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ON THE CONSTRUCTION OF CERTAIN TRANSITION FUNCTIONS[1]

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0. Summary. We consider a probabilistic method for constructing certain transition functions which have a given conservative matrix as their initial derivative matrix. The technique originated in a conjecture of Reuter (see [9]). Kingman (see [5]) has considered similar problems but no proof of the original conjecture has appeared. The methods used are intended to confirm a remark made by Chung (see [1] page 158) to the effect that it should be possible to view sticky (regular) boundary points as a suitable limiting case of non-sticky (non-regular) boundary points. The probabilistic construction in Reuter's conjecture can be used to motivate some general analytical constructions of transition functions. These constructions and the connection with modern boundary theory are discussed in the last two sections.

1. Preliminaries. Let I be a denumerable set of indices. A transition function p is a nonnegative function on $[0, \infty)\times I\times I$ such that

$$\begin{aligned}(1.1)\qquad &\textstyle\sum_j p(t, i, j) \leqq 1,\\ &\textstyle\sum_k p(s, i, k)p(t, k, j) = p(s+t, i, j),\\ &\lim_{t\to 0} p(t, i, j) = p(0, i, j) = \delta_{ij}.\end{aligned}$$

We shall write $p(t, i, I)$ for the sum in (1.1). The function p will be called a stochastic transition function if $p(t, i, I) = 1$ for $i \in I$ and $t \geqq 0$, and will be called strictly substochastic otherwise. The reader is referred to [2] for the basic properties of transition functions.

If p is a transition function, then f is a p-entrance law if f is a nonnegative function on $(0, \infty)\times I$ such that

$$\textstyle\sum_k f(s, k)p(t, k, j) = f(s+t, j).$$

We shall write $f(t, I)$ for $\sum_j f(t, j)$. A p-entrance law f is said to be bounded if $\int_0^t f(s, I)\, ds < \infty$ for every $t > 0$ and is said to be normalized if

$$\lim_{t\to 0} f(t, I) = 1.$$

If f is a normalized p-entrance law and τ is a strictly positive (possibly infinite) random variable, then a family of random variables $\{x(t), 0 < t < \tau\}$ is called a

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continuous parameter, open Markov chain with state space I, transition function p, absolute distribution function f and lifetime τ if

$$(1.2) \qquad P\{x(t_1) = i_1, \cdots, x(t_n) = i_n, t_n < \tau\} = f(t_1, i_1)p(t_2 - t_1, i_1, i_2) \cdots p(t_n - t_{n-1}, i_{n-1}, i_n)$$

for $i_1, \cdots, i_n \in I$ and $0 < t_1 < \cdots < t_n$. If p is stochastic, then (since f is assumed normalized) $\tau = \infty$ almost everywhere.

If p is a transition function, then g is a p-exit law if g is a nonnegative function on $(0, \infty) \times I$ such that

$$\textstyle\sum_k p(s, i, k)g(t, k) = g(s+t, i).$$

A p-exit law is said to bounded if $\sup_i \int_0^t g(s, i)\, ds < \infty$ for every $t > 0$. For every transition function p, there is a unique bounded p-exit law g such that

$$(1.3) \qquad \textstyle\int_0^t g(s, i)\, ds = 1 - p(t, i, I)$$

(see [6] page 360). The reader is referred to [6] and [10] for a complete discussion of entrance and exit laws.

We shall be dealing with the Laplace transforms of transition functions, entrance laws and exit laws in what follows. For simplicity of notation, λ and μ are reserved for the (strictly positive) arguments of transformed functions. For example, $p(\lambda, i, j)$, $p(\lambda, i, I)$, $f(\lambda, j)$, $f(\lambda, I)$ and $g(\lambda, i)$ will represent the Laplace transforms (evaluated at λ) of the functions $p(\cdot, i, j)$, $p(\cdot, i, I)$, $f(\cdot, j)$, $f(\cdot, I)$ and $g(\cdot, i)$. Note that, if f and g are bounded, the last three transformed quantities above are finite.

For a p-entrance law f and a p-exit law g, define a nonnegative function $[f, g]$ on $(0, \infty)$ by the equation

$$[f, g](s+t) = \textstyle\sum_k f(s, k)g(t, k).$$

The function $[f, g]$ is easily shown to be well defined (possibly infinite).

PROPOSITION 1.1. *If f is a bounded p-entrance law and if g is the unique p-exit law defined by* (1.3), *then $\int_0^\infty (1 - e^{-\lambda t})[f, g](t)\, dt < \infty$ for $\lambda > 0$ and*

$$(1.4) \qquad \lim_{t \to \infty} f(t, I) + \textstyle\int_0^\infty (1 - e^{-\lambda t})[f, g](t)\, dt = \lambda f(\lambda, I).$$

In particular, if f is normalized, then

$$(1.5) \qquad \lim_{t \to \infty} f(t, I) + \textstyle\int_0^\infty [f, g](t)\, dt = 1$$

and the function $[f, g]$ is the probability density function of the life-time τ of a continuous parameter, open Markov chain with absolute distribution function f and transition function p.

PROOF. Observe that

$$\begin{aligned} \int_0^u [f, g](s+t)\,dt &= \textstyle\sum_k f(s, k)\int_0^u g(\ , k)\,dt \\ &= \textstyle\sum_k f(s, k)(1-p(u, k, I)) \\ &= f(s, I)-f(s+u, I). \end{aligned} \tag{1.6}$$

Letting $u \to \infty$, we obtain

$$\lim_{t\to\infty} f(t, I)+\int_0^\infty [f, g](s+t)\,dt = f(s, I). \tag{1.7}$$

Taking the Laplace transform of both sides of (1.7), we obtain

$$\lim_{t\to\infty} f(t, I)+\lambda\int_0^\infty\int_0^\infty e^{-\lambda s}[f, g](s+t)\,ds\,dt = \lambda f(\lambda, I). \tag{1.8}$$

Therefore, the double integral on the left side of (1.8) is finite. Making a change of integration variables $s = (u-v)/2^{\frac{1}{2}}$ and $t = (u+v)/2^{\frac{1}{2}}$ in this double integral, we obtain (1.4). Equation (1.5) follows from (1.4) by letting $\lambda \to \infty$. The last statement of the proposition follows by letting $s \to 0$ in (1.6). The proof is complete.

The initial derivative matrix of a transition function p is defined by $Q = (q(i,j)) = (p'(0, i, j))$. Any matrix Q which satisfies

$$0 \leqq q(i, j) < \infty \quad (i \neq j), \qquad 0 \leqq -q(i, i) < \infty, \qquad \textstyle\sum_j q(i, j) = 0,$$

is called a conservative matrix. For any conservative matrix, Q, there is a unique minimal transition function p which has Q as its initial derivative matrix (see [2]).

2. Reuter's Conjecture. Throughout this section, Q will denote a given conservative matrix and p will denote the minimal transition function corresponding to Q.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space on which the processes of this section are defined. For $m \geqq 1$, let $N_m = \{N_m(t),\ t \geqq 0\}$ be a Poisson process with parameter $\lambda_m > 0$. Let σ_{mn} denote the time of the nth jump of the mth Poisson process N_m. For $m, n \geqq 1$, let $x_{mn} = \{x_{mn}(t), 0 < t < \tau_{mn}\}$ be a continuous parameter, open Markov chain with state space I, transition function p, absolute distribution function f_m and lifetime τ_{mn}. The processes

$$\{N_1, x_{11}, x_{12}, \cdots, N_2, x_{21}, x_{22}, \cdots, N_m, x_{m1}, x_{m2}, \cdots\} \tag{2.1}$$

are assumed to be mutually independent.

For each $m, n \geqq 1$, let T_{mn} denote the sum of all the lifetimes τ_{ij} such that $\sigma_{ij} < \sigma_{mn}$. For $t > 0$, define a random variable $x^\infty(t)$ on the set

$$\{T_{mn} < t < T_{mn}+\tau_{mn} \quad \text{for some} \quad m, n \geqq 1\} \tag{2.2}$$

by

$$x^\infty(t) = x_{mn}(t-T_{mn}) \quad \text{if} \quad T_{mn} < t < T_{mn}+\tau_{mn}. \tag{2.3}$$

The random intervals $[T_{mn}, T_{mn}+\tau_{mn})$, which may be of the form $[\infty, \infty) = \varnothing$, are disjoint as m and n vary (for a fixed $\omega \in \Omega$) and therefore, for each $t > 0$, $x^\infty(t)$

is uniquely defined on the set in (2.2). We leave $x^\infty(t)$ undefined on the complement of the set in (2.2). Since p is the minimal transition function corresponding to a conservative matrix Q, we may assume that each sample function $x_{mn}(\cdot)$ maps $(0, \tau_{mn})$ into I and is right continuous with left limits in I, if I is given the discrete topology. It follows that $x^\infty(t) \in I$ whenever $x^\infty(t)$ is defined. Let $x^\infty = \{x^\infty(t), 0 < t < \infty\}$.

Reuter's conjecture is contained in the following theorem.

THEOREM 1. *If the p-entrance law $f = \sum_m \lambda_m f_m$ is not bounded, then $P\{x^\infty(t)$ is defined$\} = 0$ for all $t > 0$. If f is bounded, then there is a stochastic transition function p^∞ defined on $[0, \infty) \times I \times I$ with initial derivative matrix Q and a normalized p^∞-entrance law f^∞ defined on $(0, \infty) \times I$ such that x^∞ is a continuous parameter, open Markov chain with state space I, transition function p^∞ and absolute distribution function f^∞. More precisely, for each $t > 0$, $x^\infty(t)$ is defined almost everywhere (the exceptional set depends on t) and the collection of random variables $\{x^\infty(t), 0 < t < \infty\}$ satisfies (1.2) with f and p replaced by f^∞ and p^∞ respectively and τ identically equal to ∞. The Laplace transforms of p^∞ and f^∞ are given by*

$$p^\infty(\lambda, i, j) = p(\lambda, i, j) + g(\lambda, i)[\lambda f(\lambda, I)]^{-1} f(\lambda, j), \tag{2.4}$$

$$f^\infty(\lambda, j) = [\lambda f(\lambda, I)]^{-1} f(\lambda, j), \tag{2.5}$$

where g is the unique p-entrance law defined by (1.3).

The idea of the proof is to obtain a sequence of processes x^k which approximate x^∞ and a corresponding sequence of transition functions p^k which approximate p^∞, and then to pass to the limit as $k \to \infty$. Before proceeding to the proof, we define these approximations and state some lemmas.

Fix $k \geqq 1$. For $1 \leqq m \leqq k$ and $n \geqq 1$, define T^k_{mn} to be the sum of all the lifetimes τ_{ij} such that $\sigma_{ij} < \sigma_{mn}$ and $1 \leqq i \leqq k$. For $t > 0$, define a random variable $x^k(t)$ on the set where $T^k_{mn} < t < T^k_{mn} + \tau_{mn}$ for some $1 \leqq m \leqq k$ and $n \geqq 1$ by

$$x^k(t) = x_{mn}(t - T^k_{mn}) \quad \text{if} \quad T^k_{mn} < t < T^k_{mn} + \tau_{mn}.$$

As before, $x^k(t)$ is well defined and $x^k(t) \in I$ whenever $x^k(t)$ is defined. We emphasize that the definition of $x^k(t)$ is entirely analogous to the definition of $x^\infty(t)$ except that only the processes N_m and x_{mn} for $1 \leqq m \leqq k$ are involved in the definition of $x^k(t)$. As before, let $x^k = \{x^k(t), 0 < t < \infty\}$.

LEMMA 2.1. *Let $k \geqq 1$. Theorem 1 holds with $\sum_m \lambda_m f_m$ replaced by $\sum_{m=1}^k \lambda_m f_m$ (which is automatically bounded) and with x^∞ replaced by x^k. Under these replacements we will write p^k for p^∞ and f^k and f^∞.*

PROOF. We leave it to the reader to verify that the processes x^k fall into a class of continuous parameter, open Markov chains first considered by Doob [3] and later by Chung [2] page 267. According to the latter reference, if we set $F_k = (\sum_{m=1}^k \lambda_m)^{-1} \sum_{m=1}^k \lambda_m f_m$, then

$$f^k(t, j) = F_k(t, j) + \sum_{n=1}^\infty \int_0^t F_k(t-s, j)[F_k, g]^{n*}(s)\, ds, \tag{2.6}$$

where $[F_k, g]^{n*}$ denotes the n-fold convolution of $[F_k, g]$ with itself. Taking the Laplace transform in (2.6), we obtain

$$\begin{aligned} f^k(\lambda, j) &= F_k(\lambda, j)[1-[F_k, g](\lambda)] \\ &= F_k(\lambda, j)[\lambda F_k(\lambda, I)], \end{aligned}$$

where the last equality follows from Proposition 1.1. This proves (2.5); (2.4) follows from

$$p^k(t, i, j) = p(t, i, j)+\int_0^t g(s, i) f^k(t-s, j)\, ds.$$

this completes the proof.

LEMMA 2.2. *If the p-entrance law f of Theorem* 1 *is bounded, then*

(a) *the number of pairs* (m, n) *such that* $\sigma_{mn} < t$ *and* $\tau_{mn} = \infty$ *is finite for all* $t > 0$;

(b) *the sum of all the lifetimes* τ_{mn} *with* $\sigma_{mn} < t$ *and* $\tau_{mn} < \infty$ *is finite for all* $t > 0$.

If the p-entrance law f is not bounded, then $P\{T_{mn} = \infty\} = 1$ *for all* $m, n \geqq 1$.

PROOF. For $m \geqq 1$ and $t > 0$, let $S(m, t)$ denote the sum of all the lifetimes τ_{mj} such that $\sigma_{mj} \leqq t$. It follows from the zero-one law that

$$P\{\lim_{M\to\infty} \textstyle\sum_{m=M}^{\infty} S(m, t) < \infty\} = 1 \quad \text{or} \quad 0. \tag{2.7}$$

If $\lambda > 0$, then this probability equals 1 or 0, depending on whether

$$\lim_{M\to\infty} E\{\exp[-\lambda \textstyle\sum_{m=M}^{\infty} S(m, t)]\} = 1 \quad \text{or} \quad 0.$$

Noting that, as m varies, the random variables $S(m, t)$ are mutually independent and also, that $[f_m, g]$ is the probability density of τ_{mn} for $n \geqq 1$, we can show that the quantity on the left side of (2.7) equals

$$\begin{aligned} &\lim_{M\to\infty} \exp\{-t \textstyle\sum_{m=M}^{\infty} \lambda_m(1-\int_0^\infty e^{-\lambda t}[f_m, g](t)\, dt)\} \\ &\qquad = \lim_{M\to\infty} \exp\{-t\lambda \textstyle\sum_{m=M}^{\infty} \lambda_m f_m(\lambda, j)\} \end{aligned} \tag{2.8}$$

where we have used Proposition 1.1. in the last equality. Since $f = \sum_{m=1}^{\infty} \lambda_m f_m$ is a bounded entrance law if and only if $f(\lambda, I) < \infty$, and since $\sum_{m=1}^{M} \lambda_m f_m(\lambda, I) < \infty$ for all $M \geqq 1$, we can conclude that the quantity in (2.8) is equal to 1 or 0, depending on whether f is bounded or not. Therefore, the same is true for the probability in (2.7). The lemma follows easily from the last statement.

LEMMA 2.3. If the p-entrance law f of Theorem 1 is bounded, then the set

$$\{t > 0: t \notin (T_{mn}, T_{mn}+\tau_{mn}) \quad \text{for any} \quad m, n \geqq 1\} \tag{2.9}$$

has Lebesgue measure 0 almost everywhere on Ω.

PROOF. Choose a fixed $\omega_0 \in \Omega$ such that $\sup_{m,n} T_{mn}(\omega_0) = \infty$ and which satisfies (a) and (b) of Lemma 2.2. Since $\sup_{m,n} T_{mn} = \infty$ almost everywhere on Ω, it suffices to show that for ω_0, the set in (2.9) has Lebesgue measure 0. If $T_{mn}(\omega_0) < \infty$ for some pair (m, n), then the sum of all the lengths of all intervals $(T_{ij}(\omega_0), T_{ij}(\omega_0)+\tau_{ij}(\omega_0))$ with $T_{ij}(\omega_0)+\tau_{ij}(\omega_0) \leqq T_{mn}(\omega_0)$ is equal to the sum of all the lifetimes $\tau_{ij}(\omega_0)$ with $\sigma_{ij}(\omega_0) < \sigma_{mn}(\omega_0)$. This last sum is by definition $T_{mn}(\omega_0)$. The lemma now follows if we observe that, as a consequence of Lemma 2.2, either $T_{mn}(\omega_0) < \infty$ for all $m, n \geqq 0$ or there is exactly one pair (m, n) such that $T_{mn}(\omega_0) < \infty$ and $\tau_{mn}(\omega_0) = \infty$.

LEMMA 2.4. *Let C be a countable, dense subset of $(0, \infty)$ and let p^* be a stochastic transition function on $[0, \infty)\times I\times I$ with no instantaneous states. Suppose $x = \{x(t), t \in C\}$ is an open Markov chain with transition function p^* in the sense that (1.2) is satisfied whenever each $t_i \in C$. Then, for every $s > 0$ and almost every $\omega \in \Omega$ (the exceptional set depends on s), there is an open interval $(S_1(\omega), S_2(\omega))$ containing s such that $x(u, \omega) = x(v, \omega) \in I$ for $u, v \in (S_1(\omega), S_2(\omega))\cap C$.*

PROOF. This result is well known. For a proof see [2] page 153.

PROOF OF THEOREM 1. The first statement of the theorem follows from Lemma 2.2. Henceforth, we shall assume that f is a bounded p-entrance law.

Let $x_0 = \{x_0(t), 0 < t < \tau_0\}$ be a continuous parameter, open Markov chain with state space I, transition function p, absolute distribution function f_0 and lifetime τ_0. Assume x_0 is independent of all the processes appearing in (2.1). For $t > 0$, let $(x_0, x^\infty)(t)$ be defined as $x_0(t)$ if $0 < t < \tau_0$, and $x^\infty(t-\tau_0)$ if $\tau_0 < t < \infty$ (and if $x^\infty(t-\tau_0)$ is defined). Leave $(x_0, x^\infty)(t)$ undefined if neither of the above conditions is satisfied. The random variable $(x_0, x^k)(t)$ is defined analogously. Notice that (x_0, x^k) is a continuous parameter, open Markov chain with transition function p^k as given in Lemma 2.1. In what follows it will be convenient to take different choices for f_0.

Consider the set $\Gamma \subset (0, \infty)\times\Omega$ consisting of all pairs (t, ω) such that $(x_0, x^\infty)(t, \omega)$ is defined. $\Gamma \in B(0, \infty)\times\mathscr{F}$, where $B(0, \infty)$ denotes the Borel subsets of $(0, \infty)$. Denote the sections of Γ at ω and t by Γ_ω and Γ_t respectively. Lemma 2.3 shows that $(0, \infty)-\Gamma_\omega$ has Lebesgue measure 0 for almost all $\omega \in \Omega$. By Fubini's theorem on the interchange of integration order, it follows that $P\{\Gamma_t\} = 1$ for almost all (Lebesgue) $t > 0$. In other words, the probability that $(x_0, x^\infty)(t)$ is defined equals 1 for almost all (Lebesgue) $t > 0$.

Choose $i \in I$ and set $f_0(\cdot, \cdot) = p(\cdot, i, \cdot)$. It follows from the sample function properties of the processes x_{mn} (see the discussion preceding Theorem 1) and from the definition of the processes x^∞ and x^k that, whenever $(x_0, x^\infty)(t)$ is defined and equal to j, the process $(x_0, x^k)(t)$ is also defined and equal to j for all sufficiently large k. Let D_i denote the subset of $(0, \infty)$ on which $P\{\Gamma_t\} = 1$. For $t \in D_i$ and $j \in I$,

$$(2.10)\qquad P\{(x_0, x^\infty)(t) = j\} = \lim_{k\to\infty} P\{(x_0, x^k)(t) = j\} = \lim_{k\to\infty} p^k(t, i, j).$$

For $t \in D_i$ and $j \in I$, set $p^\infty(t, i, j)$ equal to the limit in (2.10). To see that the definition of $p^\infty(t, i, j)$ can be extended to all $t \geqq 0$, recall that (see [2] page 129)

$$(2.11) \qquad \textstyle\sum_j |p^k(s+t, i, j) - p^k(s, i, j)| \leqq 1 - p^k(t, i, i) \leqq 1 - p(t, i, i)$$

for $s, t \geqq 0$ and $i \in I$. For fixed $i, j \in I$, $\{p^k(\cdot, i, j), k \geqq 1\}$ is therefore a uniformly equicontinuous family of functions which converge on a dense subset D_i of $[0, \infty)$. Hence, they converge uniformly on compact subsets of $[0, \infty)$. Denote the limit by $p^\infty(t, i, j)$. The function $p^\infty(\cdot, i, j)$ is continuous on $[0, \infty)$.

We now show that p^∞ is a stochastic transition function. Positivity is immediate. That $\lim_{t \to 0} p^\infty(t, i, j) = \delta_{ij}$ follows from the continuity of $p^\infty(\cdot, i, j)$ and from the fact that $p^\infty(0, i, j) = \delta_{ij}$. Fatou's Theorem applied to (2.11) yields

$$\textstyle\sum_j |p^\infty(s+t, i, j) - p^\infty(s, i, j)| \leqq 1 - p(t, i, i).$$

Hence, the function $p^\infty(\cdot, i, I)$ is continuous on $[0, \infty)$. For $t \in D_i$, $p^\infty(t, i, I) = 1$ and, since D_i is dense in $[0, \infty)$, it follows that $p^\infty(t, i, I) = 1$ for $t \geqq 0$. Fatou's Theorem also implies that

$$(2.12) \qquad \textstyle\sum_k p^\infty(s, i, k) p^\infty(t, k, j) \leqq p^\infty(s+t, i, j).$$

Summing over j in (2.12) and observing that we must have equality, we see that p^∞ satisfies the Chapman-Kolmogorov equations. It follows that p^∞ is a stochastic transition function. Equation (2.4) follows by letting $k \to \infty$ in the corresponding equation for p^k. Since

$$(1 - p^\infty(t, i, i))/t = (1 - \lim_k p^k(t, i, i))/t \leqq (1 - p(t, i, i))/t,$$

all the states of I are stable with respect to the transition function p^∞.

Let D denote the subset of $(0, \infty)$ where $x^\infty(t)$ is defined with probability 1. As before, it follows that $(0, \infty) - D$ has Lebesgue measure 0. Also,

$$\begin{aligned} P\{x^\infty(t_1) = i_1, \cdots, x^\infty(t_n) = i_n\} \\ &= \lim_k P\{x^k(t_1) = i_1, \cdots, x^k(t_n) = i_n\} \\ &= \lim_k P\{x^k(t_1) = i_1\} p^k(t_2 - t_1, i_1, i_2) \cdots p^k(t_n - t_{n-1}, i_{n-1}, i_n) \\ &= P\{x^\infty(t_1) = i_1\} p^\infty(t_2 - t_1, i_1, i_2) \cdots p^\infty(t_n - t_{n-1}, i_{n-1}, i_n), \end{aligned}$$

for $i_1, \cdots, i_n \in I$ and $0 < t_1 < \cdots < t_n$ with each $t_i \in D$. D contains a countable, dense subset of $(0, \infty)$ which we denote by C. The collection of random variables $\{x^\infty(t), t \in C\}$ satisfies the conditions of Lemma 2.4. Hence, for any $s > 0$, $x^\infty(\cdot, \omega)$ restricted to C is constant in some set $(S_1(\omega), S_2(\omega) \cap S$ which contains s. Since Q is a conservative matrix, each sample function $x_{mn}(\cdot)$ takes on an infinite number of values in I. Combining the last two statements, we see that $x^\infty(s)$ is defined almost everywhere on Ω (the exceptional set will depend on s). Hence, $D = (0, \infty)$. Therefore, $x^\infty = \{x^\infty(t), t > 0\}$ is a continuous parameter, open Markov chain with transition function p^∞. Equation (2.5) follows by letting $k \to \infty$ in the corresponding equation for f^k.

It remains only to show that p^∞ has the same initial derivative matrix as p, namely Q. By the same reasoning used for the process x^∞, it follows that the process (x_0, x^∞) is a continuous parameter, open Markov chain with transition function p^∞ (for any choice of f_0). If we observe that $(x_0, x^\infty)(t) = x_0(t)$ for $0 < t < t_0$ and use the probabilistic interpretation of the quantities q_i and q_{ij}/q_i, we obtain our result. This completes the proof of the theorem.

3. Some extensions. The expression on the right side of (2.9) makes sense for an arbitrary transition function p and a bounded, nonzero p-entrance law f, if g is defined by (1.3). The transition function p may even have instantaneous states. We show in the next theorem that the analytic content of Theorem 1 remains valid in this more general setting. In fact, even if p is the minimal transition function corresponding to a conservative matrix Q, Theorem 2 is not implied by Theorem 1 since we are not assuming f can be written in the form $f = \sum_m \lambda_m f_m$, where each f_m is a normalized p-entrance law.

THEOREM 2. *Let p be a transition function and let f be a bounded, non-zero p-entrance law. The expression*

$$p(\lambda, i, j) + g(\lambda, i)[\lambda f(\lambda, I)]^{-1} f(\lambda, j) \tag{3.1}$$

is the Laplace transform of a unique stochastic transition function p^∞, where g is the unique bounded p-exit law defined in (1.3). *Furthermore, the quantity*

$$[\lambda f(\lambda, I)]^{-1} f(\lambda, j)$$

is the Laplace transform of a unique normalized p^∞-entrance law f^∞ and the quantity

$$g(\lambda, i)[\lambda f(\lambda, I)]^{-1}$$

is the Laplace transform of a unique bounded p^∞-exit law g^∞.

PROOF. If we define $p^\infty(\lambda, i, j)$ to be the quantity in (3.1), it is easy to verify that

$$\lambda p^\infty(\lambda, i, I) = 1, \qquad \lim_{\lambda \to \infty} \lambda p^\infty(\lambda, i, j) = \delta_{ij}.$$

The resolvent equation and the existence of f^∞ and g^∞ follow from a routine calculation if we use Propositions 2.1.4 and 2.2.4 of [6] and the equality

$$\begin{aligned} \mu f(\mu, I) - \lambda f(\lambda, I) &= \textstyle\int_0^\infty (e^{-\lambda s} - e^{-\mu s})[f, g](s)\, ds \\ &= (\mu - \lambda) \textstyle\int_0^\infty \int_0^\infty e^{-\lambda s} e^{-\mu t} [f, g](s+t)\, ds\, dt \\ &= (\mu - \lambda) \textstyle\sum_k f(\lambda, k) g(\mu, k), \end{aligned} \tag{3.2}$$

where the first equality follows from Proposition 1.1 and the second equality follows from a change of integration variables. The proof is complete.

The proof of Theorem 1 involved an approximation of p^∞ by p^k where p^k can be constructed from (3.1) by using the p-entrance law $\sum_{m=1}^k \lambda_m f_m$ and

$\lim_{t\to 0}\sum_{m=1}^{k}\lambda_m f_m(t, I) < \infty$. The next proposition shows that a similar approximation is possible in our more general setting.

PROPOSITION 3.1. *Let the p-entrance law f in Theorem 2 be such that*

$$\lim_{t\to 0} f(t, I) = \infty.$$

Let f_δ be the p-entrance law defined by

$$f_\delta(t, i) = f(t+\delta, i)$$

for $t > 0$ and $i \in I$. Then $\lim_{t\to 0} f_\delta(t, I) < \infty$, and, if p_δ is the stochastic transition function constructed from f_δ by means of Theorem 2, then

$$\lim_{\delta\to 0} p_\delta(t, i, j) = p^\infty(t, i, j)$$

for $t \geqq 0$ and $i, j \in I$.

PROOF. It follows from (3.1) that

$$\lim_{\delta\to 0} p_\delta(\lambda, i, j) = p^\infty(\lambda, i, j).$$

Since $P_\delta(t, i, j) \geqq p(t, i, j)$ for $t \geqq 0$ and $i, j \in I$, the proposition follows from the following lemma.

LEMMA 3.2. *For $0 \leqq n \leqq \infty$, let p_n be a transition function. Suppose $p_n(t, i, j) \geqq p_0(t, i, j)$ for $t \geqq 0$, $1 \leqq n < \infty$ and $i, j \in I$. Also, suppose that*

$$\lim_{n\to\infty} p_n(\lambda, i, j) = p_\infty(\lambda, i, j) \tag{3.3}$$

for $\lambda > 0$ and $i, j \in I$. Then

$$\lim_{n\to\infty} p_n(t, i, j) = p_\infty(t, i, j)$$

for $t \geqq 0$ and $i, j \in I$.

PROOF. The ideas of the proof are contained in [4]. We repeat them here in our context. Since

$$p_n(\lambda, i, j) = \int_0^t e^{-\lambda t}\, d\left[\int_0^t p_n(s, i, j)\, ds\right],$$

we can conclude from (3.3) that

$$\lim_{n\to\infty} \int_0^t p_n(s, i, j)\, ds = \int_0^t p_\infty(s, i, j)\, ds$$

for $t \geqq 0$. Let $\Pi_n(t, i, j) = \int_0^t p_n(s, i, j)\, ds$ for $1 \leqq n \leqq \infty$. We have

$$\begin{aligned} &\Pi_n(s+t, i, j) - \Pi_n(t, i, j) \\ &\quad = \int_t^{s+t} p_n(u, i, j)\, du \geqq \int_t^{s+t} p_n(t, i, j) p_n(u-t, j, j)\, du \\ &\quad = p_n(t, i, j) \int_0^s p_n(u, j, j)\, du \\ &\quad = p_n(t, i, j)\Pi_n(s, j, j). \end{aligned} \tag{3.4}$$

Letting $n \to \infty$ in (3.4), we obtain

$$\Pi_\infty(s+t, i, j) - \Pi_\infty(t, i, j) \geqq \Pi_\infty(s, j, j) \limsup_{n\to\infty} p_n(t, i, j). \tag{3.5}$$

Dividing both sides of (3.5) by s and then letting $s \to 0$, we obtain

$$p_\infty(t, i, j) \geqq \limsup_{n\to\infty} p_n(t, i, j). \tag{3.6}$$

For $0 < s < t$,

$$p_n(s, i, j) \leqq p_n(t, i, j)[p_n(t-s, j, j)]^{-1}.$$

Therefore, if $\delta < t$

$$\begin{aligned} \delta^{-1}[\Pi_n(t, i, j) - \Pi_n(t-\delta, i, j)] &= \delta^{-1} \textstyle\int_{t-\delta}^{t} p_n(u, i, j)\, du \\ &\leqq \delta^{-1} \textstyle\int_{t-\delta}^{t} p_n(t, i, j)[p_n(t-u, j, j)]^{-1}\, du \\ &\leqq p_n(t, i, j) \sup_{0<s<\delta} [p_n(s, j, j)]^{-1} \\ &\leqq p_n(t, i, j) \sup_{0<s<\delta} [p_n(s, j, j)]^{-1}. \end{aligned} \tag{3.7}$$

Letting $n \to \infty$ and then $\delta \to 0$ in (3.7), we obtain

$$p_\infty(t, i, j) \leqq \liminf_{n\to\infty} p_n(t, i, j). \tag{3.8}$$

Using (3.6) and (3.8), we obtain the lemma.

4. Boundary theory. In this section we obtain some interpretation of our analytic and probabilistic constructions in terms of boundary theory. The reader is referred to [7] and [10] for background material.

PROPOSITION 4.1. *If f^∞ and g^∞ are the entrance and exit laws constructed in Theorem 2, then $[f^\infty, g^\infty](\lambda) < \infty$ for all $\lambda > 0$ and*

$$[f^\infty, g^\infty](\lambda) - [f^\infty, g^\infty](\mu) = [\lambda f(\lambda, I)]^{-1} - [\mu f(\mu, I)]^{-1} \tag{4.1}$$

for $\lambda, \mu > 0$.

PROOF. We have

$$\begin{aligned} (\mu-\lambda) \textstyle\int_0^\infty \int_0^\infty e^{-\lambda s} e^{-\mu t} [f^\infty, g^\infty](s+t)\, ds\, dt \\ = (\mu-\lambda) \textstyle\sum_k f^\infty(\lambda, k) g^\infty(\mu, k) \\ = [\lambda f(\lambda, I) \mu f(\mu, I)]^{-1} (\mu-\lambda) \textstyle\sum_k f(\lambda, k) g(\mu, k) \\ = [\lambda f(\lambda, I)]^{-1} - [\mu f(\mu, I)]^{-1}, \end{aligned} \tag{4.2}$$

where the last equality follows from (3.2). Since the first term (4.2) is finite,

$$\textstyle\int_v^\infty e^{-\lambda s} [f^\infty, g^\infty](s)\, ds < \infty \tag{4.3}$$

for all $\lambda > 0$ and $v > 0$. Also,

$$(4.4) \qquad \int_0^v [f^\infty, g^\infty](s+t)\,dt$$
$$= \sum_k f^\infty(s, k) \int_0^v g^\infty(t, k)\,dt \leqq \sup_k \int_0^V g^\infty(t, k)\,dt < \infty,$$

since g^∞ is bounded and f^∞ is normalized. Letting $s \to 0$ in (4.4), we obtain

$$(4.5) \qquad \int_0^v [f^\infty, g^\infty](t)\,dt < \infty.$$

Combining (4.3) and (4.5), we see that $[f^\infty, g^\infty](\lambda) < \infty$ for $\lambda > 0$. A routine change of integration variables in the first term of (4.1) yields the proposition.

In case $\lim_{t\to 0} f(t, I) = \infty$, then, by letting $\mu \to \infty$ in (4.1), we obtain

$$[f^\infty, g^\infty]\,(\lambda) = [\lambda f(\lambda, I)]^{-1}$$

Substituting for $[\lambda f(\lambda, I)]^{-1}$ in (3.1), we obtain

$$(4.6) \qquad p^\infty(\lambda, i, j) = p(\lambda, i, j) + g^\infty(\lambda, i)([f^\infty, g^\infty](\lambda))^{-1} f^\infty(\lambda, j).$$

Equation (4.6) implies, in the terminology of [7] and [10], that f^∞ is coupled to g^∞.

Following the notation of Doob [4], we let $K \supset I$ be the compactified (metric) state space with respect to the transition function p^∞ of Theorem 2. It is shown in [4] that, for each $j \in I$, the function $p(\cdot, \cdot, j)$ on $(0, \infty) \times I$ has a continuous extension to $(0, \infty) \times K_0$, where K_0 is a certain Borel subset of K containing I. The set K_e consists of all $\xi \in K_0$ such that the function $p(\cdot, \xi, \cdot)$ is an extremal element of the convex cone of normalized p^∞-entrance laws. Also, $K_e \supset I$. Any continuous parameter, open Markov chain $x = \{x(t), t > 0\}$ with transition function p^∞ has a separable modification $\hat{x} = \{\hat{x}(t), t \geqq 0\}$ with values in K such that (1) almost every sample function $\hat{x}(\cdot)$ has values in K_e and is right continuous on $[0, \infty)$ with left limits (in K_0) on $(0, \infty)$, and (2) $\hat{x}$ is a strong Markov process with respect to the extended transition function p^∞.

According to the work of David Williams [10], equation (4.6) implies that $f^\infty(t, i) = p(t, \xi, j)$ on $(0, \infty) \times I$ where $\xi \in K_e - I$ and ξ is regular for $\{\xi\}$ (that is, the probability that a separable process $\hat{x} = \{\hat{x}(t), t \geqq 0\}$ with $P\{\hat{x}(0) = \xi\} = 1$ hits ξ in every interval $(0, \varepsilon)$ is 1). In addition, $\int_0^t g^\infty(s, i)\,ds$ is the expected local time (suitably normalized) that a process (with $P\{\hat{x}(0) = i\} = 1$) spends in $\{\xi\}$ before time t.

We are now in a position to describe the boundary theoretic interpretation of the probabilistic construction of Theorem 1. If $\sum_{m=1}^\infty \lambda_m < \infty$, then $f = \sum_{m=1}^\infty \lambda_m f_m$ satisfies $\lim_{t\to 0} f(t, I) < \infty$, and it is not hard to see that the process $x^\infty = \{x^\infty(t), t > 0\}$ falls into the same class of processes mentioned in Lemma 2.1. In this case, the probabilistic interpretation of the process x^∞ (see [1]) shows that the p^∞-entrance law f^∞ corresponds to an initial distribution concentrated on the union of the non-regular points of $K_e - I$ and the points of I. If $\sum_{m=1}^\infty \lambda_m = \infty$, then $\lim_{t\to 0} f(t, I) = \infty$ and $f^\infty(t, j) = p(t, \xi, j)$ where $\xi \in K_e - I$ and ξ is regular for $\{\xi\}$. A separable modification $\hat{x}^\infty$ (with values in K) of the process x^∞ would satisfy $\hat{x}^{\infty}(0) = \xi$ almost everywhere and would have $\hat{x}^{\infty}(t) = \xi$ infinitely often in $(0, \varepsilon)$ for every $\varepsilon > 0$. This corresponds to the fact that, for every $\varepsilon > 0$, $T_{mn} < \varepsilon$ for infinitely many pairs (m, n).

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