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TRANSFORMATION GROUPS AND SUFFICIENT STATISTICS

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Let $(X, \mathscr{U})$ be a pathwise connected and locally connected topological space with countable base, and $(\Theta, \mathscr{W})$ be a connected and locally connected continuous transformation group on X which is Abelian. Let $\mathscr{A}$ be the smallest σ-field containing all open sets and $P \mid \mathscr{A}$ a probability measure such that $P(U) > 0$ for every open set $U \neq \varnothing$. For every $\vartheta \in \Theta$ let P_ϑ denote the probability measure generated by the transformation ϑ, i.e. $P_\vartheta(A) := P(\vartheta^{-1}A)$, $A \in \mathscr{A}$. Assume that P_ϑ admits a continuous density relative to P for every $\vartheta \in \Theta$. Assume finally that for some sample size $n > 1$ there exists a real-valued, continuous statistic T_n which is equivariant (i.e. $T_n(x_1, \cdots, x_n) = T_n(y_1, \cdots, y_n)$ implies $T_n(\vartheta x_1, \cdots, \vartheta x_n) = T_n(\vartheta y_1, \cdots, \vartheta y_n)$ for all $\vartheta \in \Theta$) and sufficient for P_ϑ^n, $\vartheta \in \Theta$.

Under these assumptions there exists a real-valued, continuous statistic S on X which is sufficient for P_ϑ, $\vartheta \in \Theta$, such that the distribution of S is either the location parameter family of normal distributions with variance 1 or a scale parameter family of gamma distributions.

In a nutshell: Among the families of distributions which are generated by Abelian transformation groups, and which fulfill certain regularity conditions, the location parameter family of normal distributions and the scale parameter families of gamma distributions are essentially the only ones admitting for some sample size greater than one a sufficient statistic which is real valued, continuous and equivariant.

0. Summary. Any family of probability measures with continuous densities which is generated by an Abelian transformation group and which admits an equivariant, real-valued and continuous sufficient statistic for some sample size greater than 1, is either equivalent to the translation parameter family of normal distributions with variance 1 or to a scale parameter family of gamma distributions. Using a theorem of Borges and Pfanzagl (1965), this result is established under natural assumptions on the topological structure of the basic space and the transformation group.

1. The main result. Let $(X, \mathscr{U})$ be a topological space.[1] A topological transformation group $(\Theta, \mathscr{W})$ on X is *continuous* if the maps $(\vartheta, \tau) \to \vartheta\tau$, $\vartheta \to \vartheta^{-1}$ and $(\vartheta, x) \to \vartheta x$ are continuous.

Let $\mathscr{A} \subset \mathscr{P}(X)$ be the σ-field generated by $\mathscr{U}$ and $P \mid \mathscr{A}$ a p-measure (=probability measure). We remark that $x \to \vartheta x$ is continuous and therefore $\mathscr{A}, \mathscr{A}$-measurable for every $\vartheta \in \Theta$. This implies in particular that $\vartheta A \in \mathscr{A}$ for every $\vartheta \in \Theta$, $A \in \mathscr{A}$.

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[1] All topological spaces considered in this paper are Hausdorff spaces.

For every $\vartheta \in \Theta$ let $P_\vartheta \mid \mathscr{A}$ denote the p-measure defined by

$$P_\vartheta(A) := P(\vartheta^{-1}A)\,, \qquad A \in \mathscr{A}.$$

In particular: $P_\varepsilon = P$, if ε denotes the unit element of Θ. We shall say that the family P_ϑ, $\vartheta \in \Theta$, is *generated by the transformation group* Θ.

A map $T: X^n \to Y$ is *equivariant* if $T(x_1', \cdots, x_n') = T(x_1'', \cdots, x_n'')$ implies $T(\vartheta x_1', \cdots, \vartheta x_n') = T(\vartheta x_1'', \cdots, \vartheta x_n'')$ for all $\vartheta \in \Theta$. (In a similar, but not equivalent sense, the word "equivariant" is used by Berk [1].) The word "sufficient" will be used in the same sense as in [10] (pages 47, 48).

THEOREM. *Assume that*

(i) $(X, \mathscr{U})$ *is pathwise connected and locally connected with countable base;*

(ii) $(\Theta, \mathscr{W})$ *is a connected and locally connected continuous transformation group on X which is Abelian;*

(iii) *P is a p-measure on the σ-field* $\mathscr{A}$ *such that* $P(U) > 0$ *for every* $U \in \mathscr{U}$, $U \neq \emptyset$;

(iv) *the generated family* $P_\vartheta \mid \mathscr{A}$, $\vartheta \in \Theta$, *contains more than two elements;*

(v) *for each* $\vartheta \in \Theta$, *the p-measure* $P_\vartheta \mid \mathscr{A}$ *admits a continuous density relative to* $P \mid \mathscr{A}$, *say* $h(x, \vartheta)$;

(vi) *for some sample size* $n > 1$ *there exists an equivariant, real-valued, and continuous statistic which is sufficient for* $P_\vartheta^n \mid \mathscr{A}^n$, $\vartheta \in \Theta$.

Then there exists an equivariant, real-valued and continuous statistic S on X which is sufficient for $P_\vartheta \mid \mathscr{A}$, $\vartheta \in \Theta$, *such that the family of induced distributions, say* $P_\vartheta * S$, $\vartheta \in \Theta$, *is either the location parameter family of normal distributions with variance* 1 *or a scale parameter family of gamma distributions.*

More precisely: There exists a continuous function $w: \Theta \to \mathbb{R}$ *such that* $P_\vartheta * S$ *has one of the following densities relative to the Lebesgue measure:*

(a) $$r \to (2\pi)^{-\frac{1}{2}} \exp[-(r - w(\vartheta))^2/2] \quad \textit{for} \quad r \in \mathbb{R}\,;$$
$$w(\vartheta\tau) = w(\vartheta) + w(\tau)$$

(b) $$r \to \Gamma(p) w(\vartheta)^p r^{p-1} \exp[-w(\vartheta)r] \quad \textit{for} \quad r > 0\,;$$
$$p > 0,\ w(\vartheta) > 0 \quad \textit{and} \quad w(\vartheta\tau) = w(\vartheta)w(\tau)\,.$$

This implies in particular that an equivariant, real-valued, and continuous statistic which is sufficient for $P_\vartheta^n \mid \mathscr{A}^n$, $\vartheta \in \Theta$, *exists for every* $n \in \mathbb{N}$, *namely* $(x_1, \cdots, x_n) \to \sum_{i=1}^n S(x_i)$.

PROOF. Let T_n denote the sufficient statistic with the properties specified in (vi). Let $\mathscr{U}^n$ denote the product topology on X^n and $\mathscr{A}^n$ the product σ-field. As $\mathscr{U}$ has a countable base, $\mathscr{A}^n$ is the Borel field of $\mathscr{U}^n$. In the following we shall apply Lemma 7 for X^n, $\mathscr{U}^n$, $\mathscr{A}^n$, P^n instead of X, $\mathscr{U}$, $\mathscr{A}$, P. To this aim, we first remark that $(X^n, \mathscr{U}^n)$ fulfills assumption (i) and $P^n \mid \mathscr{A}^n$ fulfills assumption (iii) of Lemma 7.

The family $P_\vartheta^n | \mathscr{A}^n$, $\vartheta \in \Theta$, is generated by the transformations on X^n defined by

$$\vartheta(x_1, \cdots, x_n) = (\vartheta x_1, \cdots, \vartheta x_n) , \qquad \vartheta \in \Theta .$$

The set of these transformations, endowed with the given topology $\mathscr{W}$, is a continuous transformation group fulfilling (ii) of Lemma 7. The densities

$$(x_1, \cdots, x_n) \to \prod_{i=1}^n h(x_i, \vartheta)$$

of $P_\vartheta^n | \mathscr{A}^n$ relative to $P^n | \mathscr{A}^n$ are continuous by assumption (v) of the Theorem. Assumption (v) of Lemma 7 is assumption (vi) of the Theorem.

Therefore, by Lemma 7, there exists a function $\psi : \mathbb{R} \times \Theta \to [0, \infty)$ such that

$$\prod_{i=1}^n h(x_i, \vartheta) = \psi(T(x_1, \cdots, x_n), \vartheta) \qquad \text{for all} \quad (x_1, \cdots, x_n) \in X^n, \ \vartheta \in \Theta .$$

By Borges and Pfanzagl [2] (page 263, (2.3)) we have

$$h(x, \vartheta)h(\vartheta^{-1}x, \vartheta^{-1}) = 1 \quad P\text{–a.e.}$$

As the densities are continuous in x, this equation holds everywhere by assumption (iii). This implies $h(x, \vartheta) \neq 0$ for all $x \in X$, $\vartheta \in \Theta$. As $h(x, \vartheta) \geqq 0$ P-a.e., a continuity argument reveals that $h(x, \vartheta) > 0$ for all $x \in X$, $\vartheta \in \Theta$. By the same argument as in the proof of the corollary in [13] we get from the Theorem in [9] that there exist functions $\alpha, c : \Theta \to \mathbb{R}$ and a continuous function $g : X \to \mathbb{R}$ such that

$$h(x, \vartheta) = c(\vartheta) \exp [\alpha(\vartheta)g(x)]$$

for all $x \in X$, $\vartheta \in \Theta$. (This is the point where the assumption enters that $(X, \mathscr{U})$ is pathwise connected.)

By Borges and Pfanzagl [2] (page 263, Theorem) for any one-dimensional exponential family generated by a transformation group there exists a sufficient statistic $S: X \to \mathbb{R}$ and an invariant measure $\lambda | \mathscr{A}$ such that the λ-densities of P_ϑ, $\vartheta \in \Theta$, are of one of the following two types:

(a) $\exp [-(S - w(\vartheta))^2/2]$ with $w(\vartheta\tau) = w(\vartheta) + a(\vartheta)w(\tau)$, $a(\vartheta\tau) = a(\vartheta)a(\tau)$, $a(\vartheta) = \pm 1$;

(b) $(w(\vartheta)S)^p \exp [-w(\vartheta)S]$ with $p > 0$, $w(\vartheta\tau) = w(\vartheta)w(\tau)$, $w(\vartheta) > 0$.

As S is a linear function of g (see [2] page 268, (5.4) for case (a) and [2] page 269, (6.5) for case (b)), S is continuous.

Now we shall prove part (a) of the assertion. We have (see [2] page 268, (5.8))

$$S(\vartheta^{-1}x) = a(\vartheta)(S(x) - w(\vartheta)) \quad \text{for } P\text{–a.e.} \qquad x \in X$$

and all $\vartheta \in \Theta$. Since S is continuous, $x \to S(\vartheta^{-1}x)$ and $x \to a(\vartheta)(S(x) - w(\vartheta))$

are continuous functions. As above we conclude from assumption (iii) that

$$S(\vartheta^{-1}x) = a(\vartheta)(S(x) - w(\vartheta)) \quad \text{for all} \quad x \in X,\ \vartheta \in \Theta .$$

Now we shall show that the functions a and w are continuous. As $P_\vartheta \,|\, \mathscr{A}$, $\vartheta \in \Theta$, contains more than one element, S is not constant. Let $x_1, x_2 \in X$ be such that $S(x_1) \neq S(x_2)$. As $a(\vartheta) = [S(\vartheta^{-1}x_1) - S(\vartheta^{-1}x_2)]/[S(x_1) - S(x_2)]$, $\vartheta \in \Theta$, the function a is continuous. As Θ is connected, this implies that $\{a(\vartheta) : \vartheta \in \Theta\}$ is connected. As $\{a(\vartheta) : \vartheta \in \Theta\} \subset \{-1, 1\}$ by Borges and Pfanzagl ([2] page 263) and as $a(\varepsilon) = 1$, we have $a(\vartheta) = 1$ for all $\vartheta \in \Theta$. Hence $w(\vartheta\tau) = w(\vartheta) + w(\tau)$ and $S(\vartheta^{-1}x) = S(x) - w(\vartheta)$. The continuity of w now follows immediately from the continuity of S.

By Borges and Pfanzagl ([2] page 268) we have $\lambda(\vartheta A) = \lambda(A)$ for every $\vartheta \in \Theta$, $A \in \mathscr{A}$. This implies $\lambda * S(B + w(\vartheta)) = \lambda(S^{-1}(B + w(\vartheta))) = \lambda(\vartheta S^{-1}(B)) = \lambda(S^{-1}(B)) = \lambda * S(B)$ for every $B \in \mathscr{B}$, (the Borel field of $\mathbb{R}$). Hence the induced measure $\lambda * S \,|\, \mathscr{B}$ is invariant under the translation group $\{w(\vartheta) : \vartheta \in \Theta\}$.

As w is continuous and not constant by assumption (iv) and as Θ is connected, $\{w(\vartheta) : \vartheta \in \Theta\}$ is connected and nondegenerate and therefore the full translation group of $\mathbb{R}$. Hence $\lambda * S \,|\, \mathscr{B}$ is invariant under the full translation group, and therefore proportional to the Lebesgue measure.

As $P_\vartheta \,|\, \mathscr{A}$ has density $\exp[-(S - w(\vartheta))^2/2]$ with respect to λ, $P_\vartheta * S \,|\, \mathscr{B}$ has density $r \to \exp[-(r - w(\vartheta))^2/2]$ with respect to $\lambda * S \,|\, \mathscr{B}$ and therefore density $r \to (2\pi)^{-\frac{1}{2}} \exp[-(r - w(\vartheta))^2/2]$ with respect to the Lebesgue measure.

Finally, we shall prove part (b) of the assertion. We have (see [2] page 270, (6.7)) $S(\vartheta^{-1}x) = w(\vartheta)S(x)$ for P–a.a. $x \in X$ and all $\vartheta \in \Theta$. As S is continuous, the pertaining P-null set is open and therefore empty by assumption (iii). Furthermore $S(x) > 0$ for P–a.a. $x \in X$ (see [2] page 270, (6.8)). Let $x_0 \in X$ be such that $S(x_0) > 0$. Then

$$w(\vartheta) = S(\vartheta^{-1}x_0)/S(x_0)$$

which implies the continuity of w. As in case (a) we obtain:

$$\lambda * S(w(\vartheta)B) = \lambda(B) \quad \text{for all} \quad \vartheta \in \Theta,\ B \in \mathscr{B} .$$

Hence the induced measure $\lambda * S \,|\, \mathscr{B}$ is invariant under the group of similarity transformation $\{w(\vartheta) : \vartheta \in \Theta\}$. As w is continuous and not constant by assumption (iv) and as Θ is connected, $\{w(\vartheta) : \vartheta \in \Theta\}$ is the full group of similarity transformations in $\mathbb{R}$. Hence $\lambda * S \,|\, \mathscr{B}$ is invariant under this group, and therefore proportional to the Haar measure on the multiplicative group of positive real numbers which has density $r \to r^{-1} 1_{(0,\infty)}(r)$ with respect to the Lebesgue measure on $\mathscr{B}$. As $P_\vartheta \,|\, \mathscr{A}$ has density $(w(\vartheta)S)^p \exp[-w(\vartheta)S]$ for all $\vartheta \in \Theta$, $P_\vartheta * S$ has density $r \to (w(\vartheta)r)^p \exp[-w(\vartheta)r]$ with respect to $\lambda * S \,|\, \mathscr{B}$ and therefore density $r \to \Gamma(p)(w(\vartheta))^p r^{p-1} \exp[-w(\vartheta)r]$ with respect to the Lebesgue measure on $\mathscr{B} \cap (0, \infty)$ for all $\vartheta \in \Theta$.

2. Discussion of the results. It is well known that, under appropriate regularity conditions, any family of p-measures admitting a real-valued sufficient statistic for some sample size greater than 1 is an exponential family. It is therefore obvious that, by combining this result with the characterization of the one-dimensional exponential families generated by transformation groups given in [2], one can obtain a result like that stated in the Theorem. The only problem is whether this is possible under natural conditions on the space, the transformation group, and the sufficient statistic. This is not clear in advance, since the conditions needed to obtain exponentiality from the existence of a sufficient statistic are rather artificial (see Section 3). The Theorem demonstrates that more natural conditions can be stated in the particular case of a family of p-measures which is generated by a transformation group.

In our Theorem, the emphasis is placed upon minimizing the conditions on the sufficient statistic. Example 1 and Proposition 1 show that none of the conditions on the sufficient statistic is dispensable.

Probably, some of the conditions on X and Θ can be weakened or replaced by other more natural conditions. Of particular interest is the problem whether the conclusion of the Theorem can be obtained without assuming Θ to be Abelian in advance.

In the author's opinion, the relevance of the Theorem does not consist in possible applications (as a tool for proving that certain statistical problems can be reduced to problems concerning families of normal or gamma distributions) but in the statement that for families generated by transformation groups sufficiently regular real-valued sufficient statistics do exist in exceptional cases only.

EXAMPLE 1. Let $X = \mathbb{R}$, Θ be the group of all translations on $\mathbb{R}$ and $P \mid \mathscr{B}$ the Cauchy distribution, say.

(i) For every $n \in \mathbb{N}$ there exists a bimeasurable 1—1 map $T_n : \mathbb{R}^n \to \mathbb{R}$. (Hint: Apply the Isomorphism Theorem in [12] page 14, Theorem 2.12 for $X_1 = E_1 = \mathbb{R}$, $X_2 = E_2 = \mathbb{R}^n$.) As T_n is 1—1, it is trivially equivariant. As it is bimeasurable, it is sufficient for $P_\vartheta{}^n \mid \mathscr{B}^n$, $\vartheta \in \Theta$. Hence the continuity condition cannot be omitted.

(ii) For every $n \in \mathbb{N}$ there exists a continuous map $T_n : \mathbb{R}^n \to \mathbb{R}$ whose restriction to the complement of an appropriate Lebesgue null set in $\mathscr{B}$ is 1—1 (see Denny [5]). T_n is sufficient for the family of all p-measures which are dominated by the n-dimensional Lebesgue measure on $\mathscr{B}^n$ and therefore, in particular, sufficient for $P_\vartheta{}^n \mid \mathscr{B}^n$, $\vartheta \in \Theta$. Hence the equivariance condition cannot be omitted.

The following proposition shows that similar results for families of p-measures admitting $\mathbb{R}^2$-valued sufficient statistics cannot be expected unless the equivariance condition is replaced by a more stringent condition.

PROPOSITION 1. *Let Θ be the group of translations on $\mathbb{R}$ and $P \mid \mathscr{B}$ any p-measure having a continuous and positive density with respect to the Lebesgue measure on $\mathscr{B}$. Then the family $P_\vartheta{}^n$, $\vartheta \in \Theta$, admits an equivariant, $\mathbb{R}^2$-valued, and continuous sufficient statistic for every sample size n.*

PROOF. For $n \leqq 2$ the assertion is trivial. Let $n > 2$ in the following. By Denny [5] there exists a continuous map $S_{n-1}: \mathbb{R}^{n-1} \to \mathbb{R}$ and a Lebesgue null set $N_{n-1} \in \mathscr{B}^{n-1}$ such that the restriction of S_{n-1} to $\bar{N}_{n-1}$ is 1—1.

Let $N_n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n: (x_2 - x_1, \cdots, x_n - x_1) \in N_{n-1}\}$. We have $N_n \in \mathscr{B}^n$ and $\lambda^n(N_n) = \int \lambda^{n-1}(N_{n-1} + x_1)\, d\lambda(x_1) = 0$ (where $\lambda^k \mid \mathscr{B}^k$ denotes the Lebesgue measure). The statistic $T_n: X^n \to \mathbb{R}^2$ defined by $T_n(x_1, \cdots, x_n) = (x_1, S_{n-1}(x_2 - x_1, \cdots, x_n - x_1))$ is equivariant and continuous, and the restriction of T_n to $\bar{N}_n$ is 1—1. Hence T_n is sufficient for the family $P_\vartheta{}^n \mid \mathscr{B}^n$, $\vartheta \in \Theta$.

We shall not enter into the question of whether the conditions on the family P_ϑ, $\vartheta \in \Theta$, (i.e. conditions (iii) and (v)), can be relaxed. The following example shows, however, that the conclusion of the Theorem cannot be obtained without any such conditions.

EXAMPLE 2. Let $X = (0, \infty)$, let Θ be the group of similarity transformations and let $P \mid \mathscr{B}$ be defined by having density $1_{(0,1)}$ relative to the Lebesgue measure. The pertaining family $P_\vartheta \mid \mathscr{B}$, $\vartheta \in \Theta$, admits an equivariant, real valued and continuous sufficient statistic for every sample size n, namely $T_n(x_1, \cdots, x_n) := \max \{x_1, \cdots, x_n\}$.

3. Historical remarks. As already mentioned above it is well known in literature that the existence of a real-valued sufficient statistic characterizes the one-parameter exponential family. The most recent result in this field is Theorem 4.1. of Denny ([6] page 408), referring to arcwise connected and locally connected spaces. He assumes that (i) the densities are continuous, (ii) the sufficient statistic preserves ample sets (i.e. for each pair of nonvoid open sets U_1, U_2 the relation $T(U_1) = T(U_2)$ implies that $T(U_1 \cap \bar{N}) \cap T(U_2 \cap \bar{N}) \neq \emptyset$ for every P-null set N), (iii) there is a measure P_0 of the family and a nonvoid arcwise connected open set U_0 such that the continuous density of P_0 relative to P is not constant on some open $V \subset U_0$.

A condition on T of a different nature occurs in Brown ([3] page 1458). Regrettably, his main result (Theorem 2.1, page 458) is wrong. In the revisited and revised version of his paper (see [4]) Brown uses conditions similar to those of Denny.

If the set S (see Lemma 4) would be empty, then the conditions of our Theorem would imply that T_n preserves ample sets. Since $S \neq \emptyset$ cannot be excluded in advance, Denny's result is not applicable. Furthermore, the application of the Theorem of Laube and Pfanzagl [9] yields exponentiality without a condition like Denny's condition (iii), mentioned above. For these reasons we have made no immediate use of the results of Denny. The study of the proof of Lemma 7 reveals, however, that the techniques are strongly influenced by the techniques applied earlier by Brown and Denny.

Another pertinent reference is Lindley ([11] page 107) who states that the translation parameter normal and the scale parameter gamma families are the only families of p-measures which admit a real-valued sufficient statistic and which can be transformed into a location parameter family. Despite the additional condition that the family be transformable into a location parameter family (which originates from Lindley's interest in fiducial distributions), his paper is irrelevant for our purposes because he completely neglects questions of a more technical nature. So, for instance, the fact that the existence of a real-valued sufficient statistic for some sample size greater than 1 implies exponentiality, is used without reference to any regularity conditions.

Our Theorem is also closely related to the theorem that, on $X = \mathbb{R}$, any location parameter family such that $(x_1, \cdots, x_n) \to \sum_1^n x_i$ is sufficient for some sample size $n > 1$ is a family of normal distributions (with constant variance). This result was first obtained by Koopman [8] under the assumption that the family has differentiable densities with respect to the Lebesgue measure. The most recent version of this theorem, which may be obtained as a special case of Theorem 1 of Kelker and Matthes [7], yields the same result without any further assumptions on the location parameter family. Compared with our Theorem this demonstrates that assumptions on the family of distributions (i.e. (iii) and (v)) can be relaxed at the cost of more restrictive assumptions on the sufficient statistic (which, in fact, imply (iii) and (v)). (This is related to the fact that any measurable solution φ of the functional equation $\varphi(x_1) \cdot \varphi(x_2) = \psi(T(x_1, x_2))$ is necessarily continuous in the particular case $T(x_1, x_2) = x_1 + x_2$, whereas measurable solutions other than continuous ones do exist in the general case.)

4. Lemmas. The main result of this section is Lemma 7 which is needed for the proof of the Theorem. All other Lemmas are auxiliary results, needed for the proof of Lemma 7.

If Θ is a transformation group on X, and T an equivariant function on X, the transformations $\vartheta : X \to X$ induce in a natural way transformations $\boldsymbol{\vartheta} : T(X) \to T(X)$, defined by $\boldsymbol{\vartheta} t : = T\vartheta T^{-1}\{t\}$. (It is easy to see that for $t \in T(X)$, $T\vartheta T^{-1}\{t\}$ consists of a single element only, so that this definition is meaningful.) The transformations $\boldsymbol{\Theta} : = \{\boldsymbol{\vartheta} : \vartheta \in \Theta\}$ form a group which is homomorphic to $\boldsymbol{\Theta}$ under $\vartheta \to \boldsymbol{\vartheta}$.

In addition to the notations introduced in Section 1 we shall use $\mathscr{O}$ to denote the topology of $\mathbb{R}$ and $\mathscr{B}$ to denote its Borel field. For any p-measure $Q \mid \mathscr{A}$, let $Q * T|\mathscr{B}$ denote the induced p-measure defined by $Q * T(B) := Q(T^{-1}B)$, $B \in \mathscr{B}$.

LEMMA 1. *Let* $(X, \mathscr{U})$ *be a Hausdorff space and* $(\Theta, \mathscr{W})$ *a continuous transformation group on* $(X, \mathscr{U})$. *If* $T: X \to \mathbb{R}$ *is continuous and equivariant, then* $\vartheta \to \boldsymbol{\vartheta} t$ *is continuous for every* $t \in T(X)$.

PROOF. Let $x \in T^{-1}\{t\}$ be arbitrary. As T is equivariant, we have $\{\vartheta \in \Theta : \boldsymbol{\vartheta} t \in O\} = \{\vartheta \in \Theta : T(\vartheta x) \in O\} = \{\vartheta \in \Theta : \vartheta x \in T^{-1}O\} \in \mathscr{W}$ for every $O \in \mathscr{O}$.

LEMMA 2. *Let* $(X, \mathscr{U})$ *be connected and locally connected; let* $T: X \to \mathbb{R}$ *be continuous. If* $A \subset T(X)$ *and* $T^{-1}A \in \mathscr{U}$, *then* $A \in \mathscr{O} \cap T(X)$.

PROOF. (i) At first we shall prove the assertion for the special case that A is an interval.

If A is an open interval, the assertion is obvious.

If $A = [a, b)$ (including the case $b = \infty$), $T^{-1}A \in \mathscr{U}$ implies $T^{-1}[a, \infty) \in \mathscr{U}$ (since $T^{-1}[a, \infty) = T^{-1}[a, b) \cup T^{-1}(a, \infty)$). As $T^{-1}[a, \infty)$ is closed, too, connectedness of X implies $X = T^{-1}[a, \infty)$, i.e. $T(X) \subset [a, \infty)$. Then $A \subset (-\infty, b) \cap T(X) \subset [a, b) = A$ and hence $A = (-\infty, b) \cap T(X) \in \mathscr{O} \cap T(X)$.

If $A = [a, b]$, it can be seen as above that $X = T^{-1}[a, b]$, hence $T(X) \subset [a, b]$. This implies $A = T(X) \in \mathscr{O} \cap T(X)$.

The case $A = (a, b]$ is analogous to the case $A = [a, b)$.

(ii) Now we shall prove the assertion for arbitrary $A \subset T(X)$ with $T^{-1}A \in \mathscr{U}$.

At first we define an equivalence relation on A by $s \sim t$ if and only if there exists $U \in \mathscr{U}$ such that $\{s, t\} \subset T(U) \subset A$ and $T(U)$ is an interval. The relation is obviously symmetric and transitive. As X is locally connected, the relation is reflexive, too. The pertaining equivalence classes are intervals. Let I be one of them and let $x \in T^{-1}I$ be arbitrary. As $T^{-1}A \in \mathscr{U}$ and as X is locally connected, there exists an open and connected $U \subset T^{-1}A$ such that $x \in U$. As $t \sim T(x)$ for all $t \in T(U)$ and as $T(x) \in I$, we have $T(U) \subset I$ and therefore $U \subset T^{-1}I$. This implies $T^{-1}I \in \mathscr{U}$ and by (i) $I \in \mathscr{O} \cap T(X)$. As A is the union of all equivalence classes, we have $A \in \mathscr{O} \cap T(X)$.

LEMMA 3. *Let* $(X, \mathscr{U})$ *be connected and locally connected and let* $(\Theta, \mathscr{W})$ *be a continuous transformation group on* $(X, \mathscr{U})$. *If* $T: X \to \mathbb{R}$ *is continuous and equivariant, then* $t \to \boldsymbol{\vartheta} t$ *is* $\mathscr{O} \cap T(X)$*-continuous for every* $\vartheta \in \Theta$.

PROOF. It suffices to show that $A := \{t \in T(X) : \boldsymbol{\vartheta} t \in O\} \in \mathscr{O} \cap T(X)$ for every $O \in \mathscr{O}$. As $T^{-1}A = \vartheta^{-1}T^{-1}O \in \mathscr{U}$, the assertion follows immediately from Lemma 2.

LEMMA 4. *Let* $(X, \mathscr{U})$ *be a topological space,* $(\Theta, \mathscr{W})$ *a connected, continuous transformation group on* $(X, \mathscr{U})$, *and* $T: X \to \mathbb{R}$ *continuous and equivariant. Let* $S := \{t \in T(X) : \boldsymbol{\Theta} t = \{t\}\}$. *Then*

(i) *S is invariant and $\mathscr{O} \cap T(X)$-closed.*

(ii) *if for some $W \in \mathscr{W}$ and some $t \in T(X)$ the set $\mathbf{W}t$ consists of a single point, then $t \in S$.*

PROOF. (i) is straightforward.

(ii) $\Theta_0 := \{\vartheta \in \Theta : \boldsymbol{\vartheta} t = t\}$ is a subgroup of Θ. If $\mathbf{W}t$ consists of a single point, we have $\boldsymbol{\tau}^{-1}\mathbf{W}t = \{t\}$ for every $\tau \in W$. Hence $\tau^{-1}W \subset \Theta_0$ and therefore $\Theta_0 = \Theta_0\tau^{-1}W$. As $\tau^{-1}W$ is open, this implies that Θ_0 itself is open. As any open subgroup is closed (see [14] page 102, B), the connectedness of Θ implies $\Theta_0 = \Theta$.

By definition of sufficiency, for every $A \in \mathscr{A}$ there exists a conditional expectation of 1_A, given T, relative to P_ϑ which is independent of $\vartheta \in \Theta$, say $p(A, \cdot)$. We have

$$\int p(A, t) 1_B(t) P_\vartheta * T(dt) = P_\vartheta(A \cap T^{-1}B)$$

for every $B \in \mathscr{B}$, $\vartheta \in \Theta$. Whenever we speak of a "conditional expectation" in the following we mean a fixed version of such a "conditional expectation, given T, relative to P_ϑ which is independent of $\vartheta \in \Theta$."

LEMMA 5. *Assume that*

(i) *$(X, \mathscr{U})$ is connected and locally connected;*

(ii) *$(\Theta, \mathscr{W})$ is a connected and locally connected continuous transformation group which is Abelian;*

(iii) *$P(U) > 0$ for every $U \in \mathscr{U}$, $U \neq \emptyset$;*

(iv) *there exists an equivariant, real-valued, and continuous statistic T which is sufficient for the generated family of p-measures $P_\vartheta | \mathscr{A}$, $\vartheta \in \Theta$.*

Let S be defined as in Lemma 4. Let $t_0 \in \bar{S}$ and $U \in \mathscr{U}$ with $t_0 \in T(U)$ be given. For any conditional expectation $p(U, \cdot)$ there exists a $P * T$-null set M and a non-degenerate interval I containing t_0 such that

(i) *$p(U, t) > 0$ for every $t \in I \cap \bar{M}$,*

(ii) *$I \cap (t_0, \infty) \neq \emptyset$ unless $\mathbf{W}t_0 \cap (t_0, \infty) = \emptyset$ for some open neighborhood W of ε, and $I \cap (-\infty, t_0) \neq \emptyset$ unless $\mathbf{W}t_0 \cap (-\infty, t_0) = \emptyset$ for some open neighborhood W of ε.*

PROOF. (A) Let $\Theta_0 := \{\vartheta \in \Theta : \boldsymbol{\vartheta} t_0 = t_0\}$. As Θ is Abelian, we have $\{\vartheta \in \Theta : \boldsymbol{\vartheta} t = t\} = \Theta_0$ for every $t \in \boldsymbol{\Theta} t_0 =: B_0$. As $t_0 \in \bar{S}$ and as S is invariant (Lemma 4 (i)) we have $B_0 \subset \bar{S}$.

Let $\mathscr{W}_0 := \{\{\vartheta \in \Theta : \boldsymbol{\vartheta} t_0 \in O\} : O \in \mathscr{O}\}$. As $\mathscr{O}$ has a countable base, so has $\mathscr{W}_0$. By Lemma 1, $\mathscr{W}_0 \subset \mathscr{W}$.

The map $\vartheta \to \boldsymbol{\vartheta} t$ is $\mathscr{W}_0$-continuous on Θ for every $t \in B_0$: Using that Θ is commutative, we obtain with $t = \boldsymbol{\tau} t_0$ that $\{\vartheta \in \Theta : \boldsymbol{\vartheta} t \in O\} = \{\vartheta \in \Theta : \boldsymbol{\vartheta\tau} t_0 \in O\} = \{\vartheta \in \Theta : \boldsymbol{\vartheta} t_0 \in \boldsymbol{\tau}^{-1} O\} \in \mathscr{W}_0$ for every $O \in \mathscr{O}$ (since $\boldsymbol{\tau}^{-1} O \in \mathscr{O} \cap T(X)$) by Lemma 3).

Furthermore, $\mathbf{W}t \in \mathscr{B}$ for every $W \in \mathscr{W}$, $t \in B_0$: As $(\Theta, \mathscr{W})$ is locally connected, W is the union of open and connected sets, say $W = \bigcup_{\lambda \in \Lambda} C_\lambda$. By Lemma 1, $\mathbf{C}_\lambda t$ is an interval. If $\mathbf{C}_\lambda t$ would consist of a single point, we would have $t \in S$ by Lemma 4 (ii). As this contradicts the assumption $t \in B_0 \subset \bar{S}$, each $\mathbf{C}_\lambda t$ is a nondegenerate interval. Hence $\mathbf{W}t$ is the union of nondegenerate intervals and therefore an element of $\mathscr{B}$. (Hint: Any union of nondegenerate intervals can be written as disjoint union of nondegenerate intervals; any class of pairwise disjoint nondegenerate intervals is countable.)

(B) Let $\mathscr{D}_0$ denote the σ-field generated by $\mathscr{W}_0$. As $\{D \subset \Theta : D\Theta_0 = D\}$ is a σ-field containing $\mathscr{W}_0$, we have $D\Theta_0 = D$ for every $D \in \mathscr{D}_0$. Hence

(5.1) $\quad D, D' \in \mathscr{D}_0$ and $D \cap D' = \emptyset$ implies $\mathbf{D}t \cap \mathbf{D}'t = \emptyset$ for every $t \in B_0$.

From this we easily obtain that $\{D \in \mathscr{D}_0 : \mathbf{D}t \in \mathscr{B}\}$ is a σ-field for every $t \in B_0$. As this σ-field contains $\mathscr{W}_0$ (by (A)), we have $\mathbf{D}t \in \mathscr{B}$ for every $D \in \mathscr{D}_0$, $t \in B_0$. This enables us to define a measure $\nu \mid \mathscr{D}_0$ by

$$\nu(D) := P * T(\mathbf{D}t_0), \qquad D \in \mathscr{D}_0 .$$

That ν is, in fact, a measure, follows easily from (5.1). We have

$$\nu(\Theta) = P * T(B_0) \in (0, 1]$$

and

$$\int g(\boldsymbol{\vartheta} t_0)\nu(d\vartheta) = \int g(s) P * T(ds)$$

for every $P * T$-integrable function g.

(C) As $t_0 \in T(U) \cap \bar{S}$, there exists $x_0 \in U \cap T^{-1}S$ with $T(x_0) = t_0$. As $U \cap T^{-1}\bar{S} \in \mathscr{U}$, continuity of $(\vartheta, x) \to \vartheta x$ implies the existence of connected sets $W_0 \in \mathscr{W}$, $U_0 \in \mathscr{U}$ with $\varepsilon \in W_0$, $x_0 \in U_0$ such that

(5.2) $\quad W_0 U_0 \subset U \cap T^{-1}\bar{S} \quad$ and $\quad W_0^{-1} U_0 \subset U \cap T^{-1}\bar{S}$.

Convention. If there exist open sets $U' \ni x_0$ such that $T(U') \cap (t_0, \infty) = \emptyset$ or $T(U') \cap (-\infty, t_0) = \emptyset$, then we choose W_0, U_0 such that $T(W_0 U_0) \cap (t_0, \infty) = \emptyset$ respectively $T(W_0 U_0) \cap (-\infty, t_0) = \emptyset$.

(D) Let

$$E := \{(\vartheta, t) \in W_0\Theta_0 \times B_0 : p(U, t) < p(U_0, \boldsymbol{\vartheta} t)\} .$$

In (A) it was shown that $\vartheta \to \boldsymbol{\vartheta} t$ is $\mathscr{W}_0$-continuous on Θ for every $t \in B_0$. For any $\vartheta \in \Theta$, the restriction of $t \to \boldsymbol{\vartheta} t$ to B_0 maps onto B_0. It is $\mathscr{O} \cap B_0$-continuous by Lemma 3 and therefore $\mathscr{B}_0 := \mathscr{B} \cap B_0$-measurable. As $\mathscr{W}_0$ is countably generated, we obtain from a well-known lemma that $(\vartheta, t) \to \boldsymbol{\vartheta} t$ is $\mathscr{D}_0 \times \mathscr{B}_0$-measurable on $\Theta \times B_0$. We have $W_0\Theta_0 = \{\vartheta \in \Theta : \boldsymbol{\vartheta} t_0 \in \mathbf{W}_0 t_0\}$. As $\mathbf{W}_0 t_0 \in \mathscr{B}$ by (A) and as $\{\vartheta \in \Theta : \boldsymbol{\vartheta} t_0 \in B\} \in \mathscr{D}_0$ for every $B \in \mathscr{B}$, we have $W_0\Theta_0 \in \mathscr{D}_0$ and therefore

$$E \in \mathscr{D}_0 \times \mathscr{B}_0 . \tag{5.3}$$

Let E_ϑ and E^t denote sections of E. We shall show that

$$P * T(E_\vartheta) = 0 \quad \text{for every} \quad \vartheta \in W_0\Theta_0 . \tag{5.4}$$

As $\mathscr{B}_0 \subset \mathscr{B}$, we have for all $\vartheta \in \Theta$, $B \in \mathscr{B}_0$:

$$\begin{aligned} \int p(U_0, \boldsymbol{\vartheta} t) 1_B(t) P * T(dt) &= \int p(U_0, \boldsymbol{\vartheta} t) 1_{\vartheta B}(\boldsymbol{\vartheta} t) P * T(dt) \\ &= \int p(U_0, s) 1_{\vartheta B}(s) P_\vartheta * T(ds) = P_\vartheta(U_0 \cap T^{-1}\boldsymbol{\vartheta} B) = P(\vartheta^{-1}U_0 \cap T^{-1}B) \\ &= \int p(\vartheta^{-1}U_0, t) 1_B(t) P * T(dt) . \end{aligned}$$

Hence for every $\vartheta \in \Theta$:

$$p(U_0, \boldsymbol{\vartheta} t) = p(\vartheta^{-1}U_0, t) \quad \text{for} \quad P * T\text{–a.a.} \qquad t \in B_0 ,$$

(with the $P * T$–null set depending on ϑ and U_0).

As $\vartheta^{-1}U_0 \subset U$ for every $\vartheta \in W_0$, this implies

$$p(U_0, \boldsymbol{\vartheta} t) \leqq p(U, t) \quad \text{for every } \vartheta \in W_0 \text{ and } P * T\text{–a.a.} \qquad t \in B_0 . \tag{5.5}$$

As Θ is commutative, $\Theta_0 t = t$ for any $t \in B_0$ so that (5.5) holds for every $\vartheta \in W_0\Theta_0$. This establishes (5.4).

From (5.4) we obtain

$$\nu \times P * T(E) = 0 . \tag{5.6}$$

(E) Let $M := \{t \in \mathbb{R} : \nu(E^t) > 0\}$. By Fubini's Theorem, (5.6) implies $P * T(M) = 0$. The relation $\mathbf{W}_0 t_0 \cap (-\infty, t_0) \neq \emptyset$ implies $T(U_0) \cap (-\infty, t_0) \neq \emptyset$. (For if $T(U_0) \cap (-\infty, t_0) = \emptyset$, according to the convention in (C) W_0 and U_0 would have been chosen such that $T(W_0 U_0) \cap (-\infty, t_0) = \emptyset$. As $\mathbf{W}_0 t_0 = T(W_0 x_0) \subset T(W_0 U_0)$, this contradicts the assumption $\mathbf{W}_0 t_0 \cap (-\infty, t_0) \neq \emptyset$.) As both, $T(U_0)$ and $\mathbf{W}_0 t_0$, are intervals containing t_0 with $T(U_0) \cap (-\infty, t_0) \neq \emptyset$ and $\mathbf{W}_0 t_0 \cap (-\infty, t_0) \neq \emptyset$, there exists $s' > t_0$ such that $(s', t_0) \subset T(U_0) \cap \mathbf{W}_0 t_0$. Similarly, $\mathbf{W}_0 t_0 \cap (t_0, \infty) \neq \emptyset$ implies the existence of some $s'' > t_0$ such that $(t_0, s'') \subset T(U_0) \cap \mathbf{W}_0 t_0$. As $t_0 \in \bar{S}$, $\mathbf{W}_0 t_0$ is nondegenerate by Lemma 4 (ii). Hence at least one of the relations $\mathbf{W}_0 t_0 \cap (-\infty, t_0) \neq \emptyset$ or $\mathbf{W}_0 t_0 \cap (t_0, \infty) \neq \emptyset$ holds true. Correspondingly, we define I to be $(s', t_0]$ or $[t_0, s'')$ or (s', s'') (if both relations hold true). We have $t_0 \in I \cap T(U_0) \cap \mathbf{W}_0 t_0$. Furthermore, $I \cap (t_0, \infty) = \emptyset$ implies $\mathbf{W}_0 t_0 \cap (t_0, \infty) = \emptyset$, and $I \cap (-\infty, t_0) = \emptyset$ implies $\mathbf{W}_0 t_0 \cap (-\infty, t_0) = \emptyset$. This proves part (ii) of the assertion.

Now we shall show that $p(U, t) > 0$ for every $t \in I \cap \bar{M}$. For every $t \in I$ there exists $\tau \in W_0$ such that $t = \boldsymbol{\tau} t_0$. As Θ is commutative, we have for every integrable function g:

$$\int g(\boldsymbol{\vartheta} t)\nu(d\vartheta) = \int g(\boldsymbol{\tau}\boldsymbol{\vartheta} t_0)\nu(d\vartheta) = \int g(\boldsymbol{\tau} s) P * T(ds) = \int g(s) P_\tau * T(ds) .$$

Hence we obtain for every $t \in I \cap \bar{M}$:

$$(5.7)\qquad \int p(U_0, \vartheta t) 1_{W_0\Theta_0 \cap \bar{E}^t}(\vartheta)\nu(d\vartheta) = \int p(U_0, \vartheta t) 1_{W_0\Theta_0}(\vartheta)\nu(d\vartheta)$$
$$= \int p(U_0, \vartheta t) 1_{\mathbf{W}_0 t}(\vartheta t)\nu(d\vartheta) = \int p(U_0, s) 1_{\mathbf{W}_0 t}(s) P_\tau * T(ds)$$
$$= P_\tau(U_0 \cap T^{-1}\mathbf{W}_0 t)\,.$$

Using (5.2) we obtain $t \in I \subset T(U_0) \subset \bar{S}$. Hence the set $\mathbf{W}_0 t$ is nondegenerate by Lemma 4 (ii). Therefore $I \cap \mathbf{W}_0 t$ contains a nonempty open interval, say I_0. (If $t \neq t_0$, this follows immediately from the fact that then $t \in I^0$ by the definition of I; if $t = t_0$, we have $I \subset \mathbf{W}_0 t$.) As $I_0 \subset I \subset T(U_0)$, $U_0 \cap T^{-1}I_0$ is a nonempty open set. As $U_0 \cap T^{-1}\mathbf{W}_0 t \supset U_0 \cap T^{-1}I_0$, assumption (iii) implies $P_\tau(U_0 \cap T^{-1}\mathbf{W}_0 t) > 0$.

Together with (5.7) this implies the existence of some $\vartheta \in W_0\Theta_0 \cap E^t$ such that $p(U_0, \vartheta t) > 0$. As $\vartheta \in W_0\Theta_0 \cap \bar{E}^t$ implies $p(U, t) \geqq p(U_0, \vartheta t)$, we obtain $p(U, t) > 0$.

A function $f: X \to Y$ is a *contraction* of the function $g: X \to Y$ if $g(x') = g(x'')$ implies $f(x') = f(x'')$. In this case there exists a function $\psi: g(X) \to Y$ such that $f = \psi \circ g$.

A set A is *T-saturated* if $A = T^{-1}TA$.

LEMMA 6. *Let $(X, \mathscr{U})$ be connected and locally connected, $T: X \to \mathbb{R}$ continuous and A a T-saturated closed set. Let $h: X \to \mathbb{R}$ be a continuous map with the following properties:*

(a) *$h(x) = 1$ for $x \in A^\circ$ (the interior of A),*
(b) *h is a contraction of T on $\bar{A}$.*

Then h is a contraction of T on X.

PROOF. Let $x_1, x_2 \in X$ with $T(x_1) = T(x_2) =: t_0$ be given. We have to show that $h(x_1) = h(x_2)$.

(1) If $T(U_i) \cap (-\infty, t_0) \neq \emptyset$ for all open sets $U_i \ni x_i$, $i = 1, 2$ or $T(U_i) \cap (t_0, \infty) \neq \emptyset$ for all open sets $U_i \ni x_i$, $i = 1, 2$, then $h(x_1) = h(x_2)$.

Assume that $h(x_1) \neq h(x_2)$. As h is continuous, there exist open and connected sets $U_i \ni x_i$ such that $x \in U_i$ implies

$$|h(x) - h(x_i)| < \tfrac{1}{2}|h(x_1) - h(x_2)|\,, \qquad i = 1, 2\,.$$

W.l.g.: $T(U_i) \cap (-\infty, t_0) \neq \emptyset$ for $i = 1, 2$. As $t_0 \in T(U_i)$, $i = 1, 2$, and as U_i is connected, there exists $t_1 < t_0$ such that $(t_1, t_0) \subset T(U_1) \cap T(U_2)$.

(1a) If $(t_1, t_0) \cap T(\bar{A}) \neq \emptyset$, we choose $t \in (t_1, t_0) \cap T(\bar{A})$ and $y_i \in U_i \cap T^{-1}\{t\}$, $i = 1, 2$. As $y_i \in T^{-1}\{t\} \subset T^{-1}T\bar{A} = \bar{A}$, we have $h(y_1) = h(y_2)$ by assumption (b). As $y_i \in U_i$, we have

$$|h(y_i) - h(x_i)| < \tfrac{1}{2}|h(x_1) - h(x_2)|\,, \qquad i = 1, 2\,.$$

This, however, is contradictory.

(1b) If $(t_1, t_0) \cap T(\bar{A}) = \emptyset$, we have $(t_1, t_0) \subset T(A)$. We choose $t \in (t_1, t_0)$ and $y_i \in U_i \cap T^{-1}\{t\}$, $i = 1, 2$. As $y_i \in T^{-1}\{t\} \subset T^{-1}(t_1, t_0) \subset T^{-1}TA = A$, we have $y_i \in A^\circ$ and therefore $h(y_1) = h(y_2)$ by assumption. We obtain a contradiction as in (1a).

(2) Now we shall prove the assertion for arbitrary x_i, $i = 1, 2$. It obviously suffices to consider the case $t_0 \in T(A)$.

(2a) If $(T^{-1}\{t_0\})^\circ = \emptyset$, we have $T(U_i) \neq \{t_0\}$ for all open neighborhoods $U_i \ni x_i$, $i = 1, 2$.

(2aa) The case that $T(U_i) \cap (-\infty, t_0) \neq \emptyset$ for $i = 1, 2$ or $T(U_i) \cap (t_0, \infty) \neq \emptyset$ for $i = 1, 2$ was treated in (1).

(2ab) The remaining case is (w.l.g.):

$$T(U_1) \cap (-\infty, t_0) \neq \emptyset \quad \text{for all and} \quad T(U_1) \cap (t_0, \infty) = \emptyset \quad \text{for some open} \quad U_1 \ni x_1 ;$$

$$T(U_2) \cap (t_0, \infty) \neq \emptyset \quad \text{for all and} \quad T(U_2) \cap (-\infty, t_0) = \emptyset \quad \text{for some open} \quad U_2 \ni x_2 .$$

The sets

$$U' := (T^{-1}(-\infty, t_0])^\circ$$
$$U'' := (T^{-1}[t_0, \infty))^\circ$$

are open; $x_1 \in U'$, $x_2 \in U''$ and $U' \cap U'' = (T^{-1}\{t_0\})^\circ = \emptyset$. As X is connected, this implies $U' \cup U'' \neq X$. $x_0 \in \bar{U}' \cap \bar{U}''$ implies that $x_0 \in T^{-1}\{t_0\}$ and $T(U_0) \cap (-\infty, t_0) \neq \emptyset$ and $T(U_0) \cap (t_0, \infty) \neq \emptyset$ for every open $U_0 \ni x_0$. Hence the pair x_i, x_0 fulfills the assumptions of (1) and we obtain $h(x_i) = h(x_0)$ for $i = 1, 2$; hence $h(x_1) = h(x_2)$.

(2b) If $(T^{-1}\{t_0\})^\circ \neq \emptyset$, we obtain $h(x) = 1$ for all $x \in T^{-1}\{t_0\}$.

At first we remark that $t_0 \in T(A)$ implies $(T^{-1}\{t_0\})^\circ \subset A^\circ$ so that $h(x) = 1$ for every $x \in (T^{-1}\{t_0\})^\circ$.

Assume that $h(x_1) \neq 1$ for some $x_1 \in T^{-1}\{t_0\}$. This implies $x_1 \notin (T^{-1}\{t_0\})^\circ$, so that $T(U_1) \neq \{t_0\}$ for every open $U_1 \ni x_1$.

(2ba) If $T(U_1) \cap (-\infty, t_0) \neq \emptyset$ and $T(U_1) \cap (t_0, \infty) \neq \emptyset$ for all open $U_1 \ni x_1$, we define

$$U' := T^{-1}(-\infty, t_0) \cup T^{-1}(t_0, \infty) \cup \{x \in X : h(x) \neq 1\}$$
$$U'' := (T^{-1}\{t_0\})^\circ .$$

U', U'' are open; $x_1 \in U'$ and $U'' \neq \emptyset$ by assumption. Furthermore, $U' \cap U'' = \emptyset$ (for $U'' \subset A^\circ$ implies $h(x) = 1$ for all $x \in U''$). As X is connected, this implies $U' \cup U'' \neq X$. Let $x_0 \in \bar{U}' \cap \bar{U}''$. We have $x_0 \in T^{-1}\{t_0\}$ and $h(x_0) = 1$. Furthermore, $x_0 \in \bar{U}''$ implies $T(U_0) \neq \{t_0\}$ for every open $U_0 \ni x_0$. Hence the assumptions of (1) are fulfilled for the pair x_1, x_0, so that $h(x_1) = h(x_0)$. This implies $h(x_1) = 1$.

(2bb) The remaining case is (w.l.g.): $T(U_1) \cap (-\infty, t_0) \neq \emptyset$ for all and $T(U_1) \cap (t_0, \infty) = \emptyset$ for some open $U_1 \ni x_1$. The sets

$$U' := T^{-1}(-\infty, t_0) \cup ((T^{-1}(-\infty, t_0])^\circ \cap \{x \in X: h(x) \neq 1\})$$
$$U'' := (T^{-1}[t_0, \infty))^\circ$$

are open; $x_1 \in U'$ and $U'' \supset (T^{-1}\{t_0\})^\circ \neq \emptyset$. Furthermore, $U' \cap U'' = \emptyset$ (since $x \in U' \cap U''$ implies $x \in (T^{-1}\{t_0\})^\circ \cap \{x \in X: h(x) \neq 1\}$, which is impossible because of (a)). As X is connected, $U' \cup U'' \neq X$. For $x_0 \in \bar{U}' \cap \bar{U}''$, we have $x_0 \in T^{-1}\{t_0\}$. Furthermore, $x \in \bar{U}''$ implies $T(U_0) \cap (-\infty, t_0) \neq \emptyset$ for all open $U_0 \ni x_0$. Hence the assumptions of (1) are fulfilled for the pair x_1, x_0 so that $h(x_1) = h(x_0)$. It remains to be shown that $h(x_0) = 1$. If $h(x_0) \neq 1$, then $x_0 \in \bar{U}'$ implies that not only $T(U_0) \cap (-\infty, t_0) \neq \emptyset$ for all open $U_0 \ni x_0$ (which was obtained from $x_0 \in \bar{U}''$) but also $T(U_0) \cap (t_0, \infty) \neq \emptyset$ for all open $U_0 \ni x_0$, so that the assumptions of (2ba) are fulfilled for x_0 instead of x_1. These assumptions, however, imply $h(x_0) = 1$.

LEMMA 7. *Assume that*

(i) $(X, \mathscr{U})$ *is connected and locally connected;*

(ii) $(\Theta, \mathscr{W})$ *is a connected and locally connected continuous transformation group on X which is Abelian;*

(iii) *P is a p-measure on the σ-field* $\mathscr{A}$ *generated by* $\mathscr{U}$ *such that*

$$P(U) > 0 \quad \textit{for every} \quad U \in \mathscr{U}, \qquad U \neq \emptyset;$$

(iv) *each p-measure of the generated family* $P_\vartheta | \mathscr{A}$, $\vartheta \in \Theta$, *admits a continuous density relative to* $P|\mathscr{A}$, *say* $h(\cdot, \vartheta)$;

(v) *there exists an equivariant, real-valued, and continuous statistic T which is sufficient for* $P_\vartheta | \mathscr{A}$, $\vartheta \in \Theta$.

Then for every $\vartheta \in \Theta$, $h(\cdot, \vartheta)$ *is a contraction of T.*

PROOF. Let ϑ be an arbitrary element of Θ which remains fixed throughout the following proof. For $A \in \mathscr{A}$ let $p(A, \cdot)$ be a conditional expectation.

(A) At first we shall show that $t_0 := T(x_1) = T(x_2) \in \bar{S}$ implies $h(x_1, \vartheta) = h(x_2, \vartheta)$. If $h(x_1, \vartheta) < h(x_2, \vartheta)$ (w.l.g.) we choose $r \in (h(x_1, \vartheta), h(x_2, \vartheta))$ and define

$$U_1 := \{x \in X: h(x, \vartheta) < r\}, \qquad U_2 := \{x \in X: h(x, \vartheta) > r\}.$$

As U_i is open and $t_0 \in T(U_i) \cap \bar{S}$, we obtain from Lemma 5 the existence of nondegenerate intervals $I_i \ni t_0$ and of $P * T$-null sets M_i such that

$$I_i \cap \bar{M}_i \subset B_i := \{t \in T(X): p(U_i, t) > 0\}, \qquad i = 1, 2.$$

We shall show that $I_1 \cap I_2$ is a nondegenerate interval. If this were not the case, we had $I_1 \cap I_2 = \{t_0\}$ and therefore w.l.g.: $I_1 \cap (t_0, \infty) = \emptyset$ and

$I_2 \cap (-\infty, t_0) = \emptyset$. By the choice of I_i (see Lemma 5 (ii)) this would imply the existence of open neighborhoods W_1, W_2 of ε such that $\mathbf{W}_1 t_0 \cap (t_0, \infty) = \emptyset$ and $\mathbf{W}_2 t_0 \cap (-\infty, t_0) = \emptyset$. Hence $W := W_1 \cap W_2$ would be an open neighborhood of ε such that $\mathbf{W} t_0 = \{t_0\}$. This, however, contradicts the assumption $t_0 \in \bar{S}$ by Lemma 4 (ii).

Since $I_1 \cap I_2$ is nondegenerate, assumption (iii) implies $P * T(I_1 \cap I_2) > 0$. Hence we obtain

$$P * T(B_1 \cap B_2) \geqq P * T(I_1 \cap \bar{M}_1 \cap I_2 \cap \bar{M}_2) > 0 . \tag{7.1}$$

As T is sufficient for P_τ, $\tau \in \Theta$, there exist (apply Lehmann ([10] page 48, Theorem 8 for P instead of λ) $\mathscr{B} \cap T(X)$-measurable functions $g_\tau : T(X) \to [0, \infty)$ such that $h(x, \tau) = g_\tau(T(x))$ for P-a.a. $x \in X$. In particular:

$$N := \{x \in X : h(x, \vartheta) \neq g_\vartheta(T(x))\} \text{ is a } P\text{-null set.}$$

Let

$$C_1 := \{t \in T(X) : g_\vartheta(t) < r\} , \qquad C_2 := \{t \in T(X) : g_\vartheta(t) > r\} .$$

We have $U_i \cap T^{-1}\bar{C}_i \subset N$ and therefore

$$\int p(U_i, t) 1_{\bar{C}_i}(t) P * T(dt) = P(U_i \cap T^{-1}\bar{C}_i) = 0 .$$

As $p(U_i, t)1_{\bar{C}}(t) > 0$ for $t \in B_i \cap \bar{C}_i$, this implies

$$P * T(B_i \cap \bar{C}_i) = 0 , \qquad i = 1, 2 . \tag{7.2}$$

As $\bar{C}_1 \cup \bar{C}_2 = T(X)$, we obtain from (7.2)

$$P * T(B_1 \cap B_2) \leqq P * T(B_1 \cap B_2 \cap \bar{C}_1) + P * T(B_1 \cap B_2 \cap \bar{C}_2) = 0 .$$

This, however, contradicts (7.1). Therefore, $T(x_1) = T(x_2) \in \bar{S}$ implies $h(x_1, \vartheta) = h(x_2, \vartheta)$.

(B) Now we shall show that $x \in (T^{-1}S)^\circ$ implies $h(x, \vartheta) = 1$.

For all $A \in \mathscr{A}$ and any conditional expectation $p(A, \cdot)$ we have

$$P_\tau(A \cap T^{-1}B) = P_\tau * T(p(A, \cdot)1_B) \quad \text{for every} \quad \tau \in \Theta,\ B \in \mathscr{B} . \tag{7.3}$$

It follows from the definition of S that

$$P_\vartheta * T(p(A, \cdot)1_S) =' P * T(p(A, \cdot)1_S) . \tag{7.4}$$

(7.3) and (7.4) together imply

$$P_\vartheta(A \cap T^{-1}S) = P(A \cap T^{-1}S) . \tag{7.5}$$

Let $U := \{x \in (T^{-1}S)^\circ : h(x, \vartheta) > 1\}$. As $U \subset T^{-1}S$, (7.5) (applied for U instead of A) implies $P_\vartheta(U) = P(U)$. As $P_\vartheta(U) = \int h(x, \vartheta) 1_U(x) P(dx)$, this implies $P(U) = 0$. As U is open, we obtain $U = \emptyset$ by assumption (iii).

The proof for $\{x \in (T^{-1}S)^\circ : h(x, \vartheta) < 1\}$ is the same.

(C) The assertion now follows from Lemma 6 applied for $h(\cdot, \vartheta)$ instead of h and $T^{-1}S$ instead of A.

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