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DISTRIBUTION FREE TESTS FOR SYMMETRY BASED ON THE NUMBER OF POSITIVE SUMS

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Let $X_1, X_2, \ldots, X_N$ be independent identically distributed random variables with common continuous distribution function F. Designate by $\mathscr{I}$ a nonempty set of subsets of the integers $\{1, 2, \ldots, N\}$ and by $\mathscr{S} = \mathscr{S}(\mathscr{I})$ the mapping which assigns to each set $I \in \mathscr{I}$, $I = \{t_1, t_2, \ldots, t_k\}$ the partial sum $\sum_{t_j \in I} X_{t_j}$. Define the random variable $N = N(\mathscr{I})$ as the number of positive sums in the range of $\mathscr{S}$.

$N(\mathscr{I})$ has been shown to be distribution free when F is the distribution function of a symmetric random variable if $\mathscr{I} = \{1, 2, \ldots, N\}$ or $\mathscr{I} =$ power set of $\{1, 2, \ldots, N\}$. Several other nontrivial examples of this phenomenon have been discovered—all by different methods. This paper presents a unified method that derives all previously known results, provides a constructive method for obtaining infinitely many essentially different sets $\mathscr{I}$ with this property, and finally provides a powerful necessary condition on any such set $\mathscr{I}$ that yields a complete characterization of those sets $\mathscr{I}$ for which $N(\mathscr{I})$ is distribution free and $\mathscr{I}$ contains all k element subsets of $\{1, 2, \ldots, N\}$ where $k = 2, 3, \ldots, N - 1$.

1. Summary and introduction. Let $X_1, X_2, \cdots, X_K$ be independent identically distributed random variables with the common continuous distribution function F. Designate by $\mathscr{I}$ a nonempty set of subsets of the integers $1, 2, \cdots, K$ and by $\mathscr{S} = \mathscr{S}(\mathscr{I})$ the mapping which assigns to each set $I \in \mathscr{I}$, $I = \{t_1, t_2, \cdots, t_k\}$ the partial sum $\sum_{t_j \in I} X_{t_j}$. Define the random variable $N = N(\mathscr{I})$ as the number of positive sums in the range of $\mathscr{S}$. Previous authors have investigated sets $\mathscr{I}$ for which $N(\mathscr{I})$ is distribution free if X is symmetrically distributed; i.e. if $F(-x) = 1 - F(x)$. This paper presents a mapping with a simple geometrical interpretation that yields a unified proof of all known cases of sets $\mathscr{I}$ for which $N(\mathscr{I})$ is distribution free. In addition, a very restrictive necessary condition is derived from the mapping that enables the author to characterize all sets $\mathscr{I}$ for which $N(\mathscr{I})$ is distribution free that have all subsets of $1, 2, \cdots, K$ with j elements if $3 \leqq j \leqq K - 1$.

2. A theorem of Friedman, Katz and Koopmans. Let $a = (a_1, a_2, \cdots, a_K)$ be a K-tuple of real numbers. An s-permutation of a, $\varepsilon(a)$ is an assignment of signs at each coordinate of a followed by a permutation of the coordinates of the resulting vector, i.e., a vector of the form $(\varepsilon_{i_1} a_{i_1}, \varepsilon_{i_2} a_{i_2}, \cdots, \varepsilon_{i_K} a_{i_K})$ where $\varepsilon_j = \pm 1$, $j = 1, 2, \cdots, K$. This concept was introduced by E. Sparre-Andersen (1949). There are a total of $2^K K!$ s-permutations of any a and the set of points

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$s(a) = \{\varepsilon(a)\}$, with ε an s-permutation, is called the s-orbit of a. A vector a with coordinates satisfying $0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_K$ is said to be in normal form. For a rule $\mathscr{F}$ and $I \in \mathscr{F}$ let $\sigma(a, I)$ be that function which assigns to $I = \{t_1, t_2, \cdots, t_k\}$ the sum $\sum_{t_j \in I} a_{t_j}$. If $\sigma(\varepsilon a, I) \neq 0$ for all s-permutations ε and for all subsets I of the integers 1 through K then a is said to be admissible.

Let $V(a)$ denote the number of positive sums in $\{\sigma(a, I): I \in \mathscr{F}\}$. Clearly $0 \leqq V(a) \leqq$ number of elements in $\mathscr{F}$. Denote the number of elements in $\mathscr{F}$ by δ and for $0 \leqq j \leqq \delta$ set $f_a(j)$ equal to the frequency with which $V(a)$ equals j as ε ranges over all s-permutations of an admissible vector a.

THEOREM 1. (Friedman-Katz-Koopmans (1966)). *In order that $N = N(\mathscr{F})$ be distribution free over the set of all continuous symmetric distributions, it is sufficient that for all j, $0 \leqq j \leqq \delta$, $f_a(j)$ be independent of a for all admissible vectors a. In this case, if $f_a(j) \equiv f(j)$, $P(N(\mathscr{F}) = j) = f(j)$.*

The question of the necessity of the condition of this theorem was left open in [1].

3. A proof of the necessity of the condition in the theorem of Friedman, Katz, and Koopmans.

THEOREM 2. *Consider in R^K the set of K vectors $x = (x_1, x_2, \cdots, x_K)$ such that $0 < x_1 < x_2 < \cdots < x_K$ and let this region be divided into open subregions, to be called α-regions, by hyperplanes of the form $\sum \eta_i x_i = 0$, $\eta_i = 1, 0$, or -1, where not all $\eta_i = 0$. The points in these α-regions are just the admissible vectors in normal form.*

Let p_i denote the probability that the normal form of the vector $(X_1, X_2, \cdots, X_K)$ lies in the ith α-region. For a and b points of the same α-region, $f_a(j) = f_b(j)$. Consequently

$$P(N(\mathscr{F}) = k) = \textstyle\sum_{p_i} p_i f_i(k)$$

where $f_i(k)$ denotes the frequency $f_a(k)$ for a an element of the ith α-region. The p_i's are the integrals of the characteristic functions of α-regions times $2^K K!$ under an arbitrary continuous symmetric distribution function F and the $f_i(k)$'s are constants depending only on the regions. Using the completeness of the order statistic (Lehmann (1959)) and the fact each admissible order statistic is in just one α-region it follows that $f_i(k) = P(N(\mathscr{F}) = k)$ for all i.

One consequence of this proof is that the conditional expected value of $N(\mathscr{F})$ given the normal form of the sample vector is independent of the normal form.

4. The role of the N-hypercube in the combinatorial problem. As a consequence of Sections 3 and 4, the investigation of whether a test for symmetry based on a set $\mathscr{F}$ is distribution free may be reduced to the question of whether, for all admissible K vectors a, $f_a(k)$ is independent of a for each k. For a fixed vector a these frequencies can be calculated from a simple geometrical model. The set of points in the s-orbit of $(a_1, a_2, \cdots, a_K)$ are the vertices of a K dimensional box, if one ignores those points obtained by permuting the coordinates. The question of whether a sum $\sum_k \varepsilon_{i_k} a_{i_k}$, $\varepsilon_{i_k} = \pm 1, 0$, is positive becomes the question of whether a certain point of symmetry of the K-hyperbox, that is a point whose vector is $(\varepsilon_i a_i)_{i=1}^K$, $\varepsilon_1 = \pm 1, 0$ is on the positive side of the K-hyperplane $\sum_{i=1}^K x_i = 0$. The effect of permuting the coordinates of a may be thought of as arising from permuting the integers 1 through K in the sets of integers making up $\mathscr{I}$; informally one may speak of permuting the test $\mathscr{I}$ and say that the permuted test describes sums to be formed exactly as $\mathscr{I}$ did. The frequencies $f_a(k)$ which the Friedman-Katz-Koopmans theorem require to be independent may be thought of as arising first from assigning one of the 2^K possible combinations of signs to the vector a—thereby choosing a quadrant of the K-hyperbox determined by a. Second, one of the $K!$ permutations of the integers 1 through K is selected and the elements of the sets in $\mathscr{I}$ are permutated using it. Third, corresponding to each set I in the permuted test, the symmetric point of the K hyperbox which has just those nonzero coordinates corresponding to numbers in I is examined to see if it is on the positive side of the K-hyperplane $\sum_{i=1}^K x_i = 0$; each of these points on the positive side of $\sum_{i=1}^K x_i = 0$ corresponds to a positive sum for that s-permutation. Fourth, the number of such positive sums at the third step is counted—if there are k an s-permutation of the vector a has been found yielding k positive sums. Fifth, as all possible s-permutations are considered the number of times there are k positive points at step four is totaled—this number divided by $2^K K!$ is $f_a(k)$. The procedure just described will be called the *counting process determined by* $\mathscr{I}$. A set of points counted over at step four is called a *constellation determined by* $\mathscr{I}$.

Under the linear transformation $X_i' = X_i/a_i$ the K dimensional hyperbox determined by a becomes the standard hypercube. The hyperplane $\sum_{i=1}^K X_i = 0$ becomes the hyperplane $\sum_{i=1}^K a_i X_i' = 0$. The counting process determined by $\mathscr{I}$ on the hyperbox corresponds in the obvious way with a counting process on the symmetric points of the K-hypercube. All that is now variable is the plane through the center of the K-hypercube. If the counting process yields the same frequencies of k points in a constellation on the positive side of any plane through the origin of the K hypercube, it follows that the frequencies of the Friedman-Katz-Koopmans theorem (Theorem 1) are independent of a and that $N(\mathscr{I})$ is distribution free.

5. A theorem on the deformation of hyperplanes.

THEOREM 3. *Let H_0 and H_1 be two hyperplanes through the origin which do not meet any of the points of symmetry of a K hypercube. Then there exists a con-*

tinuously parametrized family of hyperplanes $H(t)$, $0 \leqq t \leqq 1$, *such that* $H(0) = H_0$ *and* $H(1) = H_1$ *for which no* $H(t)$, $0 \leqq t \leqq 1$ *contains more than one pair of opposite symmetric points of the K-hypercube.*

The proof of this theorem is omitted.

6. The method of proof of the invariance of the counting process determined by $\mathscr{I}$. The proceeding theorem makes it possible to conclude that if the counting process determined by $\mathscr{I}$ remains invariant as a $K - 1$ dimensional hyperplane is deformed into another $K - 1$ dimensional hyperplane only passing over a single pair of diametrically opposite symmetric points then $N(\mathscr{I})$ is distribution free.

The following theorem shows that the existence of certain kinds of maps guarantees that the count of the number of constellations with k positive points will be the same every time a moving hyperplane passes over symmetric points of the K hypercube.

THEOREM 4. *If for all symmetric points s of the K-hypercube there exists* ϕ_s *a one-to-one map of points on constellations containing s to points on constellations containing s which maps constellations containing s to constellations containing s such that for every point p in the domain of* ϕ_s, *p and* $\phi_s(p)$ *lie on opposite sides of any hyperplane which passes through the origin and the point s, then* $N(\mathscr{I})$ *is distribution free.*

PROOF. Consider as a moving hyperplane $H(\tau)$ just passes over s, with s assigned first to its positive side and then to its negative side. A constellation with k points on the positive side of $H(\tau)$ initially is mapped by ϕ_s to a constellation with k points, on the negative side of $H(\tau)$ after $H(\tau)$ passes over s. Project this resulting constellation diametrically through the origin to get a one to one correspondence of constellations with k points on the positive side of $H(\tau)$ initially to constellations with k points on the positive side of $H(\tau)$ after $H(\tau)$ passes over s.

7. The determination of a natural mapping. Consider the map:

$$\begin{aligned} \phi_s(z) &= -z && \text{if } z\cdot s = 0 \\ \phi_s(z) &= s - z && \text{if } z\cdot s \neq 0 \text{ and } (s - z)\cdot s \neq 0 \\ \phi_s(z) &= 2s - z && \text{if } z\cdot s \neq 0 \text{ and } (s - z)\cdot s = 0 . \end{aligned}$$

It is clear in each case that the component of the image vector normal to s is the negative of the component of z normal to s. Consequently the image vector lies on the other side from z of any hyperplane through s and the origin. It is easily verified that $\phi_s(z)$ maps points on the same constellation to symmetric points of the K-hypercube. All known $\mathscr{I}$ for which $N(\mathscr{I})$ is distribution free have the property that ϕ_s takes constellations to constellations. These include:

$\mathscr{F}_1 = \{\{1\}, \{1, 2\}, \cdots, \{1, 2, \cdots, K\}\}$ (E. Sparre Andersen (1949))

$\mathscr{F}_2 = 2^{\{1,2,\cdots,K\}}$ (Kraft and Van Eeden (1964))

$\mathscr{F}_3 =$ {all two element subsets of $\{1, 2, \cdots, K\}$}
(Friedman, Katz, Koopmans (1966))

$\mathscr{F}_4 =$ {all subsets of $\{1, 2, \cdots, K\}$ with an even number of elements}
(Hartigan (1969))

$\mathscr{F}_5 = \mathscr{F}_2 \theta\{\{1, 2, \cdots, K\}\}$ (Friedman, Katz, Koopmans (1966))

$\mathscr{F}_6 = \mathscr{F}_4 \theta\{\{1, 2, \cdots, K\}\}$ (K even)

$\mathscr{F}_7 = \{\{1\}, \{1, 2\}, \cdots, \{1, 2, \cdots, K\}, \{K\}, \{K, K-1\}, \cdots, \{K, K-1, \cdots, 2\}\}.$

The method of proof of the properties of the natural mapping on all known rules is exactly the same. Only a proof that $\mathscr{F}_7$ is distribution free will be offered here.

THEOREM 5. *$N(\mathscr{F})$, $\mathscr{F} = \mathscr{F}_7$ as listed is distribution free.*

PROOF. Represent the constellation schematically as a collection of K-tuples in the following way. The case $K = 5$ is used here for illustration:

$$\begin{array}{ccc}
(1, 0, 0, 0, 0) & & (0, 0, 0, 0, 1) \\
(1, 1, 0, 0, 0) & & (0, 0, 0, 1, 1) \\
(1, 1, 1, 0, 0) & & (0, 0, 1, 1, 1) \\
(1, 1, 1, 1, 0) & & (0, 1, 1, 1, 1) \\
(1, 1, 1, 1, 1) & = & (1, 1, 1, 1, 1) .
\end{array}$$

Let s be, say, $(1, 1, 1, 0, 0)$ then $\phi_{(1,1,1,0,0)}$ yields the image constellation:

$$\begin{array}{ccc}
(0, 0, 1, 0, 0) & & (0, 0, 0, 0, -1) \\
(0, 1, 1, 0, 0) & & (0, 0, 0, -1, -1) \\
(1, 1, 1, 0, 0) & & (1, 0, 0, -1, -1) \\
(1, 1, 1, -1, 0) & & (1, 1, 0, -1, -1) \\
(1, 1, 1, -1, -1) & = & (1, 1, 1, -1, -1) .
\end{array}$$

This image set is easily seen to be an s-permutation of the original constellation; it is easily verified that this represents the general situation.

8. An algorithm for generating distribution free rules. The easiest way to prove that $N(\mathscr{F})$ is distribution free is to show that images of constellations under the mappings ϕ_s, s on the constellation, are constellations. This is especially easy when the image of a constellation is obtained from the constellation by interchanging signs.

Given an arbitrary collection of subsets of $\{1, 2, \cdots, K\}$, it is an easy matter to construct a smallest rule $\mathscr{S}$ which has the proceeding desirable property. For example if $\{1, 2, 3\}$ and $\{3, 4, 5\}$ are to be in $\mathscr{S}$ then $\phi_{(1,1,1,0,0)}(0, 0, 1, 1, 1) = (1, 1, 0, -1, -1)$ shows that $(1, 1, 0, 1, 1)$ must also represent a point in $\mathscr{S}$; or $\{1, 2, 4, 5\} \in \mathscr{S}$. $\mathscr{S}$ may thus be closed in this fashion yielding the desired rule.

The importance of this algorithm is that it indicates the enormous variety of rules $\mathscr{S}$ for which $N(\mathscr{S})$ is distribution free. At first glance, it might be supposed that application of this algorithm could always lead to only a few rules. However, it is easily verified that if the sets of integers one begins with are all unchanged under a permutation of the integers 1 through K then the resulting rule $\mathscr{S}$ is made up of sets which are also unchanged under the same permutation. This leads to the conclusion that there are at least as many distinct rules $\mathscr{S}$ using K observations for which $N(\mathscr{S})$ is distribution free as there are non isomorphic finite products of permutation groups of t_j elements where $\sum t_j = K$.

9. A proof of the necessity of the hypercube formulation and a necessary condition that follows from it.

THEOREM 6. *If $N(\mathscr{S})$ is distribution free then the counting process determined by $\mathscr{S}$ yields the same frequencies for all hyperplanes through the origin not meeting the symmetric points of the hypercube.*

The proof of this theorem is omitted as it is a straightforward consequence of the necessity of the condition in the Friedman-Katz-Koopmans theorem and the manner in which the counting process is induced on the hypercube.

The next theorem is crucial to the arguments of Section 10 and will be referred to as the opposite point theorem.

THEOREM 7. *Let $N(\mathscr{S})$ be distribution free. Let z and s both lie on the same constellation of $\mathscr{S}$ and let ϕ_s be the natural mapping associated with s. Then $\phi_s(z)$ lies on as many constellations containing s as z does.*

PROOF. The theorem is obviously true if $\phi_s(z) = -z$ or $\phi_s(z) = 2s - z$. For the case $\phi_s(z) = s - z$ the following lemmas are needed.

LEMMA 7.1. *Let z and s be on the same constellation and let $\phi_s(z) = s - z$ then if v is a symmetric point of the hypercube that is on a constellation with s such that v and z always lie on opposite sides of any hyperplane through the origin and s then $v = \phi_s(z) = s - z$.*

PROOF. The vector v is of the form $\alpha s - \beta z$ where $\beta > 0$. It is easily shown that $\alpha = 1$ and $\beta = 1$.

LEMMA 7.2. *Let $N(\mathscr{S})$ be distribution free. Let s be a point on a constellation* *determined by $\mathscr{S}$ and H denote a hyperplane through the origin and s. If the sign of s is not counted there are as many constellations containing s with k points on the positive side of H as there are constellations containing s with k points on the negative side of H.*

PROOF OF LEMMA. The lemma is a consequence of equating the number of constellations containing s and $-s$ with k positive points if s is assigned to the positive side of H to the number of constellations containing s and $-s$ with k positive points if s is assigned to the negative side of H and manipulating the resulting linear equations.

LEMMA 7.3. *Let $N(\mathscr{S})$ be distribution free and let z and s lie on a constellation determined by $\mathscr{S}$ with $\phi_s(z) = s - z$. Let H be a hyperplane passing through the origin and the points s and z. Assume no sign has been assigned to s or z. Define variables as follows:*

$x(k)$ = *number of constellations with k points on the positive side of H containing z and s but not $\phi_s(z)$,*

$y(k)$ = *number of constellations with k points on the positive side of H containing $\phi_s(z)$ and s but not z.*

Then, if δ denotes the number of points in a constellation of $\mathscr{S}$ minus two,

$$x(k) + x(\delta - k) = y(k) + y(\delta - k)$$

for all k.

PROOF OF LEMMA. For the purposes of the bookkeeping involved in this proof it is convenient to exhibit the following table of variables.

TABLE 1

variable # of constellations	# of points on positive side of H	# of points on negative side of H	Constellation contains		
			s	z	$\phi_s(z)$
$x(k)$	k	$\delta - k$	yes	yes	no
$y(k)$	k	$\delta - k$	yes	no	yes
$\bar{x}(k)$	$\delta - k$	k	yes	yes	no
$\bar{y}(k)$	$\delta - k$	k	yes	no	yes
$u(k)$	k	$\delta - 1 - k$	yes	yes	yes
$\bar{u}(k)$	$\delta - 1 - k$	k	yes	yes	yes
$w(k)$	k	$\delta + 1 - k$	yes	no	no
$\bar{w}(k)$	$\delta + 1 - k$	k	yes	no	no

In terms of these variables the lemma states:

$$x(k) + \bar{x}(k) = y(k) + \bar{y}(k) .$$

If z is assigned a positive sign, then by the last lemma:

$$w(k) + x(k - 1) + u(k - 1) + y(k) = \bar{w}(k) + \bar{x}(k) + \bar{u}(k - 1) + \bar{y}(k - 1) .$$

If z is assigned a negative sign:

$$w(k) + x(k) + u(k-1) + y(k-1) = \bar{w}(k) + \bar{x}(k-1) + \bar{u}(k-1) + \bar{y}(k) .$$

Therefore, by subtracting the first equation from the second:

$$x(k) - x(k-1) + y(k-1) - y(k) = \bar{x}(k-1) - \bar{x}(k) + \bar{y}(k) - \bar{y}(k-1) .$$

Setting k equal to zero and using induction one obtains:

$$x(k) - y(k) = \bar{y}(k) - \bar{x}(k)$$

or

$$x(k) + \bar{x}(k) = y(k) + \bar{y}(k) .$$

Restatement and proof of the opposite point theorem (Theorem 7):

The opposite point theorem states that $\phi_s(z)$ lies on as many constellations containing s as z does. There are two cases to consider. If the δ of the last lemma is odd, then

$$\sum_{k=0}^{[\delta/2]} x(k) + x(\delta - k)$$

equals the number of constellations containing s and z but not $\phi_s(z)$ which, in turn, equals

$$\sum_{k=0}^{[\delta/2]} y(k) + \bar{y}(k) ,$$

the number of constellations containing s and $\phi_s(z)$ but not z. If the number of constellations containing z, s and $\phi_s(z)$ are added, the number of constellations containing s and z equals the number of constellations containing s and $\phi_s(z)$. The case δ is even is treated similarly. Thus the opposite point theorem is proved.

10. The determination of all of the distribution free band rules. A rule $\mathscr{S}$ is a *band rule* if $\mathscr{S}$ contains all subsets of $\{1, 2, \cdots, K\}$ with k elements where $2 < k < K$.

THEOREM 8. *The only band rules $\mathscr{S}$, for which $N(\mathscr{S})$ is distribution free are $\mathscr{S}_2$, $\mathscr{S}_4$, $\mathscr{S}_5$ and $\mathscr{S}_6$ as listed earlier.*

PROOF. The following lemma relates the opposite point theorem (Theorem 7) to set operations a band rule must be closed under. The final theorem then is obtained by closing the band rules under the indicated set operations.

LEMMA 8.1. *Let z and s be points on a constellation of a rule $\mathscr{S}$, where $N(\mathscr{S})$ is distribution free. Let z and s represent, respectively, an α and a β element subset of $\mathscr{S}$ where $\mathscr{S}$ contains all subsets of $\{1, 2, \cdots, K\}$ with α and β elements. Let $\phi_s(z)$ have γ nonzero entries. Then $\mathscr{S}$ contains all γ element subsets of $\{1, 2, \cdots, K\}$.*

PROOF OF LEMMA. $\mathscr{S}$ contains all γ element subsets of $\{1, 2, \cdots, K\}$ if and only if all rules obtained by permuting $\mathscr{S}$ contain any one specified γ element subset of $\{1, 2, \cdots, K\}$. Let s and z both lie in the positive quadrant. Identify with the vector $s - z$ the vector t that has the same zero entries as $s - z$ but lies in the positive quadrant of R^K. Obviously t and $s - z$ lie on the same number of constellations containing s. A constellation containing s and t contains z. By the opposite point theorem, t lies on as many constellations containing s as z does. Thus each time s and z lie on the same constellation, t lies on that constellation. Restricting attention to the positive quadrant, it follows that the γ element subset of $\{1, 2, \cdots, K\}$ that corresponds to t must appear in every permutation of $\mathscr{J}$.

The following lemma follows by repeated application of the last lemma.

LEMMA 8.2. *Let $N(\mathscr{J})$ be distribution free and let $\mathscr{J}$ contain all ρ element subsets of $\{1, 2, \cdots, K\}$ where $2 < \rho < K$, then $\mathscr{J}$ contains all η element subsets of $\{1, 2, \cdots, K\}$ where η is an integer of the form $\rho \pm 2\gamma$ (γ an integer) which does not exceed $K - 1$. If ρ is odd, $\mathscr{J}$ also contains all subsets of $\{1, 2, \cdots, K\}$ with an even number of elements except possibly $\{1, 2, \cdots, K\}$.*

PROOF. Since $\rho < K$ and $\mathscr{J}$ contains all ρ element subsets of $\{1, 2, \cdots, K\}$ by hypothesis it is possible to choose a K-tuple s with ρ nonzero entries and then another, z, that differs from it in only two places that lie on the same constellation. By computing $\phi_s(z) = s - z$ one concludes from the previous lemma that $\mathscr{J}$ contains all two element subsets of $\{1, 2, \cdots, K\}$. As $\mathscr{J}$ contains all two element subsets of $\{1, 2, \cdots, K\}$ it is possible to choose a K-tuple, t, with only two nonzero entries, that are in the same places as nonzero entries of s, that lies on the same constellation as s. By computing $\phi_s(t) = s - t$ one concludes that $\mathscr{J}$ contains all subsets of $\{1, 2, \cdots, K\}$ with $\rho - 2$ elements. The result that $\mathscr{J}$ contains all subsets with $\rho - 2\gamma$ (γ an integer) elements follows by induction. The remainder of the proof follows from similar considerations and is omitted here.

Lemma 8.2 is now applied to determine possible band rules, $\mathscr{J}$, that yield distribution free statistics, $N(\mathscr{J})$. The simplest consequence of this lemma is that if $N(\mathscr{J})$ is distribution free and $\mathscr{J}$ contains all subsets of $\{1, 2, \cdots, K\}$ with ρ elements, where ρ is an odd integer such that $2 < \rho < K$ then $\mathscr{J}$ is either the Kraft-Van Eeden rule $\mathscr{J}_2$ or the Friedman-Katz-Koopmans rule $\mathscr{J}_5$. If $\mathscr{J}$ contains only sets with an even number of elements and is a band rule, the sweep-out lemma also applies directly and it follows that $\mathscr{J}$ must be Hartigan's rule $\mathscr{J}_4$ or the rule $\mathscr{J}_6$.

The case when $\mathscr{J}$ contains a set with an odd number of elements and ρ is even requires further argument.

LEMMA 8.3. *If $\mathscr{J}$ contains a subset I with an odd number of elements then $\mathscr{J}$ contains all one element subsets of I.*

This lemma is illustrated in the following example.

EXAMPLE. $K = 6$ $I = \{1, 2, 3, 4, 5\}$.

Let $s = (1, 1, 1, 1, 1, 0)$. By the sweep-out lemma $(1, 2, 3, 4) \in \mathscr{J}$. Therefore $z = (1, 1, 1, 1, 0, 0)$ lies on a constellation with s. Now $\phi_s(z) = s - z = (0, 0, 0, 0, 1, 0)$. As z lies on every constellation containing s, by the opposite point theorem $s - z$ lies on every constellation containing s. Thus $\{5\} \in I$. Considering $\mathscr{J}$'s arising from permuting the integers 1 through 5, it follows that $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$ are all elements of $\mathscr{J}$.

From the preceding lemma, it follows that $\mathscr{J}$ contains a one element subset J. The proof is best continued by example.

EXAMPLE.

$$K = 6\ J = \{1\}$$
$$z = (1, 1, 1, 1, 0, 0)$$
$$t = (1, 0, 0, 0, 0, 0)$$
$$\phi_z(t) = z - t = (0, 1, 1, 1, 0, 0)\,.$$

Now $z - t$ lies on as many constellations containing z as t does. It is easily seen that $v = (0, 1, 0, 0, 0, 0)$ lies on as many constellations containing z as t does. Thus v lies on as many constellations containing z as $z - t$ does. By the last lemma if $z - t$ is on a constellation in the positive quadrant then v is on that constellation. Thus everytime v and z lie on the same constellation $z - t$ lies on that constellation. That is if $\{2\}$ is in a rule obtained by permuting $\mathscr{J}$ then $\{2, 3, 4\}$ is in that rule. This shows by the last lemma that if $\{2\}$ is in a rule obtained by permuting $\mathscr{J}$ then $\{4\}$ is in the rule. From this it readily follows that $\mathscr{J}$ contains every one element subset of $\{1, 2, 3, 4, 5, 6\} \cdot \mathscr{J}$ then contains all subsets of $\{1, 2, 3, 4, 5, 6\}$ with three elements by looking at $z - t = \{0, 1, 1, 1, 0, 0\}$ and applying Lemma 8.1. This reduces the problem to a case already considered where ρ is odd and the theorem is proved.

A rule $\mathscr{J}$ is said to be *symmetric* if it is invariant under all permutations.

COROLLARY. *The only symmetric rules, $\mathscr{J}$, for which $N(\mathscr{J})$ is distribution free are:*

$$\mathscr{J} = \{\{i\} : i \in \{1, 2, \cdots, K\}\}$$
$$\mathscr{J} = \{\{1, 2, \cdots, K\}\}$$
$$\mathscr{J} = \{\{i, j\} : i, j \in \{1, 2, \cdots, K\}\}$$
$$\mathscr{J} = \{\{i, j\} : i, j \in \{1, 2, \cdots, K\} \textit{ and } i \neq j\}$$
$$\mathscr{J} = \textit{any band rule } \mathscr{J} \textit{ for which } N(\mathscr{J}) \textit{ is distribution free.}$$

PROOF. If $\mathscr{J}$ is symmetric and is not a band rule and $\{1, 2, \cdots, K\} \in \mathscr{J}$ then cannot contain a singleton set or a doubleton set as the opposite point theorem would make $\mathscr{J}$ a band rule.

11. The Sparre-Andersen type rules. A rule $\mathscr{S}$ is of *Sparre-Andersen* type if the sets in $\mathscr{S}$ form a chain under set inclusion. The following theorem is an easy consequence of the opposite point theorem and has application to random walk theory.

THEOREM 9. *Let $\mathscr{S}$ be of Sparre-Andersen type. If $N(\mathscr{S})$ is distribution free there exists an integer d which is a divisor of K such that $\mathscr{S}$ is obtained by permuting*

$$\{\{1, 2, \cdots, d\}, \{1, 2, \cdots, 2d\}, \cdots, \{1, 2, \cdots, id\}\} \qquad 1 \leqq i \leqq K/d .$$

The proof is omitted here.

12. A limitation of technique used in this paper. An interesting question is if $N(\mathscr{S})$ distribution free implies that the natural mapping maps constellations to constellations. It is not possible using the technique of this paper—that is considering the linear equations that occur among the frequencies as a result of considering a finite number of different positions for a hyperplane through the origin—to derive even the result that the same number of constellations must pass through s, z, and t as through s, $\phi_s(z)$, and $\phi_s(t)$. By considering the general form of such linear equations it is possible to demonstrate the last statement cannot be derived no matter how many additional points are considered. Attempts to program a computer to search for a counter example lead to formidable storage problems. The author conjectures that either a counter example exists or the problem is undecidable.

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