The Annals of Mathematical Statistics
1971, Vol. 42, No. 4, 1316–1321

PARTIALLY BALANCED WEIGHING DESIGNS WITH TWO ASSOCIATION CLASSES[1]

BY K. V. SURYANARAYANA[2]
University of North Carolina

1. Summary. Balanced weighing designs have been investigated by Bose and Cameron [2], [3]. This is a part of the work on the extensions of balanced weighing designs to a situation of "partial balance" based on association schemes [1]. Here some methods are developed to construct these new designs (called partially balanced weighing designs) with two association classes.

2. Introduction. Bose and Cameron [2], [3] developed some methods of constructing balanced weighing designs for $p = 2$ and $v \leqq 50$ and $v \leqq 13$ and $p \leqq v/2$. In the latter case [3] they tabulated all parameter combinations in that range together with their plans. It is the purpose of this paper to construct partially balanced weighing designs (PBWD) with two association classes, which have arisen as extensions of the balanced weighing designs. Apart from two theorems (4.1 and 4.3) to construct PBWD's with any general association scheme with two classes, some specific methods of construction are given for the particular cases of (i) Group divisible scheme (ii) Latin square association scheme and (iii) Triangular association scheme. (See Bose [1] for a brief account of combinatorial properties of partially balanced incomplete block designs and association schemes).

3. Partially balanced and balanced weighing designs. (i) Balanced weighing designs: An arrangement of v objects in b-blocks of size $2p$, satisfying the conditions I and II given below, is called a balanced weighing design [2], [3] and is denoted by BWD $(v, b, r, p, \lambda_1, \lambda_2)$:

I. Each object occurs in r blocks and each block consists of two half blocks each of size p.

II. Any two objects appear in the same half block λ_1 times and in opposite half blocks λ_2 times.

(ii) Partially balanced weighing designs: An arrangement of v objects in b-blocks of size $2p$, satisfying I (vide above) and III given below, is called a partially balanced weighing design and is denoted by PBWD $(v, b, r, p; \lambda_{11}, \lambda_{21}; \lambda_{12}, \lambda_{22})$.

III. Any two objects which are ith associates occur together in the same half block λ_{1i} times and in opposite half blocks λ_{2i} times $(i = 1, 2)$.

In the next section three methods are given which are useful in the construction of PBWD's with any association scheme.

Received May 8, 1969; revised October 12, 1970.

[1] This is a part of the research submitted for a Ph.D. degree at the University of North Carolina and was supported by Grant No. DA-ARO-D-31-124-G910.

[2] Now at Fayetteville State University, Fayetteville, North Carolina.

4. Some general methods of constructing partially balanced weighing designs. (i) First method: The following result which is a generalization of that of Bose and Cameron [3] is obvious.[3]

THEOREM 4.1. *A necessary and sufficient condition for the existence of a* PBWD $(v, b, r, p; \lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22})$ *with parameters*

$$n_1, n_2, P_1 = \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ & p_{22}^1 \end{pmatrix} \quad and \quad P_2 = \begin{pmatrix} p_{11}^2 & p_{12}^2 \\ & p_{22}^2 \end{pmatrix}$$

for the association scheme, is that there exists a PBIB *design with* $v^* = v$, $b^* = b$, $r^* = r$, $k^* = 2p$, $\lambda_1{}^* = \lambda_{11}+\lambda_{21}$ *and* $\lambda_2{}^* = \lambda_{12}+\lambda_{22}$ *each of whose blocks can be divided into* 2 *half blocks such that the half blocks form a* PBIB *design with* $v_0 = v$, $b_0 = 2b$, $r_0 = r$, $k_0 = p$, $\lambda_1{}^0 = \lambda_{11}$ *and* $\lambda_2{}^0 = \lambda_{12}$.

Therefore one possible method to construct PBWD's is to start from a suitable partially balanced incomplete block design and to split each of its blocks into two halves in such a way that Theorem 4.1 is satisfied. This type of splitting might not be possible, if we start with any arbitrary PBIB design with block size $2p$.

(ii) Second method: This method uses resolvable solutions of PBIB designs to construct PBWD's. Suppose there is a resolvable solution of a PBIB design with parameters $(v, b, r, k, \lambda_1, \lambda_2)$, and with t blocks in each replication. Further, let u be an integer $\leqq t/2$. We can form all possible partitions of the t blocks in a replication R into sets "S" each containing $2u$-blocks. Again, consider all possible partitions of any such S into two sets S_1 and S_2 each containing u blocks.

Let us take all the uk-objects of S_1 as one half block and those of S_2 as the other half block. Repeating this process for each replication R, for each choice of S from out of R, and for each partition of S into S_1, S_2 we get a PBWD. This method is rather elementary and as a caution it is to be noted that this may not be economical if the number of weighings required is larger than those required by any other method.

To verify that this method gives a PBWD, the following theorem illustrates the case for $u = 1$.

THEOREM 4.2. *If there exists a resolvable* PBIB *design* $(v, rt, r, k, \lambda_1, \lambda_2)$ *with any association scheme and* $b = rt$ *blocks, then there exists a* PBWD $(v, r\binom{t}{2}, r(t-1), k, \lambda_1(t-1), (r-\lambda_1), \lambda_2(t-1), (r-\lambda_2))$.

PROOF. Let us start with a set of t-blocks of the resolvable PBIB design, which constitute a single replication. These t-blocks can be paired in all possible ways, and this results into $\binom{t}{2}$ pairs. Ignoring the order within each pair, let us take the blocks of the latter, as the two half blocks in a weighing design. In this way, it is evident that each "resolvable set" of the given PBIB design gives rise to $\binom{t}{2}$ blocks of the corresponding weighing design. Hence $b^* = r\binom{t}{2}$, where $(v^*, b^*, r^*, k^*; \lambda_{11}^*, \lambda_{21}^*; \lambda_{12}^*, \lambda_{22}^*)$ are the parameters of the weighing design.

[3] See second paragraph in Section 3 of their paper.

If we fix a particular treatment θ and consider "a resolvable set of t blocks," it is evident that the corresponding blocks of the weighing design contain this treatment (θ), $(t-1)$ times. By definition of a resolvable design, any arbitrary treatment occurs once in each of the r resolvable sets. Hence $r^* = r(t-1)$. Since, evidently $v^* = v$ and $p^* = k$, we have to verify the λ-conditions to complete the theorem.

Any complete block of the original PBIB design enters only in $(t-1)$ pairs formed from that resolvable set to which it belongs. Hence any complete block of the original design appears, as a half block, $(t-1)$ times. From this it follows that the number of times any two treatments occur together in the weighing design is $(t-1)\lambda_1$ or $(t-1)\lambda_2$ according as they are first associates or second associates respectively. Hence $\lambda_{11}^* = (t-1)\lambda_1$, $\lambda_{12}^* = (t-1)\lambda_2$.

To find other parameters λ_{21}^* and λ_{22}^*, we note that there are $(r-\lambda_1)$—"resolvable sets" in which two "first-associate" treatments θ and χ occur in different blocks rather than in the same block. By this construction method of weighing design, it is thus evident that $\lambda_{21}^* = (r-\lambda_1)$. Similar argument applies to two "second-associate" treatments, giving $\lambda_{22}^* = r-\lambda_2$. This completes the proof.

(iii) Third method: This method can be called a recursive method and it is often used (see for example, Clatworthy [6]) in the construction of designs such as BIB's and PBIB's. It involves the use of existing PBWD'S to construct those with smaller block size. The following theorem establishes this method to obtain PBWD's with $p = 2$ from out of those with $p = 3$.

THEOREM 4.3. *If there exists a* PBWD $(v, b, r, 3;\ \lambda_{11}, \lambda_{21};\ \lambda_{12}, \lambda_{22})$, *we can deduce a* PBWD $(v, 9b, 6r, 2;\ 3\lambda_{11}, 4\lambda_{21};\ 3\lambda_{12}, 4\lambda_{22})$.

PROOF. Taking any specific block, we can form 9 blocks by taking all possible pairs from each of its half blocks. Hence for the deduced design, the number of blocks $b^* = 9b$. To find r, first we notice that from out of a half block containing θ, two pairs can be formed which contain the latter. But for each such possible pair from this half block, three pairs can be formed from out of the corresponding opposite half block. Hence we conclude that whenever any treatment θ occurs in the original design, it occurs 6 times in the deduced one. Hence the number of replications is given by $r^* = 6r$.

Whenever a pair is fixed in the deduced design, it constitutes a half block three times for each original block which contains the same. Hence the λ-parameters λ_{11}^*, λ_{12}^* are given by $\lambda_{11}^* = 3\lambda_{11}$, $\lambda_{12}^* = 3\lambda_{12}$.

Similarly considering the blocks in which two treatments occur in opposite half blocks, we have $\lambda_{21}^* = 4\lambda_{21}$, $\lambda_{22}^* = 4\lambda_{22}$. Hence the theorem follows.

This method involves the use of existing partially balanced weighing designs. Thus, if a PBWD with $p = 3$ can be constructed by some methods like those proposed in the previous pages, we can deduce the corresponding PBWD with $p = 2$.

5. Construction of group divisible family of PBWD's. (i) PBWD's of singular group divisible type: The three general methods described in Section 4 are applicable for this type of association scheme. Another method which is analogous to the one used by Bose, Shrikhande and Bhattacharya [5] involves the use of BWD's to construct PBWD's of singular group divisible type. The method is proposed in the form of the following theorem.

THEOREM 5.1. *If there exists a* BWD *with parameters* $(v^*, b^*, r^*, p^*, \lambda_1^*, \lambda_2^*)$, *then by replacing each of the treatments with n treatments, we get a* PBWD *of singular-group divisible type, where the n-treatments which replace a particular one of the original design form a group.*

PROOF. Obviously $v = nv^*$. As there is no difference in the number of blocks, $b^* = b$. Since each treatment of the original design occurs r^* times, by the definition of groups introduced, $\lambda_{11} = r^*$. Since the n treatments constituting a group either all occur (in case the corresponding treatment occurs in the original BWD) in a half block or do not occur at all, it follows that $\lambda_{21} = 0$.

Two treatments are second associates if they arise from groups of two different treatments of the original set. Hence they occur together in the same half block or opposite half blocks respectively that many times, as the original treatments occur together in the same half block or in opposite half blocks. These two frequencies are λ_1^* and λ_2^*. Hence $\lambda_{12} = \lambda_1^*$ and $\lambda_{22} = \lambda_2^*$. The number of replications is unchanged and each block of p^* treatments is now replaced by np^* treatments. Hence $r = r^*$ and $p = np^*$. This proves the theorem.

(ii) PBWD's of regular group divisible type: The methods of Section 4 can be used to construct this type of design.

6. Construction of PBWD's with triangular association scheme. In addition to the general methods of Section 4, the following theorem gives another method of construction of triangular designs.

THEOREM 6.1. *From a triangular association scheme with* $v(=\frac{1}{2}n(n-1))$ *treatments, with n odd, we can construct a* PBWD *with the same association scheme and with parameters* $(v, b, r, p;\ \lambda_{11}, \lambda_{21};\ \lambda_{12}, \lambda_{22})$ *given by*

$$\left[\frac{n(n-1)}{2}, \frac{n}{2}\left[\frac{(n-1)!}{\left(\frac{n-1}{2}\right)!\left(\frac{n-1}{2}\right)!}\right], \frac{2(n-2)!}{\left(\frac{n-3}{2}\right)!\left(\frac{n-1}{2}\right)!}, \frac{n-1}{2};\right.$$
$$\left.\frac{(n-3)!}{\left(\frac{n-5}{2}\right)!\left(\frac{n-1}{2}\right)!}, \frac{(n-3)!}{\left(\frac{n-3}{2}\right)!\left(\frac{n-3}{2}\right)!};\ 0, 0\right].$$

PROOF. We adopt the following procedure to form blocks of the PBWD. Suppose we form all possible blocks from each row, by considering all possible distinct partitions of the same, into two sets of $(\frac{1}{2}(n-1))$ objects each. Let these two sets be taken as the two half blocks, the procedure being same for any other row with which we start. The rest of the proof is a mere verification of the parameters, and hence it is omitted.

7. PBWD with latin square association scheme.

THEOREM 7.1. A PBWD (s^2, $s(s-1)$, $2(s-1)$, s; $s-1$, 1; 0, 2) *of an* L_2*-association scheme on s-symbols, with* $n_1 = 2(s-1)$ *and* $n_2 = (s-1)^2$ *can be constructed directly from the association scheme.*

PROOF. Let us take each row or each column as a half block and let us form all possible pairs of rows and all possible pairs of columns to form the blocks. All blocks of this design can be formed in $\binom{s}{2}$ ways from rows and $\binom{s}{2}$ ways from columns and since the two sets of blocks thus formed are distinct, it is clear that $b = \binom{s}{2}+\binom{s}{2} = s(s-1)$.

If we fix a treatment i, it must belong to a specific row and a specific column of the Latin square formed below for the association scheme:

$$(1) \qquad \begin{bmatrix} 1 & 2 & \cdots & s \\ (s+1) & (s+2) & \cdots & 2s \\ (2s+1) & (2s+2) & \cdots & 3s \\ \cdots & \cdots & \cdots & \cdots \\ (s-1)s+1 & (s-1)s+2 & \cdots & s^2 \end{bmatrix}$$

But there are $(s-1)$ blocks of the weighing design, containing any specific row as a half block. Similarly the number is $(s-1)$ for blocks formed from columns. Hence $r = 2(s-1)$. By the method of construction, it is clear that $p = s$. If we fix two treatments θ and χ which are first associates, they must belong to a single row or a single column of (1). But as θ and χ determine the half block uniquely, in either case, there are $(s-1)$ blocks with θ and χ belonging to the same half block. Hence $\lambda_{11} = (s-1)$.

For the same θ and χ (mentioned above) to belong to opposite half blocks, we have to count the number of possible ways the blocks can be formed with the row (column) containing χ as one half block and the row (column) containing θ as the other half block. This is possible in one way. Hence $\lambda_{21} = 1$. By the method of construction of half-blocks, it is evident that $\lambda_{12} = 0$. Lastly consider two second associates χ and ϕ which are the (i,j)th and (i_1,j_1)th elements of the array (1), the first and second suffixes representing the row number and the column number respectively. Since χ and ϕ are second associates, it follows that $i \neq i_1, j \neq j_1$. Hence evidently there are two blocks (ith row; i_1th row) (jth column; j_1th column) in which χ and ϕ occur together in opposite half blocks. Therefore $\lambda_{22} = 2$.

Note: The type of argument can be extended to construct in general a PBWD with $p = rs$ from a Latin square association scheme with s symbols, provided $2r < s$. This can be done by taking all possible sets of $2r$-rows which are partitioned into two sets of r-rows each in all possible ways. The rs elements belonging to the r-rows of any such partitioning are the elements of a half block. This theorem together with the methods proposed in Section 4 can be used to construct PBWD's of the Latin square type.

REFERENCES

[1] Bose, R. C. (1963). Combinatorial properties of partially balanced designs and association schemes. *Sankhyā, Ser. A* **25** 109–36.

[2] Bose, R. C. and Cameron, J. M. (1965). The bridge tournament problem and calibration designs for comparing pairs of objects. *J. Res. Nat. Bur. Standards* **69B** 323–32.

[3] Bose, R. C. and Cameron, J. M. (1967). Calibration designs based on solutions to the tournament problem. *J. Res. Nat. Bur. Standards* **71B** 149–60.

[4] Bose, R. C., Clatworthy, W. H. and Shrikhande, S. S. (1954). Tables of partially balanced designs with two associate classes. *North Carolina Agr. Expt. Sta. Tech. Bull. No.* 107.

[5] Bose, R. C., Shrikhande, S. S. and Bhattacharya, K. N. (1953). On the construction of group divisible incomplete block designs. *Ann. Math. Statist.* **24** 167–195.

[6] Clatworthy, W. H. (1956). Contributions to partially balanced incomplete block designs with two associate classes. *Nat. Bur. Standards Applied Math. Series* **47**.

[7] Suryanarayana, K. V. (1969). Contributions to partially balanced weighing designs. Ph.D. dissertation submitted to the University of North Carolina.

[8] Suryanarayana, K. V. and Chakravarti, I. M. (1969). Partial difference sets and partially balanced weighing designs. Submitted to *J. Comb. Theory*.

[9] Suryanarayana, K. V. (1970). Partial difference sets and partially balanced weighing designs with three association classes. *Proceedings of the Second Chapel Hill Conference on Combinatorial Mathematics and its Applications*, Univ. North Carolina Press.