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CONTINUITY PROPERTIES OF SOME GAUSSIAN PROCESSES

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Let (S, d) be a compact metric space; let $(\Omega, \mathscr{F}, P)$ be a probability space, and for each $t \in S$ let $X_t : \Omega \to \mathbb{R}$ be a random variable, with $E(X_t) = 0$ and such that $\{X_t\}_{t \in S}$ forms a Gaussian process. In this paper we find sufficient conditions for the Gaussian process $\{X_t\}_{t \in S}$ to admit a separable and measurable model whose sample functions are continuous with probability one. The conditions involve the covariance, $E(X_s X_t)$, of the process and also the ε-entropy of S.

1. Introduction. Let (S, d) be a compact metric space, let $(\Omega, \mathscr{F}, P)$ be a probability space, and for each $t \in S$ let $X_t : \Omega \to \mathbb{R}$ be a random variable such that $\{X_t\}_{t \in S}$ forms a Gaussian process. We are interested in finding conditions under which this process has a model whose sample functions are continuous with probability one, i.e., such that $X_t(\omega)$: $S \to \mathbb{R}$ is continuous (as a function of t) for almost all $\omega \in \Omega$.

We will make the following assumptions about the process:

(i) It has mean zero, i.e. for all $t \in S$ we have $E(X_t) = 0$.

(ii) It has a continuous covariance, i.e. if we define K: $S \times S \to \mathbb{R}$ by $K(s, t) = E(X_s X_t)$ then we require that K is continuous.

A problem which is essentially the same as the one stated above is the following: Suppose we are given K: $S \times S \to \mathbb{R}$ such that K is continuous and positive definite. Then by the Kolmogorov Consistency Theorem there exists a Gaussian process $\{X_t\}_{t \in S}$ with $E(X_t) = 0$ for all $t \in S$ and $E(X_s X_t) = K(s, t)$ for any $s, t \in S$. For what functions K does there exist a model of the process whose sample functions are continuous with probability one?

In order to state our results we need to introduce some notation:

(a) For any $x \in S$ and $r > 0$ we let $B_r(x) = \{y \in S: d(x, y) < r\}$; $B_r(x)$ is called the open ball with radius r and centre x.

(b) For any $\varepsilon > 0$ $N(\varepsilon)$ will denote the minimum number of open balls of radius ε required to cover S.

(c) We let $H(\varepsilon) = \log N(\varepsilon)$; $H(\varepsilon)$ is called the ε-entropy of S.

(d) $r_0(S)$, the exponent of entropy of S, is given by

$$r_0(S) = \limsup_{\varepsilon \to 0} \left\{ \frac{\log H(\varepsilon)}{\log (1/\varepsilon)} \right\}.$$

(e) Let $\alpha = \sup \{d(x, y): x, y \in S\}$.

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The main result of this paper is:

I. A sufficient condition for a Gaussian process $\{X_t\}_{t \in S}$ to admit a separable and measurable model whose sample functions are continuous with probability one is that if we set

$$v(u) = \sup_{d(s,t) \leqq 2u} \{E(|X_s - X_t|^2)\}^{\frac{1}{2}}$$

then we have

$$\int_0^1 [\log (r^{-1}) H(2\alpha r)]^{\frac{1}{2}} \, dv(r) < \infty \, .$$

Furthermore there is a model in which

$$|X_s(\omega) - X_t(\omega)| \leqq 20 \int_0^{d(s,t)} [\log B(\omega) + 2(\log (r^{-1}) + 2) H(2\alpha r)]^{\frac{1}{2}} \, dv(r)$$

where $B\colon \Omega \to \mathbb{R}$ is a random variable with $E(B) \leqq 4(2)^{\frac{1}{2}}$. (Note: we can express this result in terms of the covariance, since $E(|X_s - X_t|^2) = K(s, s) - 2K(s, t) + K(t, t)$.) Applying I to some particular cases we get.

II. Suppose S is a compact subset of a real Hilbert space $\mathscr{H}$ (with the metric on S induced by the norm on $\mathscr{H}$). Let (x, y) denote the inner product of x and y in $\mathscr{H}$; it is simple to verify that $K\colon S \times S \to \mathbb{R}$, given by $K(x, y) = (x, y)$, is continuous and positive definite on S. If $r_0(S) < 2$ there exists a Gaussian process $\{X_t\}_{t \in S}$ with $E(X_s X_t) = (s, t)$ and such that $\{X_t\}_{t \in S}$ has a separable and measurable model whose sample functions are continuous with probability one. This result was proved by Dudley in [1]. We prove further here that there is a model such that given any δ with $0 < \delta < 1 - \frac{1}{2} r_0(S)$ then there exists $D\colon \Omega \to \mathbb{R} \cup \{\infty\}$ such that D is finite with probability one and

$$|X_s(\omega) - X_t(\omega)| \leqq D(\omega)[d(s, t)]^{\delta} \qquad \text{for all} \quad s, t \in S \, .$$

(Note: In [1] Dudley conjectures that if $r_0(S) < 2$ then $\{X_t\}_{t \in S}$ cannot have a model whose sample functions are continuous with probability one, and this conjecture is still unsettled.)

III. Suppose S is any compact metric space with $r_0(S) < 2$. Then there exists a Gaussian process $\{X_t\}_{t \in S}$ such that

(a) $E(X_s X_t) = \exp[-\frac{1}{2}(d(s, x_0) - d(t, x_0))^2]$ for some point $x_0 \in S$.

(b) $\{X_t\}_{t \in S}$ admits a separable and measurable model whose sample functions are continuous with probability one.

The method of obtaining these results is a generalization of that used by Garsia et al in [2], [3], and [4] to obtain similar results in the case that either S is $[0, 1]$ or S is a cube in $\mathbb{R}^n$. We use an extension of a real variable lemma of Garsia, Rodemich and Rumsey to show that the Karhunen-Loève expansion of the Gaussian process converges uniformly with probability one.

2. The proof of the main result. Let $\mathscr{B}$ denote the Borel subsets of S, and let μ be a finite, nonnegative Borel measure on $(S, \mathscr{B})$, with the property that

$\mu(B_r(x)) > 0$ for all $x \in S$ and $r > 0$. Let $K: S \times S \to \mathbb{R}$ be continuous and positive definite on S. By Mercer's theorem there exist $\varphi_n : S \to \mathbb{R}$ and $\lambda_n > 0$ for $n = 1, 2, \cdots$ such that

(i) φ_n is continuous for each n.
(ii) $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal system in $\mathscr{L}^2(S, \mathscr{B}, \mu)$.
(iii) $\lambda_n \varphi_n(x) = \int K(x, t)\varphi_n(t)\, d\mu(t)$ for all n and all $x \in S$.
(iv) $\sum_{\nu=1}^{n} \lambda_\nu \varphi_\nu(s)\varphi_\nu(t)$ converges uniformly to $K(s, t)$ in $S \times S$.

Let $\theta_n : \Omega \to \mathbb{R}$, $n = 1, 2, \cdots$, be a sequence of independent Gaussian random variables with $E(\theta_n) = 0$ and $E(\theta_n^2) = 1$. For any $n \geqq 1$ and $t \in S$ define $X_t^{(n)}$: $\Omega \to \mathbb{R}$ by

$$X_t^{(n)}(\omega) = \sum_{\nu=1}^{n} \lambda_\nu^{\frac{1}{2}} \varphi_\nu(t)\theta_\nu(\omega) .$$

($\{X_t^{(n)}\}_{n=1}^{\infty}$ is called the Karhunen-Loève expansion of a Gaussian process having covariance K.) Note we have for fixed $t \in S$ and n that $X_t^{(n)}$ is a Gaussian random variable with mean zero, and for fixed $\omega \in \Omega$ and n that $X_t^{(n)}(\omega)$ is a continuous function of $t \in S$. Suppose that with probability one we have $\{X_t^{(n)}\}_{n=1}^{\infty}$ converges uniformly as $n \to \infty$. Then defining $X_t : \Omega \to \mathbb{R}$ by

$$\begin{aligned} X_t(\omega) &= \lim_{n\to\infty} X_t^{(n)}(\omega) && \text{if it exists} \\ &= 0 && \text{otherwise,} \end{aligned}$$

it is simple to verify that $\{X_t\}_{t \in S}$ is a Gaussian process which is separable and measurable, and has continuous sample functions with probability one. Also we have $\{X_t\}_{t \in S}$ has mean zero and $E(X_s X_t) = \lim_{n\to\infty} E(X_s^{(n)} X_t^{(n)}) = \lim_{n\to\infty} \sum_{\nu=1}^{n} \lambda_\nu \varphi_\nu(s)\varphi_\nu(t) = K(s, t)$. We will show that if the covariance K satisfies certain conditions then we do have $\{X_t^{(n)}\}_{n=1}^{\infty}$ converges uniformly with probability one.

At this point we make the following remarks:

A. The uniform convergence with probability one of the Karhunen-Loève expansion of a Gaussian process $\{X_t\}_{t \in S}$ is (from the above) clearly sufficient to establish the existence of a version of the process which has continuous sample functions with probability one. The condition is also necessary; this is proved in [5] by Jain and Kallianpur, using the ideas of Itô and Nisio [6].

B. The proof, to be given below, that for certain covariances the Karhunen-Loève expansion of the process converges uniformly with probability one, also shows that for the same covariances we have the uniform convergence with probability one of the expansion of the process with respect to any orthonormal basis for the reproducing Hilbert space of the covariance kernel.

Let μ be the Borel measure on $(S, \mathscr{B})$ as before; define $m: \mathbb{R}^+ \to \mathbb{R}^+$ by $m(r) = \inf_{x \in S} \mu(B_r(x))$. Let $\psi: \mathbb{R} \to \mathbb{R}^+$ be continuous, symmetric (i.e. $\psi(t) = \psi(-t)$), convex increasing (for $t \in \mathbb{R}^+$), with $\psi(0) \geqq 0$; let $\rho: \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and increasing, with $\rho(0) = 0$. The proof of our results depends heavily on the following lemma:

LEMMA 1. *Let $f \in \mathscr{L}^1(S, \mathscr{B}, \mu)$ and suppose that*

$$\iint \psi\left(\frac{f(s) - f(t)}{\rho(d(s, t))}\right) d\mu(s)\, d\mu(t) = c_0 < \infty .$$

Suppose also that

$$\int_0^1 \psi^{-1}\left(\frac{c_0}{(m(u/2))^2}\right) d\bar{\rho}(u) < \infty , \qquad \textit{where} \quad \bar{\rho}(u) = \rho(2u) .$$

Then there exists $g: S \to \mathbb{R}$ with g continuous and $g = f$ a.e. *Furthermore we have*

$$|g(x) - g(y)| \leqq 10 \int_0^{d(x,y)} \psi^{-1}\left(\frac{c_0}{(m(u/2))^2}\right) d\bar{\rho}(u) .$$

PROOF. This is given in [7].

Let K, φ_n, λ_n and $X_t^{(n)}$ be as before. We define ΔK: $S \times S \to \mathbb{R}$ by $\Delta K(s, t) = K(s,s) - 2K(s,t) + K(t,t)$. Thus $\Delta K(s,t) = \sum_{\nu=1}^{\infty} \lambda_\nu (\varphi_\nu(s) - \varphi_\nu(t))^2$. Let σ: $\mathbb{R}^+ \to \mathbb{R}^+$ be continuous, increasing with $\sigma(0) = 0$ and such that $\sigma(u) \geqq \sup_{d(s,t) \leqq u} \{\Delta K(s,t)\}^{\frac{1}{2}}$. We now have:

LEMMA 2. *Suppose*

$$\int_0^1 \left[\log\left(\frac{1}{m(u/2)}\right)\right]^{\frac{1}{2}} d\bar{\sigma}(u) < \infty , \qquad (\textit{where } \bar{\sigma}(u) = \sigma(2u)) .$$

Then with probability one we have $\{X_t^{(n)}\}_{n=1}^{\infty}$ converges uniformly on S as $n \to \infty$. Further we have

$$|X_s^{(n)}(\omega) - X_t^{(n)}(\omega)| \leqq 20 \int_0^{d(s,t)} \left[\log\left(\frac{B(\omega)}{(m(u/2))^2}\right)\right]^{\frac{1}{2}} d\bar{\sigma}(u) ,$$

$$B(\omega) = \sup_n \iint \exp \tfrac{1}{4} \left\{\frac{X_s^{(n)}(\omega) - X_t^{(n)}(\omega)}{\sigma(d(s, t))}\right\}^2 d\mu(s)\, d\mu(t)$$

and $E(B) \leqq 4(2)^{\frac{1}{2}}[\mu(S)]^2$.

PROOF. The proof is exactly the same as in [3] or [4] and so we omit it.

We now construct a suitable measure μ on S that will enable us to get results involving the ε-entropy of S. Recall that $\alpha = \sup\{d(x, y): x, y \in S\}$; $\alpha < \infty$ since S is compact.

LEMMA 3. *There exists a nonnegative Borel measure μ on $(S, \mathscr{B})$ with $\mu(S) = 1$ and such that if B is any open ball with radius r then $\mu(B) \geqq [\prod_{k=1}^{\gamma(r)+2} N(2^{-k}\alpha)]^{-1}$, where $\gamma(r)$ is the integer part of* $(\log(r^{-1})/\log 2)$.

PROOF. For any $n = 1, 2, \cdots$, and for $1 \leqq i_j \leqq N(2^{-j}\alpha)$ for $j \leqq n$ we define $B(i_1, i_2, \cdots, i_n)$ to be an open ball of radius $2^{-n}\alpha$ such that

(i) $B(i_1, \cdots, i_{n-1}, j) \cap B(i_1, \cdots, i_{n-1}) \cap \cdots \cap B(i_1, i_2) \cap B(i_1) \neq \emptyset$ for any $1 \leqq j \leqq N(2^{-n}\alpha)$.

(ii) $\bigcup_{j=1}^{N)2^{-n}\alpha)} B(i_1, \cdots, i_{n-1}j) \supset B(i_1) \cap B(i_1, i_2) \cap \cdots \cap B(i_1, \cdots, i_{n-1})$.

(iii) $\bigcup_{i=1}^{N(2^{-1}\alpha)} B(i) = S$.

It is clear how open balls with these properties can be constructed inductively. Choose $p(i_1, \cdots, i_n) \in B(i_1, \cdots, i_n) \cap \cdots \cap B(i_1, i_2) \cap B(i_1)$ in such a way that $p(i_1, \cdots, i_n) = p(j_1, \cdots, j_n)$ iff $i_k = j_k$ for $k = 1, \cdots, n$. Let $\mathscr{C}(n) = \{B(i_1, \cdots, i_n)$: $1 \leqq i_j \leqq N(2^{-j}\alpha), j = 1, \cdots, n\}$, and let $\mathscr{M}(n) = \{p(i_1, \cdots, i_n)$: $1 \leqq i_j \leqq N(2^{-j}\alpha)$, $j = 1, \cdots, n\}$. Then it is easy to verify that:

(i) $\mathscr{C}(n)$ consists of $\prod_{k=1}^{n} N(2^{-k}\alpha)$ open balls of radius $2^{-n}\alpha$.

(ii) The elements of $\mathscr{C}(n)$ cover S for each n.

(iii) $\mathscr{M}(n)$ consists of $\prod_{k=1}^{n} N(2^{-k}\alpha)$ points.

(iv) If $m < n$ and $B \in \mathscr{C}(m)$ then there are least $\prod_{k=m+1}^{n} N(2^{-k}\alpha)$ points from $\mathscr{M}(n)$ in B.

Let μ_n be the Borel measure on $(S, \mathscr{B})$ given by

$$\mu_n = [\prod_{k=1}^{n} N(2^{-k}\alpha)]^{-1} \sum_{x \in \mathscr{M}(n)} \mu_x \, .$$

(Here μ_x denotes the unit point mass at x.)

We now have:

(i) $\mu_n(S) = 1$ for $n = 1, 2, \cdots$,

(ii) If $m < n$ and $B \in \mathscr{C}(m)$ then $\mu_n(B) \geqq [\prod_{k=1}^{m} N(2^{-k}\alpha)]^{-1}$.

But S is separable so there exists a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ and a nonnegative Borel measure μ on $(S, \mathscr{B})$ such that $\mu_{n_k} \to_{w^*} \mu$ as $k \to \infty$. Thus we have $\mu(S) = 1$ and if $B \in \mathscr{C}(n)$ then $\mu(\bar{B}) \geqq \lim_{k\to\infty} \mu_{n_k}(B) \geqq [\prod_{k=1}^{n} N(2^{-k}\alpha)]^{-1}$. If A is any open ball in S with radius r and $2^{-n-1}\alpha \leqq r < 2^{-n}\alpha$ then there exists $B \in \mathscr{C}(n+2)$ with $\bar{B} \subset A$ and so

$$\mu(A) \geqq [\prod_{k=1}^{n+2} N(2^{-k}\alpha)]^{-1} \geqq [\prod_{k=1}^{\gamma(r)+2} N(2^{-k}\alpha)]^{-1} \, .$$

We are now in a position to prove:

THEOREM 1. *A sufficient condition for the Gaussian process $\{X_t\}_{t\in S}$ to admit a separable and measurable model whose sample functions are continuous with probability one is that if we set*

$$v(u) = \sup_{d(s,t)\leqq 2u} \{E(|X_s - X_t|^2)\}^{\frac{1}{2}}$$

then we have

$$\int_0^1 [\log (r^{-1}) H(2\alpha r)]^{\frac{1}{2}} \, dv(r) < \infty \, .$$

Furthermore there is a model in which

$$|X_s(\omega) - X_t(\omega)| \leqq 20 \int_0^{d(s,t)} [\log B(\omega) + 2(\log (r^{-1}) + 2) H(2\alpha r)]^{\frac{1}{2}} \, dv(r)$$

where B: $\Omega \to \mathbb{R}$ is a random variable with $E(B) \leqq 4(2)^{\frac{1}{2}}$.

PROOF. Let $K(s, t) = E(X_s X_t)$, so $v(u) = \sup_{d(s,t) \leqq 2u} \{\Delta K(s, t)\}^{\frac{1}{2}}$. Let μ be the Borel measure constructed in Lemma 3. Then

$$m(r) = \inf_{x \in S} \mu(B_r(x)) \geqq [\prod_{k=1}^{\gamma(r)+2} N(2^{-k}\alpha)]^{-1} \geqq [N(\alpha 2^{-(\gamma(r)+2)})]^{-(\gamma(r)+2)} .$$

Therefore $\log(1/m(r/2)) \leqq (\log(r^{-1}) + 2) \log N(2\alpha r) = (\log(r^{-1}) + 2)H(2\alpha r)$. Hence from Lemma 2 we have that if $\int_0^1 [(\log(r^{-1}) + 2)H(2\alpha r)]^{\frac{1}{2}} dv(r) < \infty$ then with probability one $\{X_t^{(n)}\}_{n=1}^{\infty}$, the Karhunen-Loève expansion of the process, converges uniformly on S. We also have

$$|X_s^{(n)}(\omega) - X_t^{(n)}(\omega)| \leqq 20 \int_0^{d(s,t)} [\log B(\omega) + 2(\log(r^{-1}) + 2)H(2\alpha r)]^{\frac{1}{2}} dv(r) .$$

Letting $n \to \infty$ completes the proof of the theorem.

REMARK. For certain metric spaces whose structure is better known Lemma 3 can be improved, and thus Theorem 1 can be improved in these cases; for example, if $S = [0, 1]^n$ then clearly the best thing to do is to put μ equal to Lebesgue measure on $[0, 1]^n$. Then in this case the condition in Theorem 1 can be replaced by the condition that

$$\int_0^1 [H(2\alpha r)]^{\frac{1}{2}} dv(r) < \infty .$$

This gives the same result as Garsia in [4].

3. Auxiliary results.

THEOREM 2. *Let $\mathscr{H}$ be a real Hilbert space, and let S be a compact subset of $\mathscr{H}$. If $r_0(S) < 2$ then there exists a Gaussian process $\{X_t\}_{t \in S}$ with $E(X_s X_t) = (s, t)$ (where (s, t) denotes the inner product of s and t in $\mathscr{H}$), and such that $\{X_t\}_{t \in S}$ has a separable and measurable model whose sample functions are continuous with probability one. Further, there is a model such that given any δ with $0 < \delta < 1 - \frac{1}{2}r_0(S)$ then there exists $D\colon \Omega \to \mathbb{R} \cup \{\infty\}$ such that D is finite with probability one and*

$$|X_s(\omega) - X_t(\omega)| \leqq D(\omega)[d(s, t)]^{\delta} \qquad \text{for all} \quad s, t \in S .$$

PROOF. If $E(X_s X_t) = (s, t)$ then $E(|X_s - X_t|^2) = (s - t, s - t)$, and so $v(u) = \sup_{d(s,t) \leqq 2u} \{E(|X_s - X_t|^2)\}^{\frac{1}{2}} = 2u$. Since $r_0(S) < 2$ there exists $\xi_0 < 2$ and $\eta_0 > 0$ such that if $\varepsilon \leqq \eta_0$ then $\log H(\varepsilon) \leqq \xi_0 \log(\varepsilon^{-1})$, i.e. $H(\varepsilon) \leqq (\varepsilon^{-1})^{\xi_0}$. Thus $\int_0^1 [\log(r^{-1})H(2\alpha r)]^{\frac{1}{2}} dv(r) < \infty$ and so we can apply Theorem 1. Hence we have there exists a model of the process in which

$$|X_s(\omega) - X_t(\omega)| \leqq 40 \int_0^{d(s,t)} [\log B(\omega) + 2(\log(r^{-1}) + 2)H(2\alpha r)]^{\frac{1}{2}} dr$$

where $B\colon \Omega \to \mathbb{R}$ is such that $E(B) \leqq 4(2)^{\frac{1}{2}}$. Take δ with $0 < \delta < 1 - \frac{1}{2}r_0(S)$ and let $\xi = 2 - 2\delta$. We thus have $r_0(S) < \delta < 2$, and so there exists $\eta > 0$ such that if

$$\varepsilon \leqq \eta \quad \text{then} \quad 2(\log(\varepsilon^{-1}) + 2)H(2\alpha\varepsilon) \leqq (\varepsilon^{-1})^{\xi} .$$

Therefore if $d(s, t) \leqq \eta$ we have

$$\begin{aligned}|X_s(\omega) - X_t(\omega)| &\leqq 40 \int_0^{d(s,t)} [\log B(\omega) + (r^{-1})^{\xi}]^{\frac{1}{2}}\, dr \\ &\leqq 40[\log B(\omega)]^{\frac{1}{2}} \int_0^{d(s,t)} dr + 40 \int_0^{d(s,t)} (r^{-1})^{\xi/2}\, dr \\ &= 40[\log B(\omega)]^{\frac{1}{2}} d(s,t) + 40(\delta^{-1})[d(s,t)]^{\delta} \\ &\leqq 40\{[\log B(\omega)]^{\frac{1}{2}} + (\delta^{-1})\}[d(s,t)]^{\delta}\,.\end{aligned}$$

We can thus clearly find $D\colon \Omega \to \mathbb{R} \cup \{\infty\}$ such that $D(\omega) < \infty$ whenever $B(\omega) < \infty$ and

$$|X_s(\omega) - X_t(\omega)| \leqq D(\omega)[d(s,t)]^{\delta} \qquad \text{for all} \quad s, t \in S\,.$$

Let S be an arbitrary compact metric space. There is an obvious way of constructing continuous positive definite functions K on $S \times S$ to be used as covariance functions for Gaussian processes on S; namely let $g\colon \mathbb{R} \to \mathbb{R}$ be continuous and positive definite, let $x_0 \in S$ and define $K(s,t) = g(d(s,x_0) - d(t,x_0))$. Clearly K is positive definite. We have $\Delta K(s,t) = 2[g(0) - g(d(s,x_0) - d(t,x_0))]$ and so

$$\sup_{d(s,t)\leqq u} [\Delta K(s,t)]^{\frac{1}{2}} \leqq \sup_{r\leqq u} [2(g(0) - g(r))]^{\frac{1}{2}}$$

since $|d(s,x_0) - d(t,x_0)| \leqq d(s,t)$. Using this observation we immediately have:

LEMMA 4. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be continuous and positive definite, and let $x_0 \in S$. Define $G(u) = \sup_{r\leqq 2u} [2(g(0) - g(r))]^{\frac{1}{2}}$ and suppose*

$$\int_0^1 [\log (r^{-1}) H(2\alpha r)]^{\frac{1}{2}}\, dG(r) < \infty\,.$$

Then there exists a Gaussian process $\{X_t\}_{t\in S}$ such that

(a) $E(X_s X_t) = g(d(s,x_0) - d(t,x_0))$.

(b) *$\{X_t\}_{t\in S}$ admits a separable and measurable model whose sample functions are continuous with probability one.*

Finally, from Lemma 4 we have:

THEOREM 3. *Suppose $r_0(S) < 2$, and let $x_0 \in S$. Then there exists a Gaussian process $\{X_t\}_{t\in S}$ such that*

(a) $E(X_s X_t) = \exp[-\frac{1}{2}(d(s,x_0) - d(t,x_0))^2]$.

(b) *$\{X_t\}_{t\in S}$ admits a separable and measurable model whose sample functions are continuous with probability one.*

PROOF. Let $g(t) = e^{-t^2/2}$; then g is continuous and positive definite and $G(u) = \sup_{r\leqq 2u} [2(g(0) - g(r))]^{\frac{1}{2}} = [2(g(0) - g(2u))]^{\frac{1}{2}}$. Now since $r_0(S) < 2$ there exists $\xi < 2$ and $\eta > 0$ such that if $\varepsilon \leqq \eta$ then $H(\varepsilon) \leqq (\varepsilon^{-1})^{\xi}$. But it is easy to check that

$$\int_0^1 [\log (r^{-1})(r^{-1})^{\xi}]^{\frac{1}{2}}\, dG(r) < \infty$$

and so

$$\int_0^1 [\log (r^{-1}) H(2\alpha r)]^{\frac{1}{2}}\, dG(r) < \infty\,.$$

The result thus follows from Lemma 4.

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