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ON PROBABILITIES OF RECTANGLES IN MULTIVARIATE STUDENT DISTRIBUTIONS: THEIR DEPENDENCE ON CORRELATIONS

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0. Summary. Recently several authors (cf. [5], [6], [8], [9]) have established for arbitrary positive numbers $c_1, \cdots, c_k$ the inequality

$$P\{|X_1| \leqq c_1, \cdots, |X_k| \leqq c_k\} \geqq \prod_{i=1}^{k} P\{|X_i| \leqq c_i\} \tag{1}$$

valid for a random vector $X = (X_1, \cdots, X_k)$ having a multivariate normal distribution with mean values 0 and with an arbitrary covariance matrix. A question then arises whether also an analogue to (1) for multivariate Student distributions holds true, i.e. the inequality

$$P\{|X_1|/S_1 \leqq c_1, \cdots, |X_k|/S_k \leqq c_k\} \geqq \prod_{i=1}^{k} P\{|X_i|/S_i \leqq c_i\} \tag{2}$$

where $X = (X_1, \cdots, X_k)$ is as before, while $S_i = (\sum_{v=1}^{p} Z_{iv}^2)^{\frac{1}{2}}$, $i = 1, \cdots, k$, where $Z_v = (Z_{1v}, \cdots, Z_{kv})$, $v = 1, \cdots, p$, is a random sample of p vectors, which are mutually independent and independent of X, and each of which has, in the simplest case, the same normal distribution as X. More generally, the Z_v's have some normal distributions with mean values 0 and with some covariance matrices which need not coincide with that of X and even need not be identical.

A certain proof of (2) was presented by A. Scott [6] but we shall give here a counterexample showing that, unfortunately, this proof is incorrect. However, if the correlations between X_i and X_j have the form $\lambda_i \lambda_j \rho_{ij}$ $(i, j = 1, \cdots, k; i \neq j)$ where $|\lambda_i| \leqq 1$ $(i = 1, \cdots, k)$ and where $\{\rho_{ij}\}$ is any fixed correlation matrix, and if the correlations between Z_{iv} and Z_{jv} have the form $\tau_{iv}\tau_{jv}$ $(i, j = 1, \cdots, k;\ i \neq j;$ $v = 1, \cdots, p)$ where $|\tau_{iv}| < 1 (i = 1, \cdots, k; v = 1, \cdots, p)$, we shall prove here that the left-hand side probability in (2) is a non-decreasing function of each $|\lambda_i|$ and each $|\tau_{iv}|$; therefore, in this case of a special correlation structure, (2) is indeed true. The general validity of (2) still remains an open question.

1. Bibliographical remarks and a counterexample. Let us begin by a few remarks on related investigations. (Here, and throughout the whole paper, $\rho(X_i, X_j)$ will denote the correlation between the variables X_i and X_j, etc.)

The proof of the inequality (1) for normal random variables was originally given by O. J. Dunn [2] only for cases where $k = 2$ or $k = 3$ or $\rho(X_i, X_j) = \lambda_i \lambda_j$ $(i, j = 1, \cdots, k; i \neq j)$ with $0 \leqq \lambda_i \leqq 1$ $(i = 1, \cdots, k)$. The general validity of (1) for the case of an arbitrary covariance matrix was established almost simultaneously, but by different methods, by three authors: A. Scott [6], the present author (the result announced in [8], the proof given in [9]), and C. G. Khatri [5]; to be more

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precise, the last author did not mention the inequality (1) explicitly, but his main result was

$$P\{(X_1, \cdots, X_m) \in D_1, (X_{m+1}, \cdots, X_k) \in D_2\} \geqq P\{(X_1, \cdots, X_m) \in D_1\} P\{(X_{m+1}, \cdots, X_k) \in D_2\} \tag{3}$$

for any convex regions D_1, D_2 symmetric about the origin, so that (1) is an immediate consequence of his inequality (3). (Cf. also Corollary 4, Corollary 5, and Corollary 5′ in [5] giving the inequality (1) as an easy specialization.) Still a different proof of (1) follows from a more general assertion presented in [11]: If X is as above, and $\rho(X_i, X_j) = \lambda_i \lambda_j \rho_{ij}$ $(i, j = 1, \cdots, k; i \neq j)$ where $0 \leqq \lambda_i \leqq 1$ $(i = 1, \cdots, k)$, and $\{\rho_{ij}\}$ is any correlation matrix, then the left-hand side in (1) is a non-decreasing function of each λ_i.

The case of multivariate "Student" variables defined in a different and simpler way, namely, the case of $X_1/S, \cdots, X_k/S$ (where all X_i's have the same variance σ^2, and S is a common single estimate of σ) was studied simultaneously. The relevant analogue of the inequality (2) was given also in [2] for the same special cases as mentioned above, and in [8], [9] for the general case; moreover, it is sufficient to suppose here only that S is any positive random variable independent of $X = (X_1, \cdots, X_k)$.

Coming to the subject of the present paper, we will now concentrate on the inequality (2). Certain special cases of this inequality were proved by the following four authors, again almost simultaneously.

M. Halperin [4] proved (2) for the case $k = 2$, but assuming completely generally that the correlations $\rho(Z_{1\nu}, Z_{2\nu})$ may be different for different ν's. For a general dimension $k \geqq 2$, but for a special correlation structure of Z_ν's, the present author [10] established (though by a somewhat lengthy method) the following result: If X and Z_ν $(\nu = 1, \cdots, p)$ have normal distributions (possibly different) described in the Summary, where the covariance matrix of X is arbitrary, the variances of all $Z_{i\nu}$'s are 1, and $\rho(Z_{i\nu}, Z_{j\nu}) = \tau_i \tau_j$ $(i, j = 1, \cdots, k; i \neq j; \nu = 1, \cdots, p)$ with $0 \leqq \tau_i \leqq 1$ $(i = 1, \cdots, k)$, then (2) is true. Essentially the same result, assuming more generally only $|\tau_i| \leqq 1$, follows by an easy specialization from C. G. Khatri [5], Corollary 8. (The precise contents of this Corollary lies somewhat far from our present interest.) These results will be further slightly generalized in Corollary 1 of the present paper.

A certain general proof of the inequality (2) under the assumption that X and all Z_ν's have the same normal distribution with mean values 0 and an arbitrary covariance matrix was presented by A. Scott [6]. (Precisely, he states explicitly only a more special inequality, in which the X_i's are the averages and the S_i's are the empirical standard deviations in a multivariate normal random sample.) Unfortunately, this proof is correct only for $k = 2$ but incorrect for $k > 2$. (This fact is also stated in the Correction to [6].) As a matter of fact, Scott's proof uses in an essential way Lemma 2 in [6] asserting (in a different notation) that

$$P\{|Z_1| \geqq d_1, \cdots, |Z_k| \geqq d_k\} \geqq \textstyle\prod_{i=1}^k P\{|Z_i| \geqq d_i\} \tag{4}$$

for any positive numbers $d_1, \cdots, d_k$, whenever $Z_1, \cdots, Z_k$ have a k-variate normal distribution with mean values 0 and with an arbitrary covariance matrix. However, the proof of (4) was given in [6] only for $k = 2$, whereas it was skipped for $k > 2$, and we are now going to show by a counterexample that, in fact, the assertion (4) generally fails to hold in this latter case. (Note, of course, that this counterexample shows only that Scott's method of proving (2) fails but it does not say anything about the validity of (2) itself.)

The inequality (4) does hold for $k \geqq 2$ provided that $\rho(Z_i, Z_j) = \tau_i \tau_j$ $(i, j = 1, \cdots, k; i \neq j)$ with $|\tau_i| \leqq 1$ $(i = 1, \cdots, k)$, as was shown by C. G. Khatri [5], Theorem 2. Moreover, Khatri in a Remark on page 1864 expressed some hope that (4) might be true for any covariance matrix; however, our counterexample shows that this hope was in vain.

In the sequel, we write $\varphi(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)$, $\Phi(x) = \int_{-\infty}^{x} \varphi(u)\, du$, and by an $N(0, 1)$ variable we understand the random variable with the density $\varphi(x)$. For the counterexample, we shall need the following

LEMMA 1. $\Phi(-2^{\frac{1}{2}}x) > 2\Phi^2(-x)$ *for all* $x > 0$.

PROOF. Writing $\psi(x) = \Phi(-2^{\frac{1}{2}}x) - 2\Phi^2(-x)$, we have

$$\psi'(x) = 2\pi^{\frac{1}{2}}\varphi(-x)[2\pi^{-\frac{1}{2}}\Phi(-x) - \varphi(-x)].$$

Put now $\omega(x) = 2\pi^{-\frac{1}{2}}\Phi(-x) - \varphi(-x)$. We get then $\omega'(x) = \varphi(-x)[-2\pi^{-\frac{1}{2}} + x]$, so that $\omega'(x) = 0$ at exactly one point $x = 2\pi^{-\frac{1}{2}}$. Since $\omega''(2\pi^{-\frac{1}{2}}) > 0$, the function $\omega(x)$ has a local minimum at $x = 2\pi^{-\frac{1}{2}}$, and has no other extremes for $x > 0$. Taking further into account that $\omega(0) > 0$, $\omega'(0) < 0$, and $\lim_{x \to \infty} \omega(x) = 0$, we see that $\omega(x) = 0$ has exactly one root for $x > 0$. Hence also $\psi'(x) = 0$ for exactly one $x > 0$, and the function $\psi(x)$ has at most one local extreme for $x > 0$. Since, moreover, $\psi(0) = 0$, $\psi'(0) > 0$, and $\lim_{x \to \infty} \psi(x) = 0$, we can conclude that $\psi(x) > 0$ for all $x > 0$, which proves Lemma 1.

The following counterexample now contradicts the inequality (4).

EXAMPLE. Let Z_1, Z_2 be two independent $N(0, 1)$ variables, and let $Z_3 = Z_1/2^{\frac{1}{2}} + Z_2/2^{\frac{1}{2}}$. Then, for any $d > 0$ and for any sufficiently small $d_3 > 0$, we have

$$P\{|Z_1| \geqq d, |Z_2| \geqq d, |Z_3| \geqq d_3\} < P\{|Z_1| \geqq d\}P\{|Z_2| \geqq d\}P\{|Z_3| \geqq d_3\}. \tag{5}$$

PROOF. We shall investigate the difference

$$\begin{aligned} &P\{|Z_1| \geqq d\}P\{|Z_2| \geqq d\}P\{|Z_3| \geqq d_3\} - P\{|Z_1| \geqq d, |Z_2| \geqq d, |Z_3| \geqq d_3\} \\ &= -P\{|Z_1| \geqq d\}P\{|Z_2| \geqq d\}P\{|Z_3| < d_3\} + P\{|Z_1| \geqq d, |Z_2| \geqq d, |Z_3| < d_3\} \qquad (6) \\ &= P\{|Z_1| \geqq d, |Z_2| \geqq d, |Z_1/2^{\frac{1}{2}} + Z_2/2^{\frac{1}{2}}| < d_3\} - 8\Phi^2(-d)[\Phi(d_3) - \tfrac{1}{2}]. \end{aligned}$$

We introduce new random variables U_1, U_2 by $Z_1 = U_1/2^{\frac{1}{2}} - U_2/2^{\frac{1}{2}}$, $Z_2 = U_1/2^{\frac{1}{2}} + U_2/2^{\frac{1}{2}}$. Since this is an orthogonal transformation, U_1 and U_2 are again independent $N(0, 1)$ variables. Then (6) equals

$$\begin{aligned} & P\{|U_1-U_2| \geqq 2^{\frac{1}{2}}d, |U_1+U_2| \geqq 2^{\frac{1}{2}}d, |U_1| < d_3\} - 8\Phi^2(-d)[\Phi(d_3)-\tfrac{1}{2}] \\ (7) \qquad & \geqq P\{|U_1| < d_3, |U_2| \geqq 2^{\frac{1}{2}}d + d_3\} - 8\Phi^2(-d)[\Phi(d_3)-\tfrac{1}{2}] \\ & = 4[\Phi(d_3)-\tfrac{1}{2}]\Phi(-2^{\frac{1}{2}}d-d_3) - 8\Phi^2(-d)[\Phi(d_3)-\tfrac{1}{2}] \\ & = 4[\Phi(d_3)-\tfrac{1}{2}][\Phi(-2^{\frac{1}{2}}d-d_3)-2\Phi^2(-d)]). \end{aligned}$$

However, by Lemma 1, this last expression is positive for any $d > 0$ and for sufficiently small $d_3 > 0$, and thus (5) is proved.

This is an example of a singular distribution for which (5) holds. However, since the inequality in (5) is strict, and since a singular distribution may be approximated by some nonsingular ones (cf. H. Cramér [1], Section 24.3), there exists also a triple Z_1, Z_2, Z_3 with a nonsingular normal distribution for which (5) is true.

2. A stronger positive result for special correlations. In this section we shall prove a positive result, which is stronger than the inequality (2) but which is, unfortunately, restricted only to cases of a special correlation structure of Z_{iv}'s. First, we prove a Lemma, generalizing Theorem 2 in C. G. Khatri [5] (which was also mentioned here just before our Lemma 1).

LEMMA 2. *If the random vector $Z = (Z_1, \cdots, Z_k)$ has a normal distribution with mean values* 0, *arbitrary variances, and $\rho(Z_i, Z_j) = \tau_i\tau_j$ $(i, j = 1, \cdots, k; i \neq j)$ where $|\tau_i| < 1$ $(i = 1, \cdots, k)$, then, for any positive numbers $d_1, \cdots, d_k$, the probability*

$$(8) \qquad P\{|Z_1| \geqq d_1, \cdots, |Z_k| \geqq d_k\}$$

as a function of τ_i $(i = 1, \cdots, k)$ is non-increasing for $-1 < \tau_i < 0$ and non-decreasing for $0 < \tau_i < 1$, so that it has a minimum for $\tau_i = 0$.

The proof follows the same lines as that of Theorem 1 in [7]. First, let us prove Lemma 2 for the case where all τ_i's are nonnegative. Clearly, we may suppose that all variances are equal to 1, and prove the assertion only for τ_1 satisfying $0 < \tau_1 \leqq C$, where C is a fixed constant, $C < 1$. Further, we may employ the device by C. W. Dunnett and M. Sobel [3] supposing that each Z_i has the form $Z_i = (1-\tau_i^2)^{\frac{1}{2}}Y_i - \tau_i Y_0$ $(i = 1, \cdots, k)$ where $Y_0, Y_1, \cdots, Y_k$ are independent $N(0, 1)$ variables. Then, writing simply $P(\tau_1)$ for the probability (8), we get

$$\begin{aligned} P(\tau_1) &= P\{(1-\tau_i^2)^{\frac{1}{2}}Y_i - \tau_i Y_0 \geqq d_i \text{ or } (1-\tau_i^2)^{\frac{1}{2}}Y_i - \tau_i Y_0 \leqq -d_i, i = 1, \cdots, k\} \\ (9) \qquad &= P\{Y_i \geqq (\tau_i Y_0 + d_i)(1-\tau_i^2)^{-\frac{1}{2}} \text{ or } Y_i \leqq (\tau_i Y_0 - d_i)(1-\tau_i^2)^{-\frac{1}{2}}, i = 1, \cdots, k\} \\ &= \int_{-\infty}^{\infty} \varphi(y) \prod_{i=1}^{k} [1 - \Phi((\tau_i y + d_i)(1-\tau_i^2)^{-\frac{1}{2}}) + \Phi((\tau_i y - d_i)(1-\tau_i^2)^{-\frac{1}{2}})]\, dy. \end{aligned}$$

It is sufficient to prove that $\partial P(\tau_1)/\partial\tau_1 \geqq 0$. However, it is easy to see that, in view of the assumption $0 < \tau_1 \leqq C < 1$, differentiation under the sign of integral in (9) is justified. Thus, after differentiation in (9) and after some manipulation with the functions φ in the resulting expression, we obtain

$$(10)\quad \begin{aligned} &\partial P(\tau_1)/\partial\tau_1 \\ &= \varphi(d_1)\textstyle\int_{-\infty}^{\infty}\varphi((y-d_1\tau_1)(1-\tau_1^2)^{-\frac{1}{2}})(y-d_1\tau_1)(1-\tau_1^2)^{-\frac{3}{2}} \\ &\quad\cdot\textstyle\prod_{i=2}^{k}[1-\Phi((\tau_i y+d_i)(1-\tau_i^2)^{-\frac{1}{2}})+\Phi((\tau_i y-d_i)(1-\tau_i^2)^{-\frac{1}{2}})]\,dy \\ &\quad-\varphi(d_1)\textstyle\int_{-\infty}^{\infty}\varphi((y+d_1\tau_1)(1-\tau_1^2)^{-\frac{1}{2}})(y+d_1\tau_1)(1-\tau_1^2)^{-\frac{3}{2}} \\ &\quad\cdot\textstyle\prod_{i=2}^{k}[1-\Phi((\tau_i y+d_i)(1-\tau_i^2)^{-\frac{1}{2}})+\Phi((\tau_i y-d_i)(1-\tau_i^2)^{-\frac{1}{2}})]\,dy\cdot \end{aligned}$$

Substituting now $z=(y-d_1\tau_1)(1-\tau_1^2)^{-\frac{1}{2}}$ in the first integral in (10), $z=(y+d_1\tau_1)(1-\tau_1^2)^{-\frac{1}{2}}$ in the second integral in (10), and writing then briefly $p_i=d_i(1-\tau_i^2)^{-\frac{1}{2}}, q_i=d_1\tau_1\tau_i(1-\tau_i^2)^{-\frac{1}{2}}, r_i=\tau_i(1-\tau_1^2)^{\frac{1}{2}}(1-\tau_i^2)^{-\frac{1}{2}}, A=\varphi(d_1)(1-\tau_1^2)^{-\frac{1}{2}}$, we get

$$(11)\quad \begin{aligned} \partial P(\tau_1)/\partial\tau_1 &= A\textstyle\int_{-\infty}^{\infty}\varphi(z)z\prod_{i=2}^{k}[1-\Phi(zr_i+q_i+p_i)+\Phi(zr_i+q_i-p_i)]\,dz \\ &\quad -A\textstyle\int_{-\infty}^{\infty}\varphi(z)z\prod_{i=2}^{k}[1-\Phi(zr_i-q_i+p_i)+\Phi(zr_i-q_i-p_i)]\,dz \\ &= A\textstyle\int_{-\infty}^{\infty}\varphi(z)z\{\prod_{i=2}^{k}[1-\Phi(zr_i+q_i+p_i)+\Phi(zr_i+q_i-p_i)] \\ &\quad -\textstyle\prod_{i=2}^{k}[1-\Phi(zr_i-q_i+p_i)+\Phi(zr_i-q_i-p_i)]\}\,dz. \end{aligned}$$

Further, split the last integral $\int_{-\infty}^{\infty}\cdots$ in (11) into the sum of two integrals $\int_0^{\infty}\cdots$ and $\int_{-\infty}^{0}\cdots$, denoting them by I_1 and I_2, respectively. Beginning by the integral I_1, observe that we have here $zr_i \geqq 0$, $q_i \geqq 0$, $p_i > 0$. Hence $|zr_i-q_i| \leqq zr_i+q_i$, which evidently implies

$$\Phi(zr_i-q_i+p_i)-\Phi(zr_i-q_i-p_i) \geqq \Phi(zr_i+q_i+p_i)-\Phi(zr_i+q_i-p_i).$$

Therefore

$$\begin{aligned} &\textstyle\prod_{i=2}^{k}[1-\Phi(zr_i-q_i+p_i)+\Phi(zr_i-q_i-p_i)] \\ &\quad \leqq \textstyle\prod_{i=2}^{k}[1-\Phi(zr_i+q_i+p_i)+\Phi(zr_i+q_i-p_i)] \end{aligned}$$

so that clearly $I_1 \geqq 0$. In an analogous manner we find also $I_2 \geqq 0$, and the proof of Lemma 2 for nonnegative τ_i's is finished.

If some τ_i's are negative, it suffices to use the assertion just proved for the variables $Z_1^*, \cdots, Z_k^*$ defined so that $Z_i^* = Z_i$ whenever $\tau_i \geqq 0$, $Z_i^* = -Z_i$ whenever $\tau_i < 0$.

The following main Theorem of this paper is a partial "Student" analogue to Theorem 1, or rather to Corollary 1, in [11].

THEOREM. *Let the random vector $X=(X_1, \cdots, X_k)$ have a normal distribution with mean values* 0, *arbitrary variances, and $\rho(X_i, X_j)=\lambda_i\lambda_j\rho_{ij}$ $(i,j=1,\cdots,k;\ i\neq j)$ where $|\lambda_i| \leqq 1$ $(i=1,\cdots,k)$ and where $\{\rho_{ij}\}$ is some fixed correlation matrix. Further, let $Z_\nu=(Z_{1\nu},\cdots,Z_{k\nu})$, $\nu=1,\cdots,p$, be a sample of p random vectors, which are mutually independent and also independent of X, and let Z_ν have a normal distribution with mean values* 0, *arbitrary variances, and $\rho(Z_{i\nu}, Z_{j\nu})=\tau_{i\nu}\tau_{j\nu}$*

$(i, j = 1, \cdots, k;\ i \neq j;\ v = 1, \cdots, p)$ where $|\tau_{iv}| < 1$ $(i = 1, \cdots, k;\ v = 1, \cdots, p)$. Put $S_i = (\sum_{v=1}^{p} Z_{iv}^2)^{\frac{1}{2}}$, $i = 1, \cdots, k$. Then, for any positive numbers $c_1, \cdots, c_k$, the probability

$$P\{|X_1|/S_1 \leqq c_1, \cdots, |X_k|/S_k \leqq c_k\} \tag{12}$$

as a function of λ_i $(i = 1, \cdots, k)$ is non-increasing for $-1 \leqq \lambda_i < 0$ and non-decreasing for $0 < \lambda_i \leqq 1$, so that it has a minimum for $\lambda_i = 0$, and, as a function of τ_{iv} $(i = 1, \cdots, k;\ v = 1, \cdots, p)$ is non-increasing for $-1 < \tau_{iv} < 0$ and non-decreasing for $0 < \tau_{iv} < 1$, so that it has a minimum for $\tau_{iv} = 0$.

PROOF. (I) First, let us prove the assertion for λ_i's in the case where all λ_i's are nonnegative. Clearly, it is sufficient to prove it for the conditional probability analogous to (12) for given $Z_{iv} = z_{iv}$ $(i = 1, \cdots, k;\ v = 1, \cdots, p)$. However, this immediately follows from Corollary 1 in [11] (quoted also in the present paper after formula (3)), and it then suffices to take expectations.

If some λ_i's are negative, we apply the assertion just proved for the vector $X^* = (X_1^*, \cdots, X_k^*)$ defined so that $X_i^* = X_i$ whenever $\lambda_i \geqq 0$, $X_i^* = -X_i$ whenever $\lambda_i < 0$.

(II) The proof of the assertion for τ_{iv}'s is based again on a conditional argument. It is sufficient to carry out the proof for $v = 1$ and for the conditional probability analogous to (12) for given $X_i = x_i$ $(i = 1, \cdots, k)$, $Z_{iv} = z_{iv}$ $(i = 1, \cdots, k;\ v = 2, \cdots, p)$. Now, this amounts to proving that the probability

$$P\{x_1^2 c_1^{-2} - \textstyle\sum_{v=2}^{p} z_{1v}^2 \leqq Z_{11}^2, \cdots, x_k^2 c_k^{-2} - \sum_{v=2}^{p} z_{kv}^2 \leqq Z_{k1}^2\}$$

as a function of τ_{i1} is non-increasing for $-1 < \tau_{i1} < 0$ and non-decreasing for $0 < \tau_{i1} < 1$, which is an obvious consequence of our Lemma 2.

COROLLARY 1. *If $X = (X_1 \cdots, X_k)$ has a normal distribution with mean values 0, arbitrary variances, and arbitrary correlations, and if $Z_v = (Z_{1v}, \cdots, Z_{kv})$, $v = 1, \cdots, p$, is a sample specified in the Theorem, then*

$$P\{|X_1|/S_1 \leqq c_1, \cdots, |X_k|/S_k \leqq c_k\} \geqq \textstyle\prod_{i=1}^{k} P\{|X_i|/S_i \leqq c_i\}.$$

This Corollary 1 shows the validity of the inequality (2) in certain special cases, and gives a generalization of the results proved by M. Halperin [4], the present author [10], and of the result obtained from C. G. Khatri [5], Corollary 8. (For details, cf. Section 1.)

COROLLARY 2. *Let $Y_v = (Y_{1v}, \cdots, Y_{kv})$, $v = 1, \cdots, n$, be a random sample of n independent vectors, each of which has the same normal distribution with mean values $\mu_1, \cdots, \mu_k$, variances $\sigma_1^2, \cdots, \sigma_k^2$, and $\rho(Y_{iv}, Y_{jv}) = \tau_i \tau_j$ $(i, j = 1, \cdots, k;\ i \neq j;\ v = 1, \cdots, n)$ where $|\tau_i| \leqq 1$ $(i = 1, \cdots, k)$. Putting $\bar{Y}_i = n^{-1} \sum_{v=1}^{n} Y_{iv}$, $s_i = [(n-1)^{-1} \sum_{v=1}^{n} (Y_{iv} - \bar{Y}_i)^2]^{\frac{1}{2}}$, $(i = 1, \cdots, k)$, we have*

$$P\{n^{\frac{1}{2}} |\bar{Y}_1 - \mu_1|/s_1 \leqq c_1, \cdots, n^{\frac{1}{2}} |\bar{Y}_k - \mu_k|/s_k \leqq c_k\} \geqq \textstyle\prod_{i=1}^{k} P\{n^{\frac{1}{2}} |\bar{Y}_i - \mu_i|/s_i \leqq c_i\}.$$

This last Corollary may be used in constructing conservative confidence rectangular regions for unknown mean values $\mu_1, \cdots, \mu_k$ in the case of unknown variances (for details cf. [2], [8], [9], [10]). As a matter of fact, this latter problem was the first stimulus to prove inequalities like (1) and (2).

Unfortunately, the results of the present paper assume a special correlation structure of the variables in the denominators of the "Student" variables in question. It would be much desirable to get rid of this restriction but this is still an open problem.

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