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DOMAINS OF ATTRACTION OF FIRST PASSAGE TIMES

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It is well known that the first passage times for Brownian motion have stable laws with exponent $\frac{1}{2}$. It is shown here that first passage times for random walks have distributions in the domain of attraction of a stable law with exponent $\frac{1}{2}$.

Let standard Brownian motion be defined on $(\Omega; l_t, A_t t \geqq 0; P)$ where Ω is the space of continuous functions ω: $[0, \infty) \to (-\infty, +\infty)$ and $l_t(\omega) = \omega(t)$. It is well known ([3] page 171) that if $\beta_b(\omega) = \inf\{t \mid \omega(t) \geqq b\}$ then the distribution of β_b is stable with exponent $\frac{1}{2}$. We will establish a similar result for random walks. Let $X_1, X_2, \cdots$ be i.i.d. rv's with $E(X_1) = 0$ and $\sigma^2(X_1) = 1$. Define $T(0), T(1), T(2), \cdots$ inductively by $T(0) = 0$, $T(n)$ the least $k > T(n-1)$ such that $\sum_{i=T(n-1)+1}^{k} X_i > b$.

THEOREM. *The distribution of* $T(1)$ *is in the domain of attraction of a stable law with exponent* $\frac{1}{2}$.

PROOF. Let $y = \sum_{i=1}^{T(1)} X_i$. Let $B(a) =$ the least k such that $\sum_{i=1}^{k} X_i \geqq a$. We know that $E(y) = c < \infty$ (see [2] page 262). We will first show that $n^{-2}B(nc) \to_{\mathscr{L}} \beta_c$. By the invariance principle (see 1, page 70) we have that for any $t > 0$

$$(1) \qquad P[\max_{1 \leqq i \leqq [n^2 t]} n^{-1} \sum_{j=1}^{i} X_j \geqq c] \to P[\sup_{0 \leqq s \leqq t} \omega(s) \geqq c] \qquad \text{as } n \to \infty .$$

Hence we have that

$$(2) \qquad P[n^{-2}B(nc) \leqq t] \to P[\beta_c \leqq t] \qquad \text{as } n \to \infty .$$

From the weak law of large numbers, it follows that for any $\varepsilon > 0$

$$(3) \qquad P[B(n(c-\varepsilon)) \leqq T(n) \leqq B(n(c+\varepsilon))] \to 1 \qquad \text{as } n \to \infty .$$

Hence

$$(4) \quad P[\beta_{c+\varepsilon} \leqq t] \leqq \liminf P[n^{-2}T(n) \leqq t] \leqq \limsup P[n^{-2}T(n) \leqq t] \leqq P[\beta_{c-\varepsilon} \leqq t] .$$

Letting $\varepsilon \downarrow 0$ yields the theorem.

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