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AN ELEMENTARY THEOREM ON THE PROBABILITY OF LARGE DEVIATIONS

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1. Introduction. Let $\{X_n\}_{n=1}^\infty$ be a sequence of real-valued random variables and let $\{\varphi_n\}_{n=1}^\infty$ be a sequence of real numbers such that $\lim_{n\to\infty} n^{-1} \log P(X_n \geqq \varphi_n) = c$, where c is some negative constant. The evaluation of this limit is of great importance in the calculation of certain limiting efficiencies of tests of statistical hypothesis [1], [2], [6].

Several approaches have been taken in the evaluation of this limit. In some cases it has been evaluated directly, which often leads to an exceedingly difficult analysis [8]. In other cases it has been found using a moment generating function technique [2], [12]. The approach taken in this paper is entirely different and, when applicable, has important advantages over previous attempts. Using the method proposed, we are able to calculate the above limit with very little effort and the approach offers some interesting insights into the structure of the problem. In this paper, we consider conditions under which the above limit can be evaluated using the density or probability function rather than the distribution function.

For example, if X_n is normally distributed with mean 0 and variance n, then it is well known [3] that, for $a \geqq 0$,

$$\lim_{n\to\infty} n^{-1} \log P(X_n \geqq an) = -a^2/2 . \tag{1.1}$$

It is easy to verify that, likewise for $a \geqq 0$

$$\lim_{n\to\infty} n^{-1} \log f_n(an) = -a^2/2 , \tag{1.2}$$

where $f_n(x)$ is the density of X_n. We point out that the equality of the two limits in (1.1) and (1.2) holds quite generally. Sufficient conditions for equality are given and discussed in Sections 2, 3, and 4, and some examples illustrative of our method are given in Section 5.

2. The main results. Let $\{X_n\}_{n=1}^\infty$ be a sequence of random variables and $\{f_n\}_{n=1}^\infty$ a sequence of real-valued functions defined on the real line. Let $\{\delta_n\}_{n=1}^\infty$, $\{\varphi_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ be nonnegative sequences of real numbers with $n^{-1} \log \delta_n = o(1)$ and $n^{-1} \log \gamma_n = o(1)$ as $n \to \infty$.

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THEOREM 2.1. *Suppose for each* $n = 1, 2, \cdots$ *that* X_n *has an absolutely continuous distribution with density* $f_n(x)$. *If there exists an integer* N *such that for* $n \geqq N$, $f_n(x)$ *is non-increasing for* x *in* $[\varphi_n, \infty)$ *and if*

$$n^{-1} \log [f_n(\varphi_n + \delta_n)/f_n(\varphi_n)] = o(1) \quad as \quad n \to \infty \tag{2.1}$$

and

$$\limsup_{n\to\infty} n^{-1} \log [P(X_n \geqq \gamma_n)/f_n(\varphi_n)] \leqq 0 , \tag{2.2}$$

then

$$n^{-1} \log f_n(\varphi_n) - n^{-1} \log P(X_n \geqq \varphi_n) = o(1) \quad as \quad n \to \infty . \tag{2.3}$$

THEOREM 2.2. *Suppose for each* $n = 1, 2, \cdots$ *that* X_n *is an integer-valued random variable with probability function* $p_n(k) = P(X_n = k)$ *and* $f_n(x) = p_n([x])$ *where* [ ] *is the greatest integer function. If the above conditions are satisfied with* $\delta_n = 1$ *for* $n = 1, 2, \cdots$, *then* (2.3) *again follows.*

Often, in examples, we find $\lim_{n\to\infty} n^{-1} \log f_n(\varphi_n) = c > -\infty$. The sequence $\{\gamma_n\}_{n=1}^{\infty}$ can often be defined by $\gamma_n = \exp(n^{\frac{1}{2}})$. The existence of the above limit along with

$$\lim_{n\to\infty} n^{-1} \log P(X_n \geqq \exp(n^{\frac{1}{2}})) = -\infty , \tag{2.4}$$

imply condition (2.2) with $\gamma_n = \exp(n^{\frac{1}{2}})$. When the remaining conditions are satisfied, $\lim_{n\to\infty} n^{-1} \log P(X_n \geqq \varphi_n)$ exists and is equal to c.

We conclude this section with a sufficient condition for (2.4) in terms of moment generating functions.

THEOREM 2.3. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of random variables such that there exists a* $\tau > 0$ *and* $M < \infty$ *such that* $E(\exp[\tau X_n]) < M$ *for all* n. *Then* (2.4) *holds.*

3. Proof of theorems.

PROOF OF THEOREM 2.1. Using the fact that $n^{-1} \log \gamma_n = o(1)$ as $n \to \infty$ and condition (2.2) we have,

$$\begin{aligned} n^{-1} &\log f_n(\varphi_n) - n^{-1} \log P(X_n \geqq \varphi_n) \\ &= n^{-1} \log [f_n(\varphi)\gamma_n/P(X_n \geqq \varphi)] + o(1) \\ &= n^{-1} \log [(f_n(\varphi_n)\gamma_n + P(X_n \geqq \gamma_n))/P(X_n \geqq \varphi_n)] \\ &\qquad - n^{-1} \log(1 + P(X_n \geqq \gamma_n)/f_n(\varphi_n)\gamma_n) + o(1) \\ &= n^{-1} \log [(f_n(\varphi_n)\gamma_n + P(X_n \geqq \gamma_n))/P(X_n \geqq \varphi_n)] + o(1) \quad \text{as} \quad n \to \infty . \end{aligned}$$

The non-increasing property of $f_n(x)$ implies that $f_n(\varphi_n)\gamma_n + P(X_n \geq \gamma_n)$ is an upper bound for $P(X_n \geqq \varphi_n)$ for sufficiently large n. It now follows that

$$n^{-1} \log f_n(\varphi_n) - n^{-1} \log P(X_n \geqq \varphi_n) \geqq o(1) \quad \text{as} \quad n \to \infty . \tag{3.1}$$

Using the non-increasing property of $f_n(x)$ along with condition (2.1) yields,

$$n^{-1}\log f_n(\varphi_n) - n^{-1}\log P(X_n \geqq \varphi_n)$$
$$= n^{-1}\log [f_n(\varphi_n + \delta_n)\delta_n/P(X_n \geqq \varphi_n)] + o(1) \quad \text{as} \quad n \to \infty .$$

The non-increasing property of $f_n(x)$ further implies that $f_n(\varphi_n + \delta_n)\delta_n$ is a lower bound for $P(X_n \geqq \varphi_n)$ for sufficiently large n and we have,

$$\text{(3.2)} \qquad n^{-1}\log f_n(\varphi_n) - n^{-1}\log P(X_n \geqq \varphi_n) \leqq o(1) \quad \text{as} \quad n \to \infty .$$

The proof is completed by combining inequalities (3.1) and (3.2).

The proof of Theorem 2.2 follows in a similar manner.

PROOF OF THEOREM 2.3. Since $P(X_n \geqq \gamma_n) = P(\exp(\tau[X_n - \gamma_n]) \geqq 1) \leqq E(\exp \tau([X_n - \gamma_n])) \leqq M \exp[-\tau\gamma_n]$ for all n. It now follows from the definition of γ_n that (2.4) holds.

4. Examples on the necessity of the conditions. Each of the conditions places necessary restrictions on a different part of the tail of the distribution of X_n. In the examples of this section we let $\delta_n = 1$, $\gamma_n = \exp(n^{\frac{1}{2}})$, and $\varphi_n = n^{\frac{1}{2}}$.

EXAMPLE 4.1. Condition (2.2) insures that there is not too much probability in the extreme tail of the distribution of X_n, that is, for $x > \gamma_n$. For a counter-example to the theorem if this condition is relaxed, let X_n have density $f_n(x)$ defined by:

$$\begin{aligned} f_n(x) &= 0 \quad \text{if} \quad x < 0 , \\ &= (2\pi)^{-\frac{1}{2}} \exp(-x^2/2) \quad \text{if} \quad 0 \leqq x \leqq \gamma_n , \\ &= (2\pi)^{-\frac{1}{2}} \exp(-\gamma_n^2/2) \quad \text{if} \quad \gamma_n < x < \gamma_n + (\pi/2)^{\frac{1}{2}} \exp(\gamma_n^2/2) , \\ &= (2\pi)^{-\frac{1}{2}} \exp\{-[x - (\pi/2)^{\frac{1}{2}} \exp(\gamma_n^2/2)]^2/2\} \quad \text{otherwise.} \end{aligned}$$

Note that for large n we have $f_n(n^{\frac{1}{2}}) = (2\pi)^{-\frac{1}{2}} \exp(-n/2)$ and the remaining conditions are almost trivially valid. Further, $P(X_n \geqq \gamma_n) > \frac{1}{2}$ for all $n = 1, 2, \cdots$ which implies that $\lim_{n\to\infty} n^{-1}P(X_n \geqq n^{\frac{1}{2}}) = 0$ while $\lim_{n\to\infty} n^{-1} \log f_n(n^{\frac{1}{2}}) = -\frac{1}{2}$; hence, condition (2.2) and equation (2.3) fail.

EXAMPLE 4.2. The non-increasing property insures that f_n behaves properly in the medium tail, that is for $\varphi_n \leqq x < \gamma_n$. Again, the theorem is not true without this condition, as can be seen by considering X_n with density $f_n(x)$ defined by:

$$\begin{aligned} f_n(x) &= 0 \quad \text{if} \quad x < 0 , \\ &= (2\pi)^{-\frac{1}{2}} \exp(-x^2/2) + 1/2 \quad \text{if} \quad \exp(n^{\frac{1}{2}}) < x < \exp(n^{\frac{1}{2}}) + 1 , \\ &= (2\pi)^{-\frac{1}{2}} \exp(-x^2/2) \quad \text{otherwise.} \end{aligned}$$

Clearly, conditions (2.1) and (2.2) are satisfied but not the non-increasing property, and again we have $\lim_{n\to\infty} n^{-1} \log P(X_n \geqq n^{\frac{1}{2}}) = 0$ while $\lim_{n\to\infty} n^{-1} \log f_n(n^{\frac{1}{2}}) = -\frac{1}{2}$.

EXAMPLE 4.3. To demonstrate the necessity of (2.1) let X_n be a normal random variable with mean 0 and variance 1. Further, let $f_n(n^{\frac{1}{2}}) = 1$, which does not change the distribution of X_n but does contradict condition (2.1) for any $\delta_n > 0$. Now $\lim_{n\to\infty} n^{-1} \log f_n(n^{\frac{1}{2}}) = 0$.

5. Examples of large deviations. The large deviation results for a wide class of distributions may be readily obtained by direct application of Theorem 2.1 or 2.2. For example, if X_n has any one of the distributions: binominal with parameters n and p, Poisson with parameter $n\lambda$, chi square with n degrees of freedom or Student's t with n degrees of freedom, with $\delta_n = 1$ and φ_n suitably chosen, then routine calculations will yield the desired limit. Two of the more interesting examples are presented here in detail.

EXAMPLE 5.1. Let X_n have an F distribution with m and n degrees of freedom, where m and n increase in such a way that m/n tends to λ and $0 < \lambda < \infty$. The particular case where $m = n$ has already been resolved by Klotz [8]. We will let $\varphi_n = \tau$ where $\tau > 1$ and fixed. It will now be shown that conditions of Theorem 2.1 are satisfied if we let $\delta_n = n^{-\frac{1}{2}}$ and $\gamma_n = \exp(n^{\frac{1}{2}})$.

$$P(X_n \geqq \gamma_n)$$
$$= \int_{\gamma_n}^{\infty} [\Gamma((m+2)/2)(m/n)^{m/2} y^{(m-2)/2}\}/\{\Gamma(m/n)\Gamma(n/2)(1+my/n)^{(m+n)/2}\}\, dy\,.$$

Using Stirling's formula on the gamma functions in the integrand and disregarding higher order terms we get:

$$\limsup_{n\to\infty} n^{-1} \log P(X_n \geqq \gamma_n)$$
$$= \limsup_{n\to\infty} n^{-1} \log \int_{\gamma_n}^{\infty} ((m+n)/(n+my))^{(m+n)/2} y^{m/2}\, dy\,.$$

Since we have, for large n,

$$(m+n)/(n+my) \leqq (m+n)/my \leqq (m+2m\lambda^{-1})/my = (1+2\lambda^{-1})/y\,,$$

by letting $k = 1 + 2\lambda^{-1}$, it follows that

$$\limsup_{n\to\infty} n^{-1} \log P(X_n \geqq \gamma_n) \leqq \limsup_{n\to\infty} n^{-1} \log \int_{\gamma_n}^{\infty} (k/y)^{(m+n)/2} y^{m/2}\, dy$$
$$= \lim_{n\to\infty} n^{-1} \log [((n-2)/2) k^{(m+n)/2} \gamma_n{}^{-(n-2)/2}] = -\infty\,.$$

Therefore, $f_n(x)$ satisfies (2.4). It is easy to verify that $(n(m-2))/(m(n+2)) < 1$, $f_n(x)$ is decreasing for all $x > (n(m-2))/m(n+2))$, and $\lim_{n\to\infty} (n(m-2))/(m(n+2)) = 1$. This implies that $f_n(x)$ satisfies the non-increasing property. With $\delta_n = n^{-\frac{1}{2}}$ a direct calculation will verify condition (2.1). Applying Stirling's formula

$$(5.1)\quad \lim_{n\to\infty} n^{-1} \log f_n(\tau) = \lim_{n\to\infty} n^{-1} \log \{[(1+\lambda)/(1+\lambda\tau)]^{(1+\lambda)n/2} \tau^{\lambda n/2}\}$$
$$= 2^{-1}[(1+\lambda)\log[(1+\lambda)/(1+\lambda\tau)] + \lambda \log \tau]\,.$$

Now (5.1) and (2.4) imply that condition (2.2) is satisfied and hence $\lim_{n\to\infty} n^{-1} \log P(X_n \geqq \gamma)$ exists and is given by (5.1).

EXAMPLE 5.2. Let X_n have a hypergeometric distribution where

$$P(X_n = x) = \binom{r}{x}\binom{N-r}{n-x}\Big/\binom{N}{n} \qquad x = 0, 1, \cdots, \min(r, n),$$
$$= 0 \qquad \text{otherwise.}$$

Further, r and N are positive integers depending on n such that $\lim_{n\to\infty} n/N = \lambda$, $\lim_{n\to\infty} r/N = p$, and $n \leqq r$ for all n. We also assume $0 < p < 1$, $\lambda < p$ and $\lambda < 1 - p$. Let $\varphi_n = n\tau$, where $p < \tau < 1$.

Clearly (2.2) is satisfied. To check the non-increasing property, let $x > \varphi_n = n\tau$ and note we need only consider $x < n$. For sufficiently large n

$$\begin{aligned} &p(X_n = [x] + 1)/P(X_n = [x]) \\ &\quad = \{(r - [x])(n - [x])\}/\{([x] + 1)(N - r - n + [x] + 1)\} \\ &\quad < (p - \lambda\tau)/(\tau - \lambda\tau) < 1 \end{aligned}$$

and the condition is satisfied. Now

$$\begin{aligned} &\lim_{n\to\infty} n^{-1} \log \{P(X_n = [n\tau] + 1)/P(X_n = [n\tau])\} \\ &\quad = \lim_{n\to\infty} n^{-1} \log (\{(r - [n\tau])(n - [n\tau])\}/\{([n\tau] + 1)(N - r - n + [n\tau] + 1)\}), \\ &\quad = \lim_{n\to\infty} n^{-1} \log [\{(p - \lambda\tau)(\lambda - \lambda\tau)\}/\{\lambda\tau(1 - p - \lambda + \lambda\tau)\}] = 0, \end{aligned}$$

and (2.1) is satisfied. Therefore (2.3) holds. Now it remains to show $\lim_{n\to\infty} n^{-1} \log f_n(n\tau)$ exists and to calculate it. We use the following: If $\lim_{n\to\infty} a/n = \alpha$, $\lim_{n\to\infty} b/n = \beta$, $0 < \beta < \alpha < \infty$ where a, b are integers, then it follows from Stirling's formula that

$$(5.2) \qquad \lim_{n\to\infty} n^{-1} \log \binom{a}{b} = \beta \log (\alpha/\beta) + (\alpha - \beta) \log (\alpha/(\alpha - \beta)).$$

We wish to evaluate

$$\lim_{n\to\infty} n^{-1} \log P(X_n = [n\tau]) = \lim_{n\to\infty} n^{-1} \log\left\{\binom{r}{[n\tau]}\binom{N-r}{n-[n\tau]}\Big/\binom{N}{n}\right\}.$$

By (5.2)

$$(5.3) \qquad \lim_{n\to\infty} n^{-1} \log \binom{r}{[n\tau]} = -\tau \log (\lambda\tau/p) - ((p/\lambda) - \tau) \log (1 - \lambda\tau/p),$$

$$(5.4) \qquad -\lim_{n\to\infty} n^{-1} \log \binom{N}{n} = \log \lambda + \{(1 - \lambda)/\lambda\} \log (1 - \lambda),$$

$$\begin{aligned} (5.5) \qquad \lim_{n\to\infty} n^{-1} \log \binom{N-r}{n-[n\tau]} &= (1 - \tau) \log \{\lambda(1 - \tau)/(1 - p)\} \\ &\quad - \{((1 - p)/\lambda) - 1 + \tau\} \log (1 - \lambda(1 - \tau)/(1 - p)). \end{aligned}$$

Summing (5.3)—(5.5) gives

$$\begin{aligned} (5.6) \quad \lim_{n\to\infty} n^{-1} \log P(X_n \geqq n\tau) &= -\tau \log (\lambda\tau/p) - ((p/\lambda) - \tau) \log (1 - \lambda\tau/p) \\ &\quad + \log \lambda + ((1 - \lambda)/\lambda) \log (1 - \lambda) - (1 - \tau) \log (\lambda(1 - \tau)/(1 - p)) \\ &\quad - \{((1 - p)/\lambda) - 1 + \tau\} \log (1 - \lambda(1 - \tau)/(1 - p)). \end{aligned}$$

6. An application of large deviations. In this section we apply the methods of Example 5.2 to determine the Bahadur exact efficiency of Mathisen's median test [9] with respect to Mood's median test [10]. The Bahadur exact efficiency is the ratio of exact slopes; for a discussion and definitions pertaining to Bahadur efficiency see reference [1].

Let $Y_1 < \cdots < Y_j$ and $Z_1 < \cdots < Z_k$ be the order statistics of samples of size j and k from absolutely continuous distributions with distribution functions $G(y)$ and $G(z-\theta)$, respectively. For testing the null hypothesis H_0: $\theta = 0$ against the alternative hypothesis H_1: $\theta > 0$, Mood proposed the statistic M, the number of items of the Z sample that exceed the median of the combined sample, and Mathisen suggested the statistic S_z, the number of items of the Z sample that exceed the median of the Y sample. When the sample size is even, the sample median is taken to be the average of the two middle order statistics.

The test defined by M is asymptotically optimum [5, p. 88] when G is the double exponential distribution. Exact slopes (times $\frac{1}{2}$) of M for the normal, logistic and double exponential distributions have been tabulated by Woodworth [13]. Gastwirth [4] has shown that in the case of curtailed sampling, the test based on S_z always reaches a decision before the test based on M. When the Pitman efficiency [11] of S_z with respect to M exists, it is equal to 1 and hence does not indicate which test is preferable.

Let $j = 2j^* + 1$, $j + k = N$, $k/N = \rho$, $0 < \rho < 1$. Then S_z = number of times $Z_\xi > Y_{j^*+1}$, $\xi = 1, 2, \cdots, k$ and

$$P(S_z = t) = \binom{j}{j^*}\binom{k}{t}(j^*+1)\Big/\binom{j+k}{j^*+k-t}(j^*+t+1)\,, \qquad t = 0, 1, \cdots k \tag{6.1}$$
$$= 0 \qquad \text{elsewhere.}$$

TABLE 1

$\frac{1}{2}$ Exact slope of Mathisen's test for normal shift alternatives

$\theta \backslash \rho$	1/2	1/4	1/8	1/16
0.25	0.004928	0.003710	0.002169	0.001163
0.50	0.019176	0.01460	0.008590	0.004624
0.75	0.041152	0.03189	0.01898	0.01028
1.00	0.06839	0.05426	0.03282	0.01796
1.25	0.09804	0.07997	0.04949	0.02740
1.50	0.1269	0.1066	0.06755	0.03814
1.75	0.1526	0.1319	0.08588	0.04957
2.00	0.1737	0.1541	0.1030	0.06099
2·25	0.1894	0.1715	0.1175	0.07131
2.50	0.2003	0.1842	0.1286	0.07983
2.75	0.2072	0.1925	0.1363	0.08612
3.00	0.2115	0.1979	0.1415	0.09058
3.25	0.2136	0.2007	0.1442	0.09302
3.50	0.2148	0.2021	0.1458	0.09444
3.75	0.2154	0.2029	0.1466	0.09517
4.00	0.2156	0.2033	0.1469	0.09550
∞	0.2158	0.2035	0.1472	0.09574

Using the techniques of Example 5.2,

(6.2) $$\lim_{n\to\infty} - N^{-1}\log P(S_z \geqq N\tau)$$
$$= -(1-\rho)\log 2 - \rho\log\rho + \tau\log\tau + (\rho-\tau)\log(\rho-\tau)$$
$$- [1-\tau-(1-\rho)/2]\log[1-\tau-(1-\rho)/2]$$
$$- [\tau+(1-\rho)/2]\log[\tau+(1-\rho)/2]\,.$$

Further, when $\theta > 0$ obtains, S_z/N converges in probability to $\rho(1 - G(-\theta))$. In the following discussion we assume G is a symmetric distribution function and hence $\rho(1 - G(-\theta)) = \rho G(\theta)$.

If we denote the exact slope of S_z by $C_z(\theta, \rho)$, then substituting $\rho G(\theta)$ for τ in equation (6.2) yields, after some manipulation

(6.3) $$C_z(\theta,\rho) = \rho h(2G(\theta)-1) - h(\rho[2G(\theta)-1])$$

where $h(x) = (1+x)\ln(1+x) + (1-x)\ln(1-x)$ and $0 < x < 1$.

TABLE 2

$\frac{1}{2}$ Exact slope of Mathisen's test for logistic shift alternatives

$\theta\backslash\rho$	1/2	1/4	1/8	1/16
0.50	0.007633	0.005758	0.003371	0.001809
1.00	0.02854	0.02188	0.01294	0.006983
1.50	0.05774	0.04538	0.02727	0.01486
2.00	0.08954	0.07245	0.04447	0.02456
2.50	0.1194	0.09957	0.06263	0.03518
3.00	0.1449	0.1241	0.08013	0.04591
3.50	0.1651	0.1449	0.09579	0.05607
4.00	0.1804	0.1614	0.1090	0.06513
4.50	0.1915	0.1739	0.1195	0.07281
5.00	0.1993	0.1830	0.1276	0.07901
5.50	0.2048	0.1895	0.1335	0.08380
6.00	0.2085	0.1941	0.1378	0.08738
6.50	0.2110	0.1972	0.1408	0.08999
7.00	0.2126	0.1993	0.1429	0.09183
7.50	0.2137	0.2008	0.1443	0.09311
8.00	0.2144	0.2017	0.1453	0.09399
8.50	0.2149	0.2023	0.1459	0.09458
9.00	0.2152	0.2027	0.1464	0.09498
9.50	0.2154	0.2030	0.1466	0.09525
10.00	0.2155	0.2032	0.1468	0.09542
10.50	0.2156	0.2033	0.1469	0.09554
11.00	0.2157	0.2034	0.1470	0.09561
11.50	0.2157	0.2034	0.1471	0.09566
12.00	0.2157	0.2034	0.1471	0.09569
12.50	0.2157	0.2035	0.1471	0.09571
13.00	0.2157	0.2035	0.1471	0.09572
13.50	0.2158	0.2035	0.1471	0.09573
∞	0.2158	0.2035	0.1472	0.09574

Let S_y = number of times $Y_\xi < Z_{k^*+1}$, $\xi = 1, 2, \cdots, j$ where $k = 2k^* + 1$. Then a similar argument shows the exact slope of S_y to be

$$(6.4)\qquad C_y(\theta, \rho) = (1 - \rho)h(2G(\theta) - 1) - h([1 - \rho][2G(\theta) - 1]) = C_z(\theta, 1 - \rho) .$$

It is shown in the following paragraph that if G is a symmetric distribution function and $0 < \rho < \frac{1}{2}$, then

$$(6.5)\qquad C_y(\theta, \rho) > C_y(\theta, 1 - \rho) .$$

It follows from (6.4) and (6.5), under the same assumptions, that

$$(6.6)\qquad C_y(\theta, \rho) > C_z(\theta, \rho) .$$

TABLE 3

$\frac{1}{2}$ *Exact slope of Mathisen's test for double exponential shift alternatives*

$\theta\backslash\rho$	1/2	1/4	1/8	1/16
0.50	0.02029	0.01546	0.009103	0.004902
1.00	0.05710	0.04485	0.02694	0.01468
1.50	0.09422	0.07658	0.04717	0.02611
2.00	0.1262	0.1059	0.06706	0.03784
2.50	0.1516	0.1309	0.08510	0.04907
3.00	0.1708	0.1509	0.1005	0.05925
3.50	0.1848	0.1662	0.1130	0.06803
4.00	0.1947	0.1776	0.1227	0.07526
4.50	0.2016	0.1857	0.1300	0.08096
5.00	0.2063	0.1915	0.1353	0.08529
5.50	0.2095	0.1954	0.1391	0.08848
6.00	0.2117	0.1981	0.1417	0.09077
6.50	0.2131	0.2000	0.1435	0.09238
7.00	0.2140	0.2012	0.1448	0.09349
7.50	0.2146	0.2020	0.1456	0.09426
8.00	0.2150	0.2025	0.1461	0.09475
8.50	0.2153	0.2029	0.1465	0.09510
9.00	0.2155	0.2031	0.1467	0.09532
9.50	0.2156	0.2032	0.1469	0.09547
10.00	0.2156	0.2033	0.1470	0.09556
10.50	0.2157	0.2034	0.1470	0.09563
11.00	0.2157	0.2034	0.1471	0.09567
11.50	0.2157	0.2034	0.1471	0.09569
12.00	0.2157	0.2035	0.1471	0.09571
12.50	0.2157	0.2035	0.1471	0.09572
13.00	0.2158	0.2055	0.1471	0.09573
13.50	0.2158	0.2035	0.1471	0.09573
14.00	0.2158	0.2035	0.1471	0.09574
14.50	0.2158	0.2035	0.1471	0.09574
15.00	0.2158	0.2035	0.1471	0.09574
∞	0.2158	0.2035	0.1472	0.09574

Hence, the more efficient of the two forms of Mathisen's test uses the median of the smaller sample.

Define for $\theta > 0$, fixed,

$$e(\theta, \rho) = C_y(\theta, \rho) - C_y(\theta, 1 - \rho)$$
$$= (1 - 2\rho)h(\beta) - ([1 - \rho]\beta) + h(\rho\beta)$$

where $\beta = 2G(\theta) - 1$ is fixed between 0 and 1. Now $e(\theta, 0^-) = e(\theta, \frac{1}{2}^+) = 0$ and $h''(x) = (1 - x^2)^{-1}$ imply that $e(\theta, \rho)$ is a strictly concave function of ρ for every fixed $\theta > 0\cdot$ Hence $0 \leqq e(\theta, \rho)$ for all $0 < \rho < \frac{1}{2}$, for every fixed $\theta > 0$.

Tables 1—3 give the exact slope (times $\frac{1}{2}$) of Mathisen's S_y for various

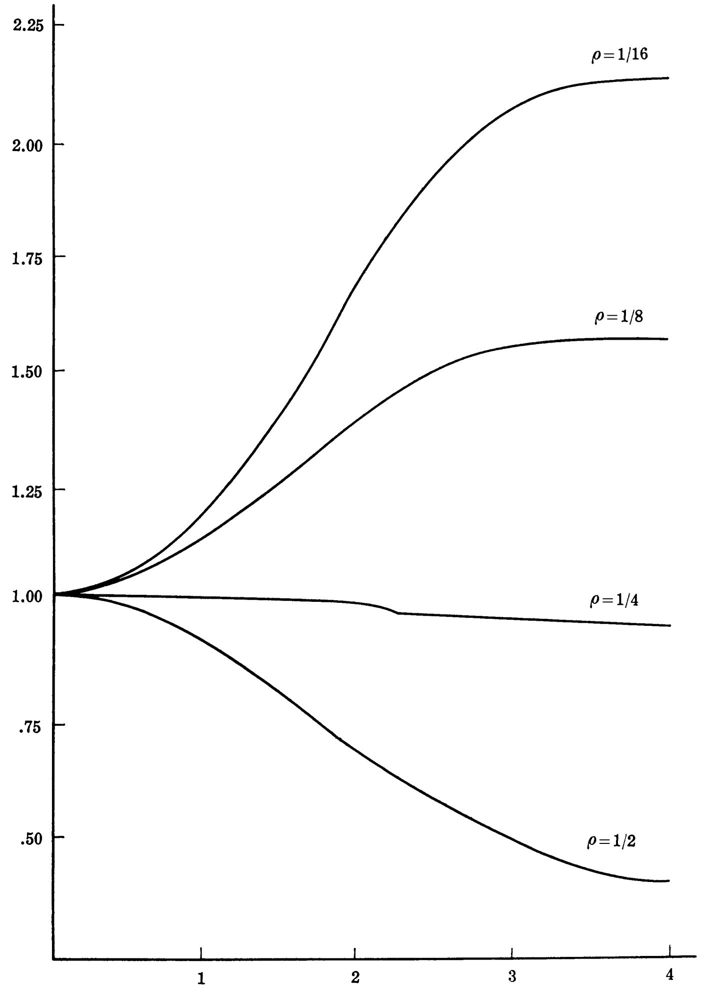

FIG. 1. Bahadur efficiencies for normal shift alternatives

values of $\rho < \frac{1}{2}$ corresponding to Woodworth's tables [13]. The exact Bahadur efficiency of the appropriate form of Mathisen's test with respect to Mood's M is shown in graphs 1—3 and indicates for the normal, logistic and double exponential distributions that M is preferable for values of ρ around $\frac{1}{2}$ but the appropriate form of Mathisen's test is preferable for the more extreme values of ρ.

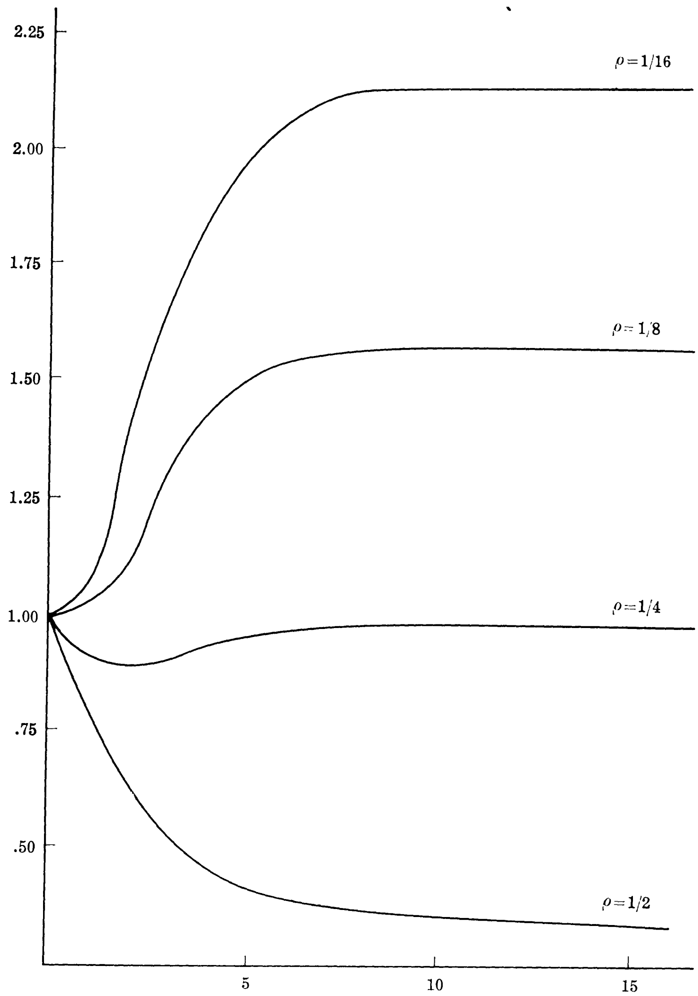

FIG. 2. Bahadur efficiencies for logistic shift alternatives

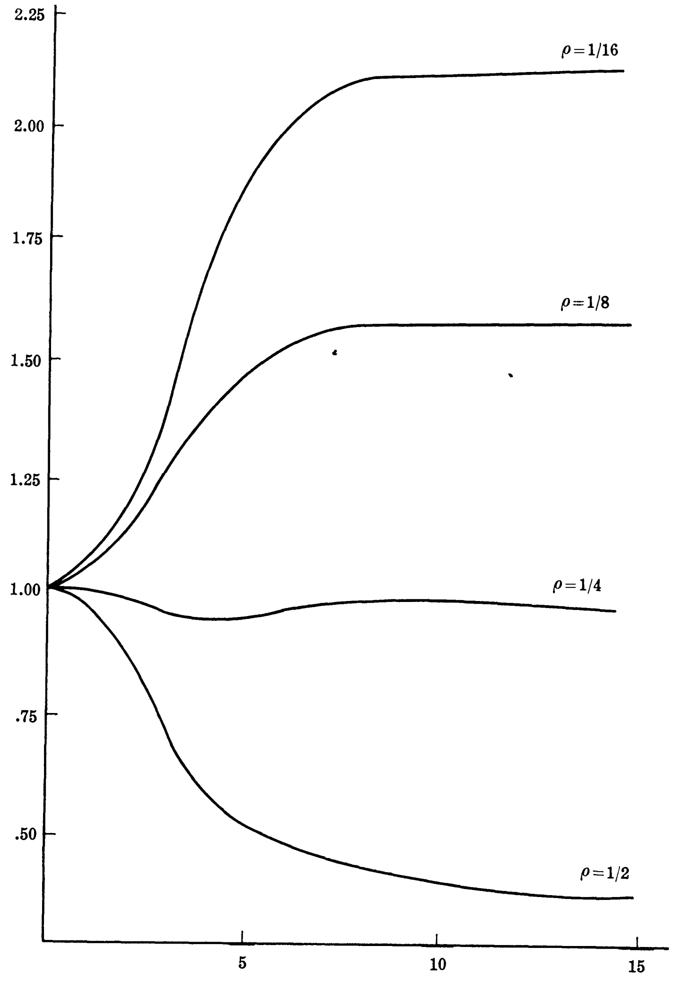

FIG. 3. Bahadur efficiencies for double exponential shift alternatives

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