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# LINEARIZED RANK ESTIMATES AND SIGNED-RANK ESTIMATES FOR THE GENERAL LINEAR HYPOTHESIS[1]

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Linearized estimates, as functions of the ranks, are proposed for the general linear hypothesis. These estimates can be computed after a single ranking of the "centered" observations. The asymptotic distribution of the estimates is shown to be the same as the maximum likelihood estimates for fairly general sequences of design matrices.

**1. Introduction.** The development of methods of estimation from ranks for the parameters of the general linear hypothesis has proceeded rapidly since the work of Hodges and Lehmann (1963) on estimates for one-sample and two-sample problems. Univariate extensions of these estimates to $k$-sample problems have been given by Lehmann (1963), and Bhuchongkul and Puri (1965); to linear regression by Adichie (1967); and to regression on monotone functions by Rao and Thornby (1969). Koul (1971) studied rank estimates for a wide class of sequences of design matrices which are assumed to be perpendicular to a vector of constants. He used an approximation theorem of Jurečková (1969) for some of the asymptotic properties. In (1969), (1971) the present authors utilized the theorem of Jurečková to study linearized versions of rank estimates for one- and two-, sample problems. These linearized versions are, in most cases, simpler to compute as well as asymptotically equivalent to the non-linearized versions.

In the present paper linearized rank estimates are described for a sub-class of the sequence of design matrices studied by Koul (1971). When Koul's estimates exist the estimates here can be considered as their linearized versions. However, the proofs given here do not require their existence. Linearized signed-rank estimates are given for an analogous sequence of designs and supposing the observations have symmetric distributions. Koul (1969) has studied estimates based on signed-rank statistics for more general sequences of designs but with stronger assumptions on the distributions of the observations.

The sequences of design matrices considered here have, at least asymptotically, fixed rank. Thus, the results do not apply to sequences of designs in which there are an increasing number of nuisance parameters as well as a fixed

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number of parameters of interest. Some of the recent results concerning rank estimates for these more complicated designs can be found in Lehmann (1964), Greenberg (1966), and Puri and Sen (1967).

Section 2 contains the assumptions and theorems concerning estimates based on rank statistics. Section 3 contains the same for estimates based on signed-rank statistics. The results of these two sections require certain initial estimates and estimates of scale. Theorems establishing the existence and construction of such estimates are given in Section 4. Section 5 contains the proofs of the theorems in Section 2 and of those in Section 4 concerning estimates based on rank statistics. Section 6 contains the proofs of the theorems in Sections 3 and 4 concerning estimates based on signed-rank statistics.

The basis of linearized estimates for a single parameter is the fundamental theorem of Jurečková (1969). Section 7 gives a particular extension of this theorem to multiparameter problems for rank-statistics and a multiparameter extension of van Eeden's (1971) analogue, for signed-rank statistics, of Jurečková's theorem.[2]

**2. Linearized rank estimates.** Suppose that, for each $\nu = 1, 2, \cdots$, for an $n_\nu \times 1$ vector of observations $Y^{(\nu)} = (Y_1^{(\nu)}, \cdots, Y_{n_\nu}^{(\nu)})'$, there exists an $n_\nu \times (p+q)$ design matrix, $Z^{(\nu)}$, of known constants and a $(p+q) \times 1$ vector $\beta$ of unknown constants such that the components of $Y^{(\nu)} - Z^{(\nu)}\beta$ are independently and identically distributed as $F(y/b)$ $(b > 0)$ where $F(y)$ is a completely specified distribution function. $p$ and $q$ will be fixed and limits will be as $\nu \to \infty$. (Super- and subscripts $\nu$ will not be written).

The following standard reduction of the parameters will be convenient. For the sequence of design matrices, $Z$, let $Z - \bar{Z} = (Z_{ij} - n^{-1} \sum_{i=1}^n Z_{ij})$ and let $p$ be the rank of $Z - \bar{Z}$. Then, if $Z_1 - \bar{Z}_1$ is a set of $p$ linearly independent columns of $Z - \bar{Z}$ and $Z_2 - \bar{Z}_2$ is the rest of the columns of $Z - \bar{Z}$, $Z - \bar{Z}$ can (after, if necessary, rearranging some of the columns) be written as $Z - \bar{Z} = (Z_1 - \bar{Z}_1, Z_2 - \bar{Z}_2)$ where $Z_1 - \bar{Z}_1$ is of size $n \times p$ and rank $p$ and $Z_2 - \bar{Z}_2 = (Z_1 - \bar{Z}_1)c$, where $c$ is a $p \times q$ matrix. Hence $Z\beta = (Z_1 - \bar{Z}_1) \times (\beta_1 + c\beta_2) + (\bar{Z}_1\beta_1 + \bar{Z}_2\beta_2)$, where $\beta = (\beta_1', \beta_2')'$ corresponds to $Z = (Z_1, Z_2)$. Let $(Z_1 - \bar{Z}_1)(\beta_1 + c\beta_2) + (\bar{Z}_1\beta_1 + \bar{Z}_2\beta_2) = (Z_1 - \bar{Z}_1)\theta + \theta_0$ with $\theta_0$ a vector of constants and $\theta$ the parameter to be estimated.

The distribution function $F$ of single observations will be assumed to satisfy the regularity conditions of Hájek and Šidák [4], namely

[2] We thank the referee for pointing out to us that Jurečková (1971) has recently obtained extensions, to multiparameter models, of her theorem and used these extensions to find estimates based on rank tests. Her conditions for rank tests are more general than those for rank tests given here. The linearizations given here do obtain under her conditions.

ASSUMPTION A.

(i) $f(y) = dF(y)/dy$ exists and is absolutely continuous on $(-\infty, \infty)$

(ii) the function $\varphi_F(u) = -(f'/f)(F^{-1}(u))$ can be written as the sum of two monotone functions each of which is square integrable on $0 < u < 1$.

Let any two vectors $u$ and $v$ be called similarly ordered if $(u_i - u_j)(v_i - v_j) \geqq 0$ for all $i$, $j$. For the sequence $Z$ of design matrices let $z = Z_1 - \bar{Z}_1$. It is supposed that the sequence $\{z = (z_{ij}^{(\nu)})\}$ satisfies

ASSUMPTION B.

(i) $\dfrac{\max_{1 \leq i \leq n} z_{ij}^2}{\sum_{i=1}^n z_{ij}^2} \to 0 \quad j = 1, \cdots, p,$

(ii) $\dfrac{1}{n} z'z \to \Sigma$ where $\Sigma$ is positive definite,

(iii) For each $j_1, j_2 (j_1 \neq j_2, j_1, j_2 = 1, \cdots, p)$ there exists a number $\gamma_{j_1 j_2} \neq 0$ such that, for $n > n_0$, $z_{j_1}$ and $z_{j_1} + \gamma_{j_1, j_2} z_{j_2}$ are similarly ordered, where $z_1, \cdots, z_p$ are the column vectors of $z$.

ASSUMPTION C. It will be supposed that there exists a sequence $\hat{\theta}_1$ of initial estimates of $\theta$ which satisfies

(i) $\hat{\theta}_1\left(\dfrac{Y - z\theta}{a}\right) = \dfrac{\hat{\theta}_1(Y) - \theta}{a}$ for all $\theta$ and all $a > 0$,

(ii) $P_\theta\{n^{\frac{1}{2}}(\hat{\theta}_1 - \theta) \in A\} \to P(A)$ for some fixed $p$-dimensional distribution $P$.

Note that C(i) is satisfied for the least squares estimates $\hat{\theta}_1$ and that, under Assumption B(i) and (ii), C(ii) will also be satisfied if $\int y^2 dF(y) < \infty$. In Section 4 a class of designs is given for which a sequence $\hat{\theta}_1$ satisfying C can be constructed from certain medians.

Define now an $n \times 1$ vector

$$\Phi_F(\theta) = \left\{\varphi_F\left(\frac{R_{(Y-z\theta)_i}}{n+1}\right)\right\},$$

where $R_{(Y-z\theta)_i}$ is the rank of the $i$th component of $Y - z\theta$ among all $n$ components. A linearized rank estimate $\hat{\theta}$ will be defined by

$$\hat{\theta} = \hat{\theta}_1 + \frac{\hat{b}}{K_{FF}} (z'z)^{-1} \Phi_F(\hat{\theta}_1), \tag{2.1}$$

where $K_{FF} = \int_0^1 \varphi_F^2(u) du$ and where $\hat{b}$ is a consistent estimate of the scale parameter $b$.

In Section 5 the following theorem will be proved.

THEOREM 2.1. *If the components of $Y - Z\beta$ have common distribution function $F(y/b)$, if $F$ satisfies* A, *if $z$ satisfies* B, *if $\hat{\theta}_1$ satisfies* C *and if $\hat{b}$ is a consistent* *estimate of b, then* $n^{\frac{1}{2}}(\hat{\theta} - \theta)$, *where* $\hat{\theta}$ *is given by* (2.1), *has asymptotically a normal distribution with mean zero and covariance* $(b^2/K_{FF})\Sigma^{-1}$.

In order to find the asymptotic distribution of the estimate (2.1) when the components of $Y - Z\beta$ are independently and identically distributed with a common distribution function $G(y)$, the following assumption $A_1$ concerning $G(y)$ and assumption D concerning the initial estimate $\hat{\theta}_1$ will be needed.

ASSUMPTION $A_1$.

(i) Assumption A (i),

(ii) $\int_0^1 \varphi_G^2(u)du < \infty$.

Let, for two distribution functions $F_1$ and $F_2$, $K_{F_1F_2} = \int_0^1 \varphi_{F_1}(u)\varphi_{F_2}(u)du$ and call two sequences of estimates $\hat{t}_1$ and $\hat{t}_2$ $G$-equivalent if $P_G\{n^{\frac{1}{2}}||\hat{t}_1 - \hat{t}_2|| > \varepsilon\} \to 0$. It will be supposed that the initial estimate $\hat{\theta}_1$ satisfies

ASSUMPTION D.

(i) $\hat{\theta}_1\left(\frac{Y - z\theta}{a}\right) = \frac{\hat{\theta}_1(Y) - \theta}{a}$ for all $\theta$ and all $a > 0$,

(ii) if $\theta = 0$, $\hat{\theta}_1$ is $G$-equivalent to $K_{SG}^{-1}(z'z)^{-1}z'\Phi_S(0)$ for some distribution function $S$ satisfying Assumption A.

THEOREM 2.2. *If the components of* $Y - Z\beta$ *have common distribution function* $G(y)$, *if F and S satisfy* A, *if G satisfies* $A_1$, *if z satisfies* B, *if* $\hat{\theta}_1$ *satisfies* D, *then, for* $\hat{\theta}$ *defined by* (2.1), $n^{\frac{1}{2}}(\hat{\theta} - \theta)$ *has asymptotically a normal distribution with mean* 0 *and covariance*

$$\left\{\frac{K_{SS}}{K_{SG}^2}\left[1 - \frac{cK_{FG}}{K_{FF}}\right]^2 + \frac{2K_{SF}c}{K_{SG}K_{FF}}\left[1 - \frac{cK_{FG}}{K_{FF}}\right] + \frac{c^2}{K_{FF}}\right\}\Sigma^{-1}, \tag{2.2}$$

where $c = P_G\text{-}\lim \hat{b}$.

Section 4 contains a method of constructing estimates $\hat{b}$ which are consistent estimates of $b$ if the components of $Y - Z\beta$ have distribution function $F(y/b)$ and for which $c$ can easily be found when the components of $Y - Z\beta$ have distribution $G(y)$. Section 4 also contains examples of initial estimates $\hat{\theta}_1$ satisfying Assumption D for wide classes of distribution functions $G$.

**3. Linearized signed-rank estimates.** Let now, for each $\nu = 1, 2, \cdots, Y^{(\nu)} = (Y_1^{(\nu)}, \cdots, Y_{n_\nu}^{(\nu)})'$ be an $n_\nu \times 1$ vector of observations, let $Z^{(\nu)}$ be an $n_\nu \times (p_1+q_1)$ design matrix and let $\beta$ be a $(p_1 + q_1) \times 1$ vector of unknown constants such that the components of $Y^{(\nu)} - Z^{(\nu)}\beta$ are independently and identically distributed as $F(y/b)$ where $F(y)$ is a completely specified distribution function. $p_1$ and $q_1$ will be fixed and limits are as $\nu \to \infty$. Super and subscripts $\nu$ will not be written.

Let $p_1$ be the rank of $Z$. Then $Z$ can be written as $Z = (x, x_1)$, where $x$ is a set of $p_1$ linearly independent columns of $Z$ and $x_1 = xd$, where $d$ is a $p_1 \times q_1$ matrix.

Let $\beta = (\beta_3', \beta_4')'$ correspond to $Z = (x, x_1)$ then $Z\beta = x(\beta_3 + d\beta_4)$. The parameter to be estimated is $\mu = \beta_3 + d\beta_4$.

Note that, in Section 2, $Z\beta = (Z_1 - \bar{Z}_1, 1)(\theta_1, \cdots, \theta_p, \theta_0)'$, where $(Z_1 - \bar{Z}_1, 1)$ is the $n \times (p+1)$ matrix consisting of the $p$ columns of $Z_1 - \bar{Z}_1$ and a column of 1's; this matrix $(Z_1 - \bar{Z}_1, 1)$ is of rank $p+1$. The estimation procedure to be given in this section can thus be used to estimate the parameter $(\theta_1, \cdots, \theta_p, \theta_0)'$ of Section 2. This leads to two different estimates for $(\theta_1, \cdots, \theta_p)'$ which, as will be seen, have asymptotically the same distribution if the underlying distributions are symmetric.

The distribution function $F$ of single observations will be assumed to satisfy

ASSUMPTION A′

(i) $f(y) = dF(y)/dy$ exists and is absolutely continuous on $(-\infty, \infty)$

(ii) $\psi_F(u) = \varphi_F((u+1)/2)$ can be written as the sum of two square integrable functions $\psi_1(u)$ and $\psi_2(u)$, where $\psi_1(u)$ is non-decreasing and nonnegative and $\psi_2(u)$ is non-increasing and nonpositive,

(iii) $f(y) = f(-y)$ for all $y$.

For the sequence of design matrices it will be supposed that $x$ satisfies

ASSUMPTION B′.

(i) $\dfrac{\max_{1 \leq i \leq n} x_{ij}^2}{\sum_{i=1}^n x_{ij}^2} \to 0$ for each $j = 1, \cdots, p_1$,

(ii) $n^{-1}x'x \to \Sigma_1$, where $\Sigma_1$ is a positive definite matrix,

(iii) for each pair $(j_1, j_2)(j_1 \neq j_2;\ j_1, j_2 = 1, \cdots, p_1)$ there exists a number $\gamma_{j_1 j_2} \neq 0$ such that, for $n > n_0$, (1.) $x_{ij_1}(x_{ij_1} + \gamma_{j_1 j_2} x_{ij_2}) \geqq 0$ for all $i$, (2.) $|x_{j_1}|$ and $|x_{j_1} + \gamma_{j_1 j_2} x_{j_2}|$ are similarly ordered, where $x_1, \cdots, x_{p_1}$ are column vectors of $x$.

ASSUMPTION C′. It will be supposed that there exists a sequence of initial estimates $\hat{\mu}_1$ of $\mu$ satisfying

(i) $\hat{\mu}_1\left(\dfrac{Y - x\mu}{a}\right) = \dfrac{\hat{\mu}_1(Y) - \mu}{a}$ for all $\mu$ and all $a > 0$,

(ii) $P_\mu(n^{\frac{1}{2}}(\hat{\mu}_1 - \mu) \in A) \to P(A)$ for some fixed $p_1$-dimensional distribution $P$.

Let $\Psi_F(\mu)$ be the $n \times 1$ vector

$$\Psi_F(\mu) = \left\{\psi_F\left(\frac{R_{|(Y-x\mu)_i|}}{n+1}\right) \operatorname{sgn}(Y - x\mu)_i\right\},$$

where $R_{|(Y-x\mu)_i|}$ is the rank of the absolute value of the $i^{\text{th}}$ component $(Y - x\mu)_i$ of $Y - x\mu$ among the absolute values of all its components and

$$\begin{aligned} \operatorname{sgn} x &= 1 \quad &&\text{if } x > 0\,; \\ &= -1 \quad &&\text{if } x < 0\,. \end{aligned}$$

A linearized estimate $\hat{\mu}$ of $\mu$ will be defined by

$$(3.1) \qquad \hat{\mu} = \hat{\mu}_1 + \frac{\hat{b}}{K_{FF}}(x' x)^{-1} x' \Psi_F(\hat{\mu}_1),$$

where $\hat{b}$ is a consistent estimate of $b$.

In Section 6 the following theorem will be proved.

THEOREM 3.1. *If the components of* $Y - Z\beta$ *have common distribution function* $F(y/b)$ *if* $F$ *satisfies* A′, *if* $x$ *satisfies* B′, *if* $\hat{\mu}_1$ *satisfies* C′, *if* $\hat{b}$ *is a consistent estimate of* $b$, *then* $n^{\frac{1}{2}}(\hat{\mu} - \mu)$, *with* $\hat{\mu}$ *given by* (3.1), *has asymptotically a normal distribution with mean* 0 *and covariance* $(b^2/K_{FF})\,\Sigma_1^{-1}$.

In order to find the asymptotic distribution of the estimate (3.1) when the components of $Y - Z\beta$ are independently and identically distributed as $G(y)$, the following assumption $A_1'$ concerning $G(y)$ and assumption D′ concerning the initial estimate $\hat{\mu}$ are needed.

ASSUMPTION $A_1'$.

(i) Assumption A′(i),

(ii) $\int_0^1 \varphi_G^2(u)\,du < \infty$,

(iii) Assumption A′(iii).

ASSUMPTION D′.

(i) $\hat{\mu}_1\left(\frac{Y - x\mu}{a}\right) = \frac{\hat{\mu}_1(Y) - \mu}{a}$ for all $\mu$ and all $a > 0$,

(ii) if $\mu = 0$, $\hat{\mu}_1$ is $G$-equivalent to $K_{SG}^{-1}(x' x)^{-1} x' \psi_S(0)$ for some distribution function $S$ satisfying A′.

THEOREM 3.2. *If the components of* $Y - Z\beta$ *have common distribution function* $G(y)$, *if* $F$ *and* $S$ *satisfy* A′, *if* $G$ *satisfies* $A_1'$, *if* $x$ *satisfies* B′, *if* $\hat{\mu}_1$ *satisfies* D′ *then, for* $\hat{\mu}$ *defined by* (3.1), $n^{\frac{1}{2}}(\hat{\mu} - \mu)$ *has asymptotically a normal distribution with mean* 0 *and covariance*

$$(3.2) \qquad \left\{\frac{K_{SS}}{K_{SG}^2}\left[1 - c\frac{K_{FG}}{K_{FF}}\right]^2 + \frac{2K_{SF}c}{K_{SG}K_{FF}}\left[1 - c\frac{K_{FG}}{K_{FF}}\right] + \frac{c^2}{K_{FF}}\right\}\Sigma_1^{-1}$$

*where* $c = P_G\text{-}\lim \hat{b}$.

Examples of estimates $\hat{b}$ which are consistent estimates of $b$ when the components of $Y - Z\beta$ have common distribution function $F(y/b)$ and for which $c$ can easily be computed when the components of $Y - Z\beta$ have common distribution function $G(y)$, are given in Section 4. Section 4 also contains examples of initial estimates $\hat{\mu}_1$ satisfying D′ for wide classes of distribution functions $G$.

**4. Initial estimates of $\theta$ and $\mu$ and estimates of the scaleparameter $b$. Initial estimates.** Perhaps the two most well-known choices for initial estimates of

$\theta$ and $\mu$ are those corresponding to the mean and the median. The resulting relatively efficiency of the linearized estimate can be found from Theorem 2.2 (resp. Theorem 3.2) if it is known that the initial estimate satisfies D (resp. D′) for some $\varphi_S$. Identifying such initial estimates $\hat{\theta}_1$ (resp. $\hat{\mu}_1$) and the corresponding $\varphi_S$ is the purpose of the following four theorems which will be proved in Section 5 for $\hat{\theta}_1$ and in Section 6 for $\hat{\mu}_1$.

THEOREM 4.1. *If the components of* $Y - Z\beta$ *have common distribution function* $G(x)$, *where* $G$ *satisfies* $A_1$ *and has a variance, if* $z$ *satisfies* B(i) *and* (ii) *then* $\hat{\theta}_1 = (z'z)^{-1} z'Y$ *satisfies* D *with* $\varphi_S(u) = G^{-1}(u)$.

A construction of an initial estimate $\hat{\theta}_1$ corresponding to the median can be most easily described for replicated designs. Suppose $Z' = (Z_1', Z_2', \cdots, Z_n')$ where, for each $i$, $Z_i = Z_0$ where $Z_0$ is a $k \times (p+q)$ matrix. Let $z_0$ span $Z_0 - \bar{Z}_0$ so that $z_0'z_0 > 0$. Then $z_0$ is $k \times p$ so that the total number of observations is $nk$. For simplicity suppose that the $k$ rows of $z_0$ are distinct. Then the $n$ observations corresponding to a given row in $z_0$ are a sample from a population with the same location (If $z_0$ has some equal rows there will be available more observations for a given "row") Let $m = (m_1, m_2, \cdots, m_k)'$ be the medians of the observations corresponding to each of the $k$ rows of $z_0$.

THEOREM 4.2. *If the components of* $Y - Z\beta$ *have common distribution function* $G(x)$, *where* $G(x)$ *satisfies* $A_1$ *and has a positive density at its median, then* $\hat{\theta}_1 = (z_0'z_0)^{-1} z_0'm$ *satisfies* D *with* $S$ *the double exponential distribution.*

The corresponding statement for an initial estimate, based on the mean, of $\mu$ is Theorem 4.3.

THEOREM 4.3. *If the components of* $Y - Z\beta$ *have common distribution* $G(x)$, *where* $G(x)$ *satisfies* $A_1'$ *and has a variance, if* $x$ *satisfies* B′(i) *and* (ii), *then* $\hat{\mu}_1 = (x'x)^{-1} x'Y$ *satisfies* D′ *with* $\psi_S(u) = G^{-1}((u+1)/2)$.

For an initial estimate $\hat{\mu}_1$ based on medians, consider again an $n$-times repeated design matrix. Let $x = (x_0', \cdots, x_0')'$ with $x_0$ a $k \times p_1$ matrix and $x_0'x_0 > 0$. Let $t = (t_1, t_2, \cdots, t_k)'$ be the medians of the observations corresponding to each of the $k$ rows of $x_0$.

THEOREM 4.4. *If the components of* $Y - Z\beta$ *have common distribution function* $G(x)$, *where* $G$ *satisfies* $A_1'$ *and has a positive density at its median, then* $\hat{\mu}_1 = (x_0'x_0)^{-1} x_0't$ *satisfies* D′ *with* $S$ *the double exponential distribution.*

*Estimates of the scale parameter.* Estimates of the scale parameter $b$ can be obtained as follows. Most measures of dispersion $D$, defined for distribution functions $H$, $H_n$ on $(-\infty, \infty)$, have the following properties

(i) $bD(H(y)) = D(H((y-a)/b))$ for all a and all $b > 0$,
(ii) $D(H_n(y)) \to D(H(y))$ whenever $\sup_y |H_n(y) - H(y)| \to 0$ and $D(H(y)) < \infty$.

Given such a measure of dispersion $D$, $\hat{b}$, in Section 2, can be taken as $D(\hat{F}_n(y))/D(F(y))$, where $\hat{F}_n(y)$ is the empirical distribution function of the components of $Y - z\hat{\Theta}_1$ and $F(y)$ is the distribution function from which $\varphi_F(u)$ is computed. Then, if the components of $Y - Z\beta$ have common distribution $F(y/b)$, if $\hat{\theta}_1$ satisfies C and if $D(F(y)) < \infty$, $\hat{b}$ is a consistent estimate of $b$. If the components of $Y - Z\beta$ have common distribution $G(y)$, if $\hat{\theta}_1$ satisfies D, if $D(F(y)) < \infty$ and $D(G(y)) < \infty$ then, in Theorem 2.2, $c = D(G(y))/D(F(y))$. The same remarks hold for estimating $b$ in Section 3. $D$ can be taken, for instance, as an interpercentile range or, if the observations have a variance, as the standard deviation.

In [11] some numerical values of the relative efficiencies of linearized estimates are given; these relative efficiencies are computed as the ratio of the Cramér-Rao lower bound $(\int_0^1 \varphi_G{}^2(u)du)^{-1}$, for the estimation problem, to

$$\frac{K_{SS}}{K_{SG}^2}\left[1 - c\frac{K_{FG}}{K_{FF}}\right]^2 + \frac{2K_{SF}c}{K_{SG}K_{FF}}\left[1 - c\frac{K_{FG}}{K_{FF}}\right] + \frac{c^2}{K_{FF}}.$$

These efficiencies are given in [11] for several choices of $F$ and $G$, for $\hat{b}$ as the standard deviation or as the interquartile range, and for both choices of the initial estimate given above.

## 5. Proof of Theorems 2.1, 2.2, 4.1, and 4.2.

PROOF OF THEOREM 2.1. Since $\hat{b}$ is a consistent estimate of $b$, it is sufficient to prove that, for the estimate (2.1) with $\hat{b}$ replaced by $b$, the distribution of $n^{\frac{1}{2}}(\hat{\theta} - \theta)$ converges to a normal distribution with mean zero and covariance $(b^2/K_{FF})\Sigma^{-1}$.

The asymptotic distribution of $n^{\frac{1}{2}}(\hat{\theta} - \theta)$ with $\hat{b}$ replaced by $b$ can be found as follows.

(a) For $c = (c_1, \cdots, c_p)' \neq 0$ and $\theta = 0$ it follows from Hájek and Šidák [4] (page 163) that $(n^{-\frac{1}{2}}b/K_{FF})c'z'\Phi_F(0)$ is asymptotically normal with mean zero and variance $(b^2/K_{FF})c'\Sigma c$ provided that $(c'z')'$ satisfies B (i) and B (ii). That it does if $z$ satisfies B (i) and B (ii) is immediate upon noting that, for

$$\frac{(1/n)\max_{1\leq i\leq n}(\sum_{j=1}^p c_j z_{ij})^2}{(1/n)\sum_{i=1}^n(\sum_{j=1}^p c_j z_{ij})^2}$$

B (ii) implies that denominator converges to $c'\Sigma c > 0$. Hence, by taking $c' = (0, \cdots, 0, 1, 0, \cdots, 0)$, it follows from B(i) that $n^{-1}\max_{1\leq i\leq n} z_{ij}^2$ approaches zero for each $j$. But $\max_{1\leq i\leq n}(\sum_{j=1}^p c_j z_{ij})^2 \leqq M^2p^2 \max_{1\leq j\leq p}\max_{1\leq i\leq n} z_{ij}^2$, where $M^2 = \max_{1\leq j\leq p} c_j{}^2$.

(b) It follows from C (i) that $\hat{\theta}(Y - z\theta) = \hat{\theta}(Y) - \theta$ so we can suppose that $\theta = 0$. If $\hat{\theta}_0$ is defined by $(b/K_{FF})(z'z)^{-1}z'\Phi_F(0)$ it is immediate from (a) $n^{\frac{1}{2}}\hat{\theta}_0$ is asymptotically normal with mean 0 and covariance $(b^2/K_{FF})\Sigma^{-1}$.

(c) Assuming $\theta = 0$ it remains to show that $n^{\frac{1}{2}}||\hat{\theta} - \hat{\theta}_0||$ converges to zero and hence that $n^{\frac{1}{2}}\hat{\theta}$ and $n^{\frac{1}{2}}\hat{\theta}_0$ have asymptotically the same distribution. However

$$(5.1) \qquad n^{\frac{1}{2}}||\hat{\theta} - \hat{\theta}_0|| = \left|\left| n^{\frac{1}{2}}\hat{\theta}_1 + \frac{bn^{\frac{1}{2}}}{K_{FF}}(z'z)^{-1}\{z'\Phi_F(\hat{\theta}_1) - z'\Phi_F(0)\}\right|\right| .$$

By C(ii) a number $d$ can be chosen so that $P\{||\hat{\theta}_1|| \leqq dn^{-\frac{1}{2}}\}$ is arbitrarily close to one for all sufficiently large $n$. Hence the right-hand side of (5.1) will be, with arbitrarily high probability, bounded by

$$\sup_{||\xi|| \leqq dn^{-\frac{1}{2}}} \left|\left| n^{\frac{1}{2}}\xi + \frac{bn^{\frac{1}{2}}}{K_{FF}}(z'z)^{-1}\{z'\Phi_F(\xi) - z'\Phi_F(0)\}\right|\right| ,$$

which can also be written as

$$\sup_{||\xi|| \leqq dn^{-\frac{1}{2}}} \left|\left| \frac{n^{\frac{1}{2}}(z'z)^{-1}}{K_{FF}} b\left\{z'\Phi_F(\xi) - z'\Phi_F(0) + z'z\frac{K_{FF}}{b}\xi\right\}\right|\right| .$$

Further, by an extension of the theorem of Jurečková [6], (see Section 7)

$$\sup_{||\xi|| \leqq dn^{-\frac{1}{2}}} \left|\left| n^{-\frac{1}{2}}\{z'\Phi_F(\xi) - z'\Phi_F(0) + \frac{K_{FF}}{b}z'z\xi\}\right|\right|$$

converges to zero in probability if $\theta = 0$. Since $n(z'z)^{-1} \to \Sigma^{-1}$, it follows that $n^{\frac{1}{2}}||\hat{\theta} - \hat{\theta}_0|| \to 0$. This completes the proof.

PROOF OF THEOREM 2.2. As in the proof of Theorem 2.1, we can suppose that $\theta = 0$. By the extension of the theorem of Jurečková [6] (see Section 7) we have, for $\theta = 0$,

$$(5.2) \qquad P_G\{\sup_{||\xi|| \leqq dn^{-\frac{1}{2}}} ||n^{-\frac{1}{2}}(z'\Phi_F(\xi) - z'\Phi_F(0) + K_{FG}z'z\xi)|| > \varepsilon\} \to 0 .$$

If $\hat{\theta}_{00} = (1 - c(K_{FG}/K_{FF}))\hat{\theta}_1 + (c/K_{FF})(z'z)^{-1}z'\Phi_F(0)$ and $\hat{\theta}_{01} = K_{SG}^{-1}(z'z)^{-1}z'\Phi_S(0)$ it follows from (5.2) and the fact that $\hat{b} \to_{P_G} c$ as in the proof of Theorem 2.1, that $P_G\{n^{\frac{1}{2}}||\hat{\theta}_{00} - \hat{\theta}|| > \varepsilon\} \to 0$. Further, by Assumption D, $P_G\{n^{\frac{1}{2}}||_{01}\hat{\theta} - \hat{\theta}_1|| > \varepsilon\} \to 0$. Hence the asymptotic distribution of $n^{\frac{1}{2}}\hat{\theta}$ is that of $n^{\frac{1}{2}}\hat{\theta}_{02}$, where

$$\hat{\theta}_{02} = \left(1 - c\frac{K_{FG}}{K_{FF}}\right)\frac{(z'z)^{-1}}{K_{SG}}z'\Phi_S(0) + \frac{c}{K_{FF}}(z'z)^{-1}z'\Phi_F(0) .$$

It follows from Hájek and Šidák [4] (page 163) that the asymptotic distribution of $n^{\frac{1}{2}}\hat{\theta}_{02}$, and hence that of $n^{\frac{1}{2}}\hat{\theta}$, is normal with mean 0 and covariance given by (2.2). □

PROOF OF THEOREM 4.1. Obviously, $\hat{\theta}_1$ satisfies D(i). Further $G^{-1}(u)$ is nondecreasing in $u$ and $\int_0^1 (G^{-1}(u))^2\,du = \int_{-\infty}^{+\infty} y^2 g(y)\,dy < \infty$ so that $S$ satisfies A if $G$ satisfies $A_1$ and has a variance. Further it follows from Hájek and Šidák [4] (page 160) that $(z'z)^{-1}z'\Phi_S(0)$ is, if $\theta = 0$, $G$-equivalent to

$$(z'z)^{-1}z'(\varphi_S(G(Y_1)), \cdots, \varphi_S(G(Y_n)))' = (z'z)^{-1}z'Y .$$

Since $K_{SG} = 1$ the result follows.

For the proof of Theorem 4.2 the following lemma is needed.

LEMMA 5.1. *If the components of* $Y - z\theta$ *have common distribution function* $G(x)$, *where* $G$ *satisfies* $A_1$ *and has a positive density at its median* $\eta$, *then, for* $\theta = 0$, *each median* $m_j$ *is G-equivalent to* $\eta + n^{-1}K_{SG}^{-1}\delta_j$ *where* $S$ *is the double exponential distribution and where* $\delta_j$ *is the sum of* $\pm$ 1's *according as the observations corresponding to the* $j$th *row of* $z_0$ *are* $\gtrless \eta$.

PROOF. By applying Markov's inequality conditionally it follows that it is sufficient to show that, assuming $\eta = 0$,

$$\mathscr{E}_G\left\{\left[n^{\frac{1}{2}}\left(2g(0)m_j - \frac{\delta_j}{n}\right)\right]^2 \middle| m_j\right\} \to_{P_G} 0 \text{ since } K_{SG} = 2g(0) > 0 .$$

Let $n_j{}^*$ be the number of observations corresponding to the $j$th row of $z_0$ which are between 0 and $m_j$. Then $\delta_j = \pm 2n_j{}^*$ according as $m_j \gtrless 0$. The conditional, given $m_j$, distribution of $n_j{}^*$ is $B(n/2, p_j)$ where $p_j = |G(m_j) - G(0)|/G(m_j)$. Hence

$$\mathscr{E}_G\left\{n\left[2g(0)m_j - \frac{\delta_j}{n}\right]^2 \middle| m_j\right\} = n[2g(0)|m_j| - p_j]^2 + 2p_j(1 - p_j)$$

which can be written as

$$\frac{nm_j{}^2}{G^2(0)}\left\{g(0) - \frac{|G(m_j) - G(0)|}{|m_j|} \cdot \frac{G(0)}{G(m_j)}\right\}^2 + 2p_j(1 - p_j) .$$

Since $n^{\frac{1}{2}}m_j$ has an asymptotic distribution and $|G(m_j) - G(0)|/|m_j| \to_{P_G} g(0)$ the result follows.

PROOF OF THEOREM 4.2. Since, for the double exponential distribution,

$$\begin{aligned} \varphi_S(u) &= 1 \qquad \text{if } u > \tfrac{1}{2} ; \\ &= -1 \qquad \text{if } u < \tfrac{1}{2} . \end{aligned}$$

$\Phi_S(0)$ is a vector of $\pm 1$'s according as $Y_i \gtrless \text{med}(Y_1, \cdots, Y_{nk})$. Letting $\delta' = (\delta_1, \cdots, \delta_k)$, with $\delta_j$ as in Lemma 5.1, it follows from the lemma that $\hat{\theta}_1$ is $G$-equivalent to [note that $z_0'(\eta, \cdots, \eta)' = (0, \cdots, 0)'$] $K_{SG}^{-1}((z_0'z_0)^{-1}/n)z_0'\delta$. However $z_0'\delta = z'\Delta$ (see Section 4) where $\Delta$ is an $nk \times 1$ vector of $\pm 1$'s according as $Y_i \gtrless \eta$. The conclusion of the theorem will follow if it is true that

$$(5.3) \qquad (nk)^{-\frac{1}{2}}||z'(\Delta - \Phi_S(0))|| \to_{P_G} 0 .$$

From Hájek and Šidák [4] (pages 61 and 160) it follows that the conditional, given $Y$, expectation of the square of each element of $z'(\Delta - \Phi_S(0))$ is bounded by

$$4\left\{\frac{\# \text{ of } Y_i \text{ between } \eta \text{ and } M}{nk}\right\} \sum_{i=1}^{nk} z_{ij}^2 ,$$

where $M = \text{med}(Y_1, \cdots, Y_{nk})$. Since $(nk)^{-1} \sum_{i=1}^{nk} z_{ij}^2 \to \Sigma_{jj}$, (5.3) and the theorem follow.

**6. Proof of Theorems 3.1, 3.2, 4.3, and 4.4.** The following proofs of Theorems 3.1 and 3.2 are the analogues for signed rank statistics to those of Theorems 2.1 and 2.2 for rank statistics. Accordingly they require a linearization theorem for signed rank statistics. Such a linearization theorem has, for $p_1 = 1$, been given in [16]; for an extension to $p_1 > 1$ see Section 7.

PROOF OF THEOREM 3.1. Since $\hat{b}$ is a consistent estimate of $b$, it is sufficient to prove that, for the estimate (3.1) with $\hat{b}$ replaced by $b$, $n^{\frac{1}{2}}(\hat{\mu} - \mu)$ has asymptotically a normal distribution with mean 0 and covariance $(b^2/K_{FF})\Sigma_1^{-1}$.

The asymptotic distribution of $n^{\frac{1}{2}}(\hat{\mu} - \mu)$ with $\hat{b}$ replaced by $b$ can be found as follows.

(a) For $c = (c_1, \cdots, c_{p_1})' \neq 0$ and $\mu = 0$, it follows from Hájek and Šidák [4] (page 166) and the Assumptions B′(i) and (ii) that $(n^{-\frac{1}{2}}b/K_{FF})c'x'\Psi_F(0)$ is asymptotically normal with mean 0 and variance $(b^2/K_{FF})c'\Sigma_1 c$.

(b) It follows from C′(i) that $\hat{\mu}(Y - x\mu) = \hat{\mu}(Y) - \mu$ so we can suppose $\mu = 0$. With $\hat{\mu}_0$ defined by $(b/K_{FF})(x'x)^{-1}x'\Psi_F(0)$ it follows from (a) that $n^{\frac{1}{2}}\hat{\mu}_0$ has asymptotically a normal distribution with mean 0 and covariance $(b^2/K_{FF})\Sigma_1^{-1}$.

(c) Assuming $\mu = 0$, it remains to show that $n^{\frac{1}{2}}||\hat{\mu} - \hat{\mu}_0||$ converges to zero. However

$$n^{\frac{1}{2}}||\hat{\mu} - \hat{\mu}_0|| = \left|\left| n^{\frac{1}{2}}\hat{\mu}_1 + \frac{bn^{\frac{1}{2}}}{K_{FF}}(x'x)^{-1}\{x'\Psi_F(\hat{\mu}_1) - x'\Psi_F(0)\}\right|\right|$$

so that, using Assumption C′(ii) it is sufficient to show that

$$\text{(6.1)} \qquad \sup_{||\xi|| \leq dn^{-\frac{1}{2}}} \left|\left| n^{-\frac{1}{2}} \left\{x'\Psi_F(\xi) - x'\Psi_F(0) + \frac{K_{FF}}{b} x'x\xi\right\}\right|\right| \to_{P_F} 0 \quad \text{if} \quad \mu = 0\,.$$

(6.1) follows from the linearization theorem for signed rank statistics proved in Section 7 (see also [16]).

PROOF OF THEOREM 3.2. As in the proof of Theorem 3.1 we can suppose that $\mu = 0$. Let

$$\hat{\mu}_{00} = \left(1 - c\frac{K_{FG}}{K_{FF}}\right)\hat{\mu}_1 + \frac{c}{K_{FF}}(x'x)^{-1}x'\Psi_F(0)$$

$$\hat{\mu}_{01} = \frac{1}{K_{SG}}(x'x)^{-1}x'\Psi_S(0)$$

$$\hat{\mu}_{02} = \left(1 - c\frac{K_{FG}}{K_{FF}}\right)\frac{(x'x)^{-1}}{K_{SG}}x'\Psi_S(0) + \frac{c}{K_{FF}}(x'x)^{-1}x'\Psi_F(0)$$

then it follows from (see Theorem 7.2)

$$P_G\{\sup_{\|\xi\|\leq dn^{-\frac12}}\|n^{-\frac12}\{(x'\Psi_F(\xi) - x'\Psi_F(0) + K_{FG}x'x\xi\}\| > \varepsilon\} \to 0 \tag{6.2}$$

and the fact that $\hat{b} \to_{P_G} c$ that the asymptotic distribution of $n^{\frac12}\hat{\mu}$ is the same as that of $n^{\frac12}\hat{\mu}_{02}$. From Hájek and Šidák [4] (page 166) it follows that the asymptotic distribution of $n^{\frac12}\hat{\mu}_{02}$ is normal with mean 0 and covariance given by (3.2).

PROOF OF THEOREM 4.3. Obviously $\hat{\mu}_1$ satisfies D′(i). That $S$ satisfies A′ follows from the fact that $G^{-1}((u+1)/2)$ is non-decreasing and nonnegative,

$$\int_0^1 \left[G^{-1}\left(\frac{u+1}{2}\right)\right]^2 du = \int_{-\infty}^{+\infty} y^2 g(y)\, dy < \infty$$

and that symmetry for $G$ implies symmetry for $S$.

From Hájek and Šidák [4] (page 166) it follows that $(x'x)^{-1}x'\Psi_S(0)$ is, if $\mu = 0$, $G$ equivalent to

$$(x'x)^{-1}x'(\psi_S(2G(Y_1) - 1), \cdots, \psi_S(2G(Y_n) - 1))' = (x'x)^{-1}x'Y\,.$$

The result then follows from the fact that $K_{SG} = 1$.

PROOF OF THEOREM 4.4. Obviously $\hat{\mu}_1$ satisfies $D'$(i) and the double exponential distribution satisfies A′.

To prove D′(ii) it needs to be shown that, if $\mu = 0$,

$$\left\|n^{\frac12}\{(x_0'x_0)^{-1}x_0't - \frac{1}{K_{SG}}(x'x)^{-1}x'\Psi_S(0)\}\right\| \to_{P_G} 0\,.$$

Let $\varepsilon_j$ be the sum of $\pm 1$'s according as the observations in the $j$th row of $x_0$ are $\gtrless 0$, let $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_k)'$, then $x'\Psi_S(0) = x_0'\varepsilon$ and

$$n^{\frac12}\{(x_0'x_0)^{-1}x_0't - \frac{1}{K_{SG}}(x'x)^{-1}x'\Psi_S(0)\} = \frac{n(x'x)^{-1}}{n^{\frac12}}x_0'\left(nt - \frac{1}{2g(0)}\varepsilon\right).$$

Hence it is sufficient to prove that

$$n^{-1}\mathscr{E}_G\left\{\left(nt_j - \frac{1}{2g(0)}\varepsilon_j\right)^2 \Big| t_j\right\} \to_{P_G} 0 \tag{6.3}$$

and (6.3) follows, as in the proof of Lemma 5.1, from the fact that the conditional, given $t_j$ distribution of $\frac12\varepsilon_j(t_j/|t_j|)$ is $B(\frac12 n, p_j)$ where

$$p_j = \frac{|G(t_j) - G(0)|}{G(t_j)}\,.$$

**7. An extension and an analogue of a Theorem of Jurečková [6].** The following theorem is an extension to more dimensions of Theorem 3.1 of Jurečková [6].

THEOREM 7.1. *If the components of Y have common distribution function* $G(x)$, *if F satisfies* A, *if G satisfies* $A_1$, *if z satisfies* B, *if*

$$S_j(\xi) = \sum_{i=1}^n z_{ij}\,\varphi_F\left(\frac{R_{Y_i-\sum_{l=1}^p z_{il}\xi_l}}{n+1}\right)$$

*then, for each* $j = 1, \cdots, p$,

(7.1) $\lim_{\nu\to\infty} P\{\sup_{||\xi||\leq d} n^{-\frac{1}{2}}|S_j(\xi n^{-\frac{1}{2}}) - S_j(0) + K_{FG}n^{-\frac{1}{2}}\sum_{l=1}^p \xi_l \sum_{i=1}^n z_{ij}z_{il}| > \varepsilon\} = 0$

*for each* $d > 0$ *and each* $\varepsilon > 0$.

PROOF. For $p = 1$ Theorem 7.1, is a special case of Theorem 3.1 of Jurečková [6]. In the following it will be supposed that $p > 1$.

The proof will be given for $j = 1$. As $\varphi_F(u)$ is the sum of two monotone square integrable functions it is sufficient to prove (7.1) for the case where $\varphi_F(u)$ is non-decreasing. The proof consists of two parts. It will first be shown that, under A and B(i) and (ii), for any fixed set of $r$ points $(\xi_1^{(k)}, \cdots, \xi_p^{(k)})$, $k = 1, \cdots, r$

(7.2) $$P\{n^{-\frac{1}{2}}|S_1(\xi^{(k)}n^{-\frac{1}{2}}) - S_1(0) + K_{FG}n^{-\frac{1}{2}}\sum_{l=1}^p \xi_l^{(k)} \sum_{i=1}^n z_{i1}z_{il}| \leqq \varepsilon \quad \text{for each} \quad k = 1, \cdots, r\} \to 1\,.$$

Jurečková [6] proves (7.2) for $p = 1$ in her Lemmas 3.1—3.8. That (7.2) holds for $p > 1$ can be seen by noting that Jurečková's lemmas 3.1—3.8 hold for $S_1(\xi n^{-\frac{1}{2}})$ if $z$ satisfies

(7.3) $$n^{-1}\max_{1\leqq i\leqq n} z_{ij}^2 \to 0 \qquad \text{for each } j = 1, \cdots, p$$

$$|n^{-1}\sum_{i=1}^n z_{ij}^2| \leqq M$$

for each $j = 1, \cdots, p$ where $M$ is a positve constant.

Then (7.2) follows from the fact that (7.3) is implied by B(i) and (ii).

In the second part of the proof it will be shown that for each $d > 0$ there exists a set of $r$ fixed points $\xi^{(k)}$, $k = 1, \cdots, r$ such that, for $n > n_0$,

(7.4) $$[n^{-\frac{1}{2}}|S_1(\xi^{(k)}n^{-\frac{1}{2}}) - S_1(0) + K_{FG}n^{-\frac{1}{2}}\sum_{l=1}^p \xi_l^{(k)} \sum_{i=1}^n z_{i1}z_{il}| \leqq \varepsilon \quad \text{for each} \quad k = 1, \cdots, r]$$

$$\Longrightarrow [\sup_{||\xi||\leqq d} n^{-\frac{1}{2}}|S_1(\xi n^{-\frac{1}{2}}) - S_1(0) + K_{FG}n^{-\frac{1}{2}}\sum_{l=1}^p \xi_l \sum_{i=1}^n z_{i1}z_{il}| \leqq 2^{p-1}\varepsilon]\,.$$

The theorem then follows from (7.2) and (7.4).

The set of points $\xi^{(k)}$, $k = 1, \cdots, r$ satisfying (7.4) can be found as follows. By B(iii) there exists, for each $j = 2, \cdots, p$, a number $\gamma_j \neq 0$ such that, for $n > n_0$.

(7.5) $$(z_{i_11} - z_{i_21})(z_{i_11} - z_{i_21} + \gamma_j(z_{i_1j} - z_{i_2j})) \geqq 0 \qquad \text{for all } i_1, i_2\,.$$

(For simplicity of notation the first subscript on $\gamma_{1,j}$ is omitted).

By the transformation

$$\eta_1 = \xi_1 - \textstyle\sum_{j=2}^{p} \dfrac{\xi_j}{\gamma_j} \tag{7.6}$$

$$n_l = \frac{\xi_l}{\gamma_l} \qquad l = 2, \cdots, p \, .$$

$S_1(\xi n^{-\frac{1}{2}})$ can be written as

$$S_{10}(\eta n^{-\frac{1}{2}}) =_{\text{def}} \textstyle\sum_{i=1}^{n} z_{i1}\varphi_F \left(\dfrac{R_{Y_i - n^{-\frac{1}{2}}(z_{i1}\eta_1 + \Sigma_{l=2}^{p}(z_{i1}+\gamma_l z_{il})\eta_l)}}{n+1}\right).$$

By (7.5) and Theorem 2.1 of Jurečková [6], $S_{10}(\eta n^{-\frac{1}{2}})$ is, for $n > n_0$, for fixed values of $\eta_1, \cdots, n_{j-1}, \eta_{j+1}, \cdots, \eta_p$, with probability one, a non-increasing step function of $\eta_j$ $(j = 1, \cdots, p)$. Now choose the $r$ fixed points $\xi^{(k)}$ as follows. Let $C$ and $\varepsilon$ be fixed positive numbers. Let $R$ be an integer and let $r = (2R+1)^p$. Divide the cube $-C \leqq \eta_j \leqq C$ $(j = 1, \cdots, p)$ into $(2R)^p$ cubes by dividing each axis into $2R$ equal pieces and choose $(2R+1)^p$ points $\eta^{(k)}$ on the corners of these cubes. These $(2R+1)^p$ points $\eta^{(k)}$ define, by (7.6), $(2R+1)^p$ points $\xi^{(k)}$. By choosing $R$ in such a way that

$$|K_{FG}| \frac{1}{n} \textstyle\sum_{i=1}^{n} z_{i1}^2 \cdot \dfrac{C}{R} \leqq \varepsilon \tag{7.7}$$

$$|K_{FG}| \left| \frac{1}{n} \textstyle\sum_{i=1}^{n} z_{i1}(z_{i1} + \gamma_l z_{il}) \right| \dfrac{C}{R} \leqq \varepsilon \qquad \text{all } l = 2, \cdots, p$$

these points $\xi^{(k)}$ satisfy, for $n > n_0$,

$$[n^{-\frac{1}{2}}|S_1(\xi^{(k)}n^{-\frac{1}{2}}) - S_1(0) + K_{FG}n^{-\frac{1}{2}} \textstyle\sum_{l=1}^{p} \xi_l^{(k)} \sum_{i=1}^{n} z_{i1}z_{il}| \leqq \varepsilon \qquad \text{for each } k = 1, \ldots, r] \tag{7.8}$$

$$\Rightarrow [\sup_{|\eta_j| \leqq C} n^{-\frac{1}{2}}[S_1(\xi n^{-\frac{1}{2}}) - S_1(0) + K_{FG}n^{-\frac{1}{2}} \textstyle\sum_{l=1}^{p} \xi_l \sum_{i=1}^{n} z_{i1}z_{il}| \leqq 2^{p-1}\varepsilon] \, . \qquad j = 1, \cdots, p$$

That (7.8) holds if $R$ satisfies (7.7) can be seen by using the above mentioned monotonicity of $S_{10}(\eta n^{-\frac{1}{2}})$ and by using the fact that (see also Jurečková [6]) if, for a monotone function $h(\xi)$ of one variable, $|h(\xi) - m\xi| \leqq \varepsilon$ for $\xi = \xi_1$ and for $\xi = \xi_2 (\xi_1 < \xi_2)$, then $\sup_{\xi_1 \leqq \xi \leqq \xi_2}|h(\xi) - m\xi| \leqq 2\varepsilon$ provided $|m|(\xi_2 - \xi_1) \leqq \varepsilon$.

That $R$ can, for $n > n_1$, be chosen such that (7.7) is satisfied can be seen as follows. Let

$$\gamma = \max_{2 \leqq j \leqq p}|\gamma_j| \, ,$$

$$\sigma = \max_{1 \leqq j \leqq p}|\Sigma_{1j}| \, , \qquad \text{where } \Sigma = (\Sigma_{ij}) \, ,$$

then, by B(ii), there exists $n_1$ such that for $n > n_1$

$$n^{-\frac{1}{2}} \textstyle\sum_{i=1}^{n} z_{i1}^2 \leqq 2\sigma \, ,$$

$$|n^{-1} \textstyle\sum_{i=1}^{n} z_{i1}(z_{i1} + \gamma_l z_{il})| \leqq 2\sigma(1+\gamma) \, ,$$

so that, by choosing $R$ such that

$$R \geqq \frac{|K_{FG}| 2\sigma C(1+\gamma)}{\varepsilon},$$

(7.7) is satisfied for $n > n_1$.

Further (7.4) follows from (7.8) by choosing $d$ such that

$$[\textstyle\sum_{i=1}^{p} \xi_i^2 \leqq d^2] \Rightarrow [|\eta_j| \leqq C \quad \text{for all} \quad j = 1, \cdots, p] \tag{7.8}$$

and a $d > 0$ satisfying (7.9) is given by

$$d^2 = C^2 \frac{[\min_{2 \leqq j \leqq p} \gamma_j]^2}{1 + [\min_{2 \leqq j \leqq p} \gamma_j]^2} .$$

The next theorem is a linearization theorem for signed rank statistics and is an extension of Theorem 3.2 in [16].

THEOREM 7.2. *If the components of $Y$ have common distribution $G(x)$,* if $F$ *satisfies* A′, *if $G$ satisfies* $A_1'$ *if $x$ satisfies* B′, *if*

$$T_j(\xi) = \textstyle\sum_{i=1}^{n} x_{ij} \psi_F \left( \frac{R_{|Y_i - \sum_{l=1}^{p} x_{il}\xi_l|}}{n+1} \right) \operatorname{sgn}(Y_i - \sum_{l=1}^{p} x_{il} \xi_l)$$

*then, for each* $j = 1, \cdots, p_1$

$$\begin{aligned} \lim_{\nu \to \infty} P\{\sup_{\|\xi\| \leqq d} n^{-\frac{1}{2}} |T_j(\xi n^{-\frac{1}{2}}) - T_j(0) \\ + K_{FG} n^{-\frac{1}{2}} \textstyle\sum_{l=1}^{p} \xi_l \sum_{i=1}^{n} x_{ij} x_{il}| > \varepsilon\} = 0 \end{aligned} \tag{7.10}$$

*for each* $d > 0$ *and each* $\varepsilon > 0$.

PROOF. The following proof is analogous to the proof of Theorem 7.1. For $p_1 = 1$ the theorem is a special case of Theorem 3.2 of [16] and in the following it will be supposed that $p_1 > 1$. The proof will be given for $j = 1$. As $\psi_F(u)$ is the sum of two square integrable functions, one non-decreasing and non-negative, the other non-increasing and nonpositive, it is sufficient to prove (7.10) for the case where $\psi_F(u)$ is non-decreasing and nonnegative.

It can be shown, analogously to Jurečková's Lemmas (3.1)—(3.8) and using the results of Hájek and Šidák [4] (page 219–221) that, under the assumptions A′, $A_1'$ and B′(i) and (ii), for any fixed set of points $\xi^{(k)}$, $k = 1, \cdots, r$,

$$\begin{aligned} P\{n^{-\frac{1}{2}} |T_1(\xi^{(k)} n^{-\frac{1}{2}}) - T_1(0) + K_{FG} n^{-\frac{1}{2}} \textstyle\sum_{l=1}^{p} \xi_l^{(k)} \sum_{i=1}^{n} x_{i1} x_{il} \leqq \varepsilon \\ \text{for each} \quad k = 1, \cdots, r\} \to 1 . \end{aligned}$$

Further, by B′(iii), there exists, for each $j = 2, \cdots, p_1$, a number $\gamma_j$ such that

$$\begin{aligned} &1. \ x_{i1}(x_{i1} + \gamma_j x_{ij}) \geqq 0 \quad \text{for all} \quad i , \\ &2. \ (|x_{i_1 1}| - |x_{i_2 1}|)(|x_{i_1 1} + \gamma_j x_{i_1 j}| - |x_{i_2 1} + \gamma_j x_{i_2 j}|) \geqq 0 \qquad \text{for all } i_1, i_2 . \end{aligned} \tag{7.11}$$

By the transformation (7.6) $T_1(\xi n^{-\frac{1}{2}})$ can be written as

$$T_{10}(\eta n^{-\frac{1}{2}}) = \sum_{i=1}^{n} x_{i1}\phi_F\left(\frac{R_{|Y_i - n^{-\frac{1}{2}}(x_{i1}\eta_1 + \sum_{l=2}^{p}(x_{i1}+\gamma_l x_{il})\eta_l)|}}{n+1}\right) \cdot \operatorname{sgn}(Y_i - n^{-\frac{1}{2}}(x_{i1}\eta_1 + \sum_{l=2}^{p}(x_{i1} + \gamma_l x_{il})\eta_l))$$

and it follows from (7.11) and Theorem 3.1 in [16] that, for $n > n_0$, $T_{10}(\eta n^{-\frac{1}{2}})$ is, for fixed values of $\eta_1, \cdots, \eta_{j-1}, \eta_{j+1}, \cdots, \eta_{p_1}$ with probability 1 a non-increasing step function of $\eta_j$ $(j = 1, \cdots, p_1)$.

The rest of the proof is identical to that of Theorem 7.1.

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