP with respect to $\{\mathscr{G}_n'\}$, it suffices by Proposition 1 to show that $\{\tilde{Y}_n, \varphi(\tilde{T}_n) - \varphi(\tilde{T}_{n-1})\}$ is an MRP with respect to $\{\tilde{\mathscr{G}}_n\}$. The latter follows since it can be shown (see [13]) that $\{\tilde{Y}_n, \tilde{W}_n\}$ is an MRP with respect to $\{\tilde{\mathscr{G}}_n\}$, and so under (A2′), (3.7) holds with Y_n, T_n and $\mathscr{G}_n$ replaced by $\tilde{Y}_n$, $\tilde{T}_n$ and $\tilde{\mathscr{G}}_n$ respectively.

4. **Examples.** We now list several classes of processes $\{\varphi_t\}$, which according to the above, determine RTT's that transform SMP's into SMP's.

Functionals of $\{X_t\}$. Using the notation of Remark 2, set $Z_t = (X_t, X_t^+, U_t, U_t^+)$ for each $t \geqq 0$, and assume $\mathscr{F}_t = \sigma(Z_s : s \leqq t)$. For each $0 \leqq s \leqq t$ let φ_t^s be a (nonnegative) random variable such that $\{\varphi_t^s \leqq u\} \in \mathscr{F}_t$. The family $\{\varphi_t^s : 0 \leqq s \leqq t\}$ is called a (nonnegative) *functional of the Markov process* $\{Z_t\}$.

We will take $\{\varphi_t^s\}$ to be continuous, that is for a.a. $\omega \in \Omega$, $t \to \varphi_t^s(\omega)$ is continuous for each s: and additive, that is for a.a. $\omega \in \Omega$

$$\varphi_t^s(\omega) + \varphi_u^t(\omega) = \varphi_u^s(\omega). \tag{4.1}$$

We will also assume that for any n and $0 \leqq s \leqq t$

$$\varphi_t^s \circ \theta_{T_n} = \varphi_{T_n+t}^{T_n+s}, \tag{4.2}$$

where $\{\theta_t\}$ is the usual family of translation operators associated with the Markov process $\{Z_t\}$. (This resembles the homogeneity property of functionals, see page 173 of [3].) We will call $\{\varphi_t^s\}$ a *nonnegative, homogeneous, continuous, additive functional of the* SMP $\{X_t\}$.

Define the process $\{\varphi_t\}$ by setting $\varphi_t = \varphi_t^0$ for each $t \geqq 0$. Clearly $\{\varphi_t\}$ satisfies (A1), and (A2) is satisfied, since by (4.1), (4.2) and the strong Markov property for $\{Z_t\}$, for any $n \geqq 0$, $x, y \in E$ and $t, u, w \geqq 0$

$$\begin{aligned} P[\varphi(T_n + t) - \varphi(T_n) \leqq u \mid \mathscr{G}_n, Y_{n+1}, W_{n+1}] &= P[\varphi_{T_n+t}^{T_n} \leqq u \mid \mathscr{G}_n] \\ &= P[\varphi_t \circ \theta_{T_n} \leqq u \mid \mathscr{F}_{T_n}] \end{aligned}$$

where the latter is a $\sigma(Y_0, Y_1, W_1)$-measurable function for each u.

The RTT's determined by functionals of this sort are analogous to the RTT's described in Dynkin [3].

An example of the above type of functional is given by

$$\varphi_t^s = \int_s^t f(X_v, X_v^+, v - U_v, U_v^+ - v)\, dv,$$

where $f(x, y, u, v)$ is a nonnegative real-valued measurable function defined for each $x, y \in E$ and $u, v \geqq 0$, and satisfies

$$(4.3) \qquad \int_0^t f(x, y, u, t-u)\, du < \infty, \qquad \text{for each } t \geqq 0.$$

That this functional is a.s. finite valued follows since for each t,

$$\varphi_t \leqq \sum_{k=1}^{N_t+1} \varphi_{T_k}^{T_{k-1}} < \infty \quad \text{a.s.},$$

the above being true since $N_t < \infty$ a.s., and by (4.3) and the fact that $W_k < \infty$ a.s.,

$$\varphi_{T_k}^{T_{k-1}} = \int_{T_{k-1}}^{T_{k-1}+W_k} f(Y_{k-1}, Y_k, u - T_{k-1}, T_k - u)\, du < \infty \quad \text{a.s.}$$

Random functions of $\{X_t\}$. Let $\{V_n(x, y): n \geqq 0, x, y \in E\}$ be a family of independent random variables which is independent of $\{Y_n, W_n\}$ and satisfies:

(i) $(x, y, \omega) \to V_n(x, y, \omega)$ is $\mathscr{E} \times \mathscr{E} \times \mathscr{F}$-measurable for each n.
(ii) For each $x, y \in E$, $V_0(x, y) = 0$ a.s. and

$$P[V_n(x, y) \leqq u] = P[V_1(x, y) \leqq u] \qquad \text{for } n \geqq 1.$$

Set $V_t = V_{n+1}(X_n, X_{n+1})$ if $T_n \leqq t < T_{n+1}$. It can be shown that $Z_t = (X_t, X_t^+, U_t, U_t^+, V_t)$, $t \geqq 0$ is a regular step Markov process. Assume that for each $t \geqq 0$

$$\mathscr{F}_t = \sigma(Z_s : s \leqq t).$$

Let $\{\varphi_t^s : 0 \leqq s \leqq t\}$ be a nonnegative, continuous additive functional of $\{Z_t\}$ satisfying (4.2) and set $\varphi_t = \varphi_t^0$ for each $t \geqq 0$. As above it can be shown that $\{\varphi_t\}$ (called a random function of $\{X_t\}$) satisfies (A1) and (A2).

The following special case of the above determines a RTT that transforms SMP's into regular step Markov processes. Suppose that the SMP $\{X_t\}$ has a kernel Q which satisfies $Q(x, \{x\} \times R_+) = 0$ for each $x \in E$. Assume that

$$\begin{aligned} P[V_1(x, y) \leqq t] &= 1 - e^{-\lambda(x)t} && \text{for } t \geqq 0, \\ &= 0 && \text{otherwise;} \end{aligned}$$

where λ is a positive $\mathscr{E}$-measurable function. For each $t \geqq 0$ set $\rho_t = V_t / W_{N_t+1}$ and $\varphi_t = \int_0^t \rho_u\, du$. This random function determines a RTT of $\{X_t\}$ such that, according to Remark 3, $\{X_t'\}$ is an SMP induced by the MRP $\{Y_n', W_n'\}$ (in this case $\{Y_n, V_n(X_{n-1}, X_n)\}$) which has kernel

$$Q'(x, A \times [0, t]) = K(x, A)(1 - e^{-\lambda(x)t}).$$

That is, $\{X_t'\}$ is a regular step Markov process. This type of RTT is presented in Yackel [15]. He uses it to show that an SMP with a countable state space (and with some instantaneous states) can be transformed into a Markov process having the same succession of states, and the same instantaneous states as the original SMP.

Random functions based on MRP'*s imbedded in* $\{Y_n, W_n\}$. As in Remark 4 we let $\{\alpha_n\}$ be a sequence of $\mathscr{G}_n$-stopping times satisfying (3.8) and let $\{\tilde{Y}_n, \tilde{W}_n\}$ be the

MRP with respect to $\tilde{\mathscr{G}}_n$ defined by (3.9) and (3.10). Setting $\tilde{X}_t = \tilde{Y}_n$ if $\tilde{T}_n \leqq t < \tilde{T}_{n+1}$, it follows that $\{\tilde{X}_t\}$ is an SMP with respect to $\{\mathscr{F}_t\}$ and induced by $\{\tilde{Y}_n, \tilde{W}_n\}$. Let $\{\tilde{X}_t, \tilde{X}_t^+, \tilde{U}_t, \tilde{U}_t^+\}$ be the Markov process associated with $\{\tilde{X}_t\}$ as in Remark 2. Let $\{V_n(x,y): n \geqq 0, x, y \in E\}$ be a family of random variables as above and set

$$\tilde{V}_t = V_{n+1}(\tilde{Y}_n, \tilde{Y}_{n+1}) \qquad \text{if } \tilde{T}_n \leqq t < \tilde{T}_{n+1}.$$

For each $t \geqq 0$ set

$$Z_t = (X_t, X_t^+, U_t, U_t^+, \tilde{X}_t, \tilde{X}_t^+, \tilde{U}_t, \tilde{U}_t^+, \tilde{V}_t)$$

and $\mathscr{F}_t = \sigma(Z_s: s \leqq t)$. It can be shown that $\{Z_t\}$ is a regular step Markov process.

Let $\{\varphi_t^s: 0 \leqq s \leqq t\}$ be a nonnegative continuous additive functional of $\{Z_t\}$, which in addition satisfies:

(i) For each $n \geqq 0$

$$\varphi_{T_{n+1}}^{T\alpha_{n+1}} = 0 \quad \text{a.s.} \tag{4.4}$$

(ii) For each $n \geqq 0$ and $0 \leqq s \leqq t$

$$\varphi_t^s \circ \mathcal{O}_{T_n} = \varphi_{T_n+t}^{T_n+s} \tag{4.5}$$

where $\{\mathcal{O}_t\}$ is the family of translation operators of $\{Z_t\}$.

Then define $\{\varphi_t\}$ by setting $\varphi_t = \varphi_t^0$ for each t. This process satisfies (A1) and it satisfies (A2′) since (3.9) follows from (4.4), and by (4.5) and the strong Markov property of $\{Z_t\}$, for any $n \geqq 0$, $x, y \in E$ and $t, u, w \geqq 0$.

$$P[\varphi(\tilde{T}_n + t) - \varphi(\tilde{T}_n) \leqq u \mid \tilde{\mathscr{G}}_n, \tilde{Y}_{n+1}, \tilde{W}_{n+1}] = P[\varphi_t \circ \theta_{T_n} \leqq u \mid \mathscr{F}_{T_n}]$$

where the latter is a $\sigma(Y_0, Y_1, W_1)$-measurable function for each u.

This type of random function can be used as follows. Given any Markov chain $\{\tilde{Y}_n\}$ (as above) imbedded in $\{Y_n\}$ and any family of distribution functions $\{G_{xy}(t): x, y \in E\}$, where $G_{xy}(0) = 0$ and $(x, y) \to G_{xy}(t)$ is $\mathscr{E} \times \mathscr{E}$-measurable for each t, one can construct an RTT of $\{X_t\}$ such that $\{X_t'\}$ is an SMP whose succession of states $\{Y_n'\}$ is given by $\{\tilde{Y}_n\}$, and whose sojourn times $\{W_n'\}$ satisfy

$$P[W'_{n+1} \leqq t \mid Y'_{n+1} = x, Y_n' = y] = G_{xy}(t).$$

A RTT of this type is determined, according to the above and Remark 3 and Remark 4, by the process $\varphi_t = \int_0^t \rho_u \, du$ where

$$\begin{aligned} \rho_t &= V_{n+1}(\tilde{Y}_n, \tilde{Y}_{n+1})/\tilde{W}'_{n+1} && \text{if } T_{\alpha_n} \leqq t < T_{\alpha_n+1} \\ &= 0 && \text{otherwise;} \end{aligned}$$

and where $P[V_1(x,y) \leqq t] = G_{xy}(t)$.

Processes $\{\varphi_t\}$ *independent of* $\{X_t\}$. Let $\{\varphi_t\}$ be a non-decreasing right continuous process (with $\varphi(0) = 0$) taking values in $(R_+, \mathscr{B}_+)$, adapted to $\{\mathscr{F}_t\}$, and which is continuous in probability and is independent of $\{Y_n, W_n\}$. Assume that $\{\varphi_t\}$ has stationary independent nonnegative increments with respect to $\{\mathscr{F}_t\}$, that is for any $s, t \geqq 0$, $\sigma(\varphi(t+s)-\varphi(s))$ is independent of $\mathscr{F}_s$ and

$$P[\varphi(t-s)-\varphi(s) \leqq u \mid \mathscr{F}_s] = P[\varphi(t) \leqq u].$$

Since $\{\varphi_t\}$ is continuous in probability, for any $t \geqq 0$

$$P[\varphi(t)-\varphi(t^-) > 0] \leqq P[\varphi(t)-\varphi(t-n^{-1}) > 0] \to 0 \qquad \text{as } n \to \infty,$$

and since $\{\varphi_t\}$ is independent of $\{Y_n, W_n\}$

$$P[\varphi(T_n)-\varphi(T_n^-) > 0 \mid Y_0 = x] = \int_{-\infty}^{\infty} P[\varphi(t)-\varphi(t^-) > 0] Q^n(x, E \times dt) = 0.$$

Thus (A1) is satisfied.

Assumption (A2) is also satisfied for it can be shown (see [1]) that if τ is any finite $\mathscr{F}_t$-stopping time, setting

$$\tilde{\varphi}_t = \varphi(t+\tau)-\varphi(\tau) \quad \text{and} \quad \tilde{\mathscr{G}}_t = \sigma(\tilde{\varphi}_s : s \leqq t) \qquad \text{for each } t \geqq 0,$$

the σ-algebra $\tilde{\mathscr{G}}_t$ is independent of $\mathscr{F}_t$ and

$$P[\tilde{\varphi}(t) \leqq u \mid \mathscr{F}_\tau] = P[\varphi(t) \leqq u] \quad \text{a.s.} \tag{4.6}$$

Then (A2) follows from (4.6) and the independence of $\{\varphi_t\}$ and $\{Y_n, W_n\}$.

RTT's determined by this class of processes are analogous to the RTT's of Markov Processes as presented in Feller [4] (which we noted in the introduction). For his RTT's, $\{\tau_t\}$ is a process with stationary independent nonnegative increments, which is independent of $\{X_t\}$; while in this case $\{\tau_t\}$ is the inverse of such a process.

5. Some other random time transformations. In this section we take $\{\tau_t\}$ to be a step function of the form

$$\tau_t = S_n \qquad \text{if } Z_n \leqq t < Z_{n+1}, \tag{5.1}$$

where $\{S_n\}$ and $\{Z_n\}$ are sequences of strictly increasing random variables such that $S_0 = Z_0 = 0$ a.s. The inverse $\{\varphi_t\}$ of $\{\tau_t\}$ obviously satisfies $\varphi_t = Z_{n+1}$ if $S_n \leqq t < S_{n+1}$. Assume the following:

(i) $\{S_n\}$ is independent of $\{Z_n\}$.
(ii) $\{\tau_t\}$ is independent of $\{Y_n, W_n\}$.
(iii) $\{\varphi_t\}$ is adapted to $\{\mathscr{F}_t\}$.
(iv) There exist distribution functions G and H such that for any n

$$P[S_{n+1}-S_n \leqq t \mid \mathscr{F}_{S_n}] = G(t)$$

and

$$P[Z_{n+2}-Z_{n+1} \leqq t \mid \mathscr{F}_{S_n}] = H(t).$$

(Note that each S_n is an $\mathscr{F}_t$-stopping time, since these are the jump points of $\{\varphi_t\}$ which is adapted to $\{\mathscr{F}_t\}$.)

Let $\{X_t'\}$ be the RTT of $\{X_t\}$ by the change of time $\{\tau_t\}$. Then we have the following.

THEOREM 2. *Suppose that* $\{X_t\}$ *is a regular step Markov process with respect to* $\{\mathscr{F}_t\}$. *Then* $\{X_t'\}$ *is an* SMP *with respect to* $\{\mathscr{F}_t'\}$, *which is induced by the* MRP $\{X(S_n), Z_{n+1}: n \geqq 0\}$ *with kernel* $\tilde{Q}(x, A \times [0,t]) = \tilde{K}(x,A)H(t)$ *where*

$$\tilde{K}(x,A) = \textstyle\int_0^\infty P[X_u \in A \mid X_0 = x]G(du).$$

Hence, if $\{\tau_t\}$ *is a compound Poisson process (i.e.,* $H(t) = 1-e^{-\mu t}$ *for* $\mu > 0$*), then* $\{X_t'\}$ *is a regular step Markov process with respect to* $\{\mathscr{F}_t'\}$.

PROOF. Just as in the proof of Theorem 1 it follows that $\{X_t'\}$ is a right continuous process adapted to $\{\mathscr{F}_t'\}$, and directly from (5.1) we have $X_t' = X(S_n)$ if $Z_n \leqq t < Z_{n+1}$. The counting process of $\{X_t'\}$ is given by $N_t' = n$ if $Z_n \leqq t < Z_{n+1}$. Each Z_n is a $\mathscr{F}_t'$-stopping time since it is a jump point of the process $\{\tau_t\}$, which is adapted to $\{\mathscr{F}_t'\}$. Thus it follows that $\{N_t'\}$ is adapted to $\{\mathscr{F}_t'\}$.

Under Assumptions (i)–(iv) and the strong Markov property for $\{X_t\}$ it follows that for each $n \geqq 0$, $x \in E$, $A \in \mathscr{E}$ and $t \geqq 0$

$$\begin{aligned} &P[X(S_{n+1}) \in A, Z_{n+2} - Z_{n+1} \leqq t \mid \mathscr{F}_{S_n}, X(S_n) = x] \\ &\quad = P[X(S_{n+1}) \in A \mid \mathscr{F}_{S_n}, X(S_n) = x] P[Z_{n+2} - Z_{n+1} \leqq t] \\ &\quad = \tilde{K}(x,A)H(t). \end{aligned}$$

Thus $\{X(S_n), Z_{n+1}\}$ is an MRP with respect to $\mathscr{F}_{S_n}$ and this proves the first statement of the theorem. The second statement follows from Remark 1. □

Note that the second statement in the above theorem is a special case of the above mentioned result in Feller. It can be shown (see [13]) that the result in Feller for Markov processes does not hold for SMP's. However, we do have the following special case.

THEOREM 3. *Suppose that the jump points of the* SMP $\{X_t\}$ *occur* a.s. *at integer time points, i.e., each* $F_{xy}(t)$ *is an arithmetic distribution with an integer valued span. Let* (τ_t) *be a Poisson process where, using the above notation, for each* $t \geqq 0$, $\tau_t = n$ *if* $Z_n \leqq t < Z_{n+1}$ *and*

$$P[Z_1 \leqq t] = 1 - e^{-\lambda t} \qquad \textit{for some } \lambda > 0.$$

Set $\tilde{Z}_n = Z_{T_n}$ *for each n. Then* $\{X_t'\}$ *is an* SMP *with respect to* $\{\mathscr{F}_t'\}$ *and is induced by the* MRP $\{Y_n, \tilde{Z}_n\}$ *which has the kernel*

$$\tilde{Q}(x, A \times [0,t]) = \textstyle\sum_{k=1}^\infty Q(x, A \times \{k\}) H_{\lambda,k}(t)$$

where $H_{\lambda,k}$ *is a gamma distribution with parameter* λ *and order k.*

PROOF. Just as in the proof of Theorem 1 it follows that $\{X_t'\}$ is a right continuous process adapted to $\{\mathscr{F}_t'\}$, and that

$$X_t' = Y_n \qquad \text{if } T_n \leqq \tau_t < T_{n+1}. \tag{5.2}$$

Under the hypotheses it follows that

$$\begin{aligned}\{T_n \leqq \tau_t < T_{n+1}\} &\\ &= \{T_n = \tau_t\} \cup \{T_n + 1 = \tau_t\} \cup \cdots \cup \{T_{n+1} - 1 = \tau_t\}\\ &= \{Z_{T_n} \leqq t < Z_{T_n+1}\} \cup \{Z_{T_n+1} \leqq t < Z_{T_n+2}\} \cup \cdots \cup \{Z_{T_{n+1}-1} \leqq t < Z_{T_{n+1}}\}\\ &= \{\tilde{Z}_n \leqq t < Z_{n+1}\}.\end{aligned}$$

Thus (5.2) is equivalent to $X_t' = Y_n$ if $\tilde{Z}_n \leqq t < \tilde{Z}_{n+1}$.

Just as in the proof of Theorem 2, it follows that the counting process associated with $\{X_t'\}$ is adapted to $\{\mathscr{F}_t'\}$. In addition, each $\tilde{Z}_n$ is a $\mathscr{F}_t$-stopping time. Thus $\{Y_n, \tilde{Z}_n\}$ is an MRP with respect to $\{\mathscr{F}_{\tilde{Z}_n}\}$, since for any $x \in E$, $A \in \mathscr{E}$ and $t \geqq 0$

$$\begin{aligned}P[Y_{n+1} \in A, \tilde{Z}_{n+1} - \tilde{Z}_n \leqq t \mid \mathscr{F}_{\tilde{Z}_n}, Y_n = x] &\\ &= \textstyle\sum_{k=1}^{\infty} P[Y_{n+1} \in A, T_{n+1} - T_n = k, Z_{T_n+k} - Z_{T_n} \leqq t \mid \mathscr{F}_{\tilde{Z}_n}, Y_n = x]\\ &= \textstyle\sum_{k=1}^{\infty} Q(x, A \times \{k\}) P[Z_k \leqq t]\\ &= \textstyle\sum_{k=1}^{\infty} Q(x, A \times \{k\}) H_{\lambda,t}(t).\end{aligned}$$

This completes the proof. ▯

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