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ON SUBJECTIVE PROBABILITY AND EXPECTED UTILITIES

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The Anscombe and Aumann double relation approach for defining subjective probabilities and utilities in terms of a person's preferences is generalized for the case in which the set of states of the world is unrestricted (finite or not). A monotone continuity condition enables us to prove σ-additivity; the necessity of this condition is also proved if our other assumptions hold.

Although the single relation approach used by Fishburn appears to be more elegant, the present approach has the advantage of showing how the subjective probabilities arise.

1. Introduction. A theory about subjective probability and expected utilities for decision under uncertainty has been presented by Fishburn [3]. Let X be the set of consequences and $\mathscr{P}$ the set of all simple probability distributions over X, that is probability distributions that give all their mass to a finite subset of X. Let Θ be the set of states of the world and $\mathscr{H}$ the set of all functions on Θ to $\mathscr{P}$. Fishburn proved that certain axioms imply that there exists a real valued function u, on $\mathscr{P}$, and a finitely-additive probability π, on the set $\mathscr{S}$ of all subsets of Θ, such that, for all $P, Q \in \mathscr{H}$

$$(1) \qquad P \prec_h Q \quad \text{if and only if} \quad \int u(P(\theta))\, d\pi(\theta) \leqq \int u(Q(\theta))\, d\pi(\theta)$$

where $\prec_h$ is a binary preference-indifference relation assumed on $\mathscr{H}$. He also proved that each $P \in \mathscr{H}$ is bounded in the sense that there exist numbers a, b, depending on P, such that

$$(2) \qquad \pi\{\theta : a \leqq u(P(\theta)) \leqq b\} = 1 .$$

In a recent paper [4], the same author generalizes the approach of [3] and, in particular, he introduces a preference axiom that implies that π in (1) is countably additive.

In this paper, on the basis of an axiomatic method characterized by a double use of von Neumann and Morgenstern theory of utility, [7], and a monotone continuity condition on $\prec_h$, similar to the one used by Villegas [6], we give a constructive definition of subjective probability like Anscombe and Aumann [1], and prove that (1) holds. Monotone continuity enables us to prove σ-additivity. The necessity of this condition is also proved.

2. Basic assumptions. We assume the existence of a weak order (transitive and linear, or complete) $\prec$, on $\mathscr{P}$, that reflects a "rational" man preference pattern among simple distributions over consequences.

Let us use p, q, p_1, q_1, and so on, to denote elements of $\mathscr{P}$. If $p_1 \prec p_2$ holds,

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but $p_2 \prec p_1$ does not, we shall say that p_2 is preferred to p_1 and write $p_1 \prec p_2$. If $p_1 \preccurlyeq p_2$ and $p_2 \preccurlyeq p_1$, we say that p_1 and p_2 are equivalent and write $p_1 \sim p_2$.

A *utility* on $\mathscr{P}$ is a real-valued function u, defined on $\mathscr{P}$, and linear, that is, if $p_1, p_2 \in \mathscr{P}$, $0 \leqq \lambda \leqq 1$, then

$$u(\lambda p_1 + (1 - \lambda)p_2) = \lambda u(p_1) + (1 - \lambda)u(p_2) .$$

We say that a weak order $\preccurlyeq$, and a utility u, on $\mathscr{P}$, *agree* if for all $p_1, p_2 \in \mathscr{P}$

$$p_1 \preccurlyeq p_2 \quad \text{if, and only if} \quad u(p_1) \leqq u(p_2) .$$

From the expected utility theorem proved by Jensen [5] it follows that a weak order $\preccurlyeq$, as the one assumed on $\mathscr{P}$, has an agreeing utility u, uniquely determined up to a positive linear transformation, if the following hypotheses hold

(H1) If $\lambda \in (0, 1]$, and $p_1 \prec p_2$, then $\lambda p_1 + (1 - \lambda)q \prec \lambda p_2 + (1 - \lambda)q$.

(H2) If $p_1 \prec p_2 \prec p_3$, then there exist numbers $\lambda, \mu \in (0, 1)$ such that $\lambda p_3 + (1 - \lambda)p_1 \prec p_2 \prec \mu p_3 + (1 - \mu)p_1$.

Let us consider now the set $\mathscr{H}$ of all functions on Θ to $\mathscr{P}$, which we shall call horse lotteries, as do Anscombe and Aumann [1], and denote them by P, Q, P_1, Q_1, and so on. To understand the meaning of a horse lottery P, we must consider an experiment whose space of outcomes is Θ, and the gamble whose payoff is $P(\theta)$ when θ occurs.

Let $\mathscr{H}^*$ be the set of all simple probability distributions over $\mathscr{H}$ and denote by $(f_1P_1, \cdots, f_kP_k)$ that element of $\mathscr{H}^*$ which chooses P_i with probability f_i. Although $\mathscr{H}$ is not, properly speaking a subset of $\mathscr{H}^*$, we shall abuse the notation by treating $\mathscr{H}$ as a subset of $\mathscr{H}^*$ under the obvious isomorphism that maps $P \in \mathscr{H}$ into $(1P) \in \mathscr{H}^*$.

We assume the existence of a weak order $\preccurlyeq_h$, on $\mathscr{H}^*$ that reflects the rational man's preferences, and that both $\preccurlyeq$ and $\preccurlyeq_h$ satisfy H1, H2, and are connected by the following reasonable hypotheses

(C1) If $P(\theta) \preccurlyeq Q(\theta)$ for all θ, then $P \preccurlyeq_h Q$.

(C2) If $p_1 \prec p_2$ and $P_1(\theta) = p_1$, $P_2(\theta) = p_2$, for all θ, then $P_1 \prec_h P_2$.

(C3) $(f_1P_1, \cdots, f_kP_k) \sim_h P$, where $P(\theta) = f_1P_1(\theta) + \cdots + f_kP_k(\theta)$ for all θ.

Under the given assumptions, there exist utilities u and u_h agreeing with $\preccurlyeq$ and $\preccurlyeq_h$. The idea of subjective probability arises when we ask for the relation between u_h and u in light of the concept of horse lottery explained above.

We shall prove that if a monotone continuity condition on $\preccurlyeq_h$ is added, then there exist utilities u_h and u, and a σ-additive probability measure $\pi : \mathscr{S} \to [0, 1]$, such that for all $P \in \mathscr{H}$, the following relation holds

$$u_h(P) = \int u(P(\theta))\, d\pi(\theta) . \tag{3}$$

We say that the sequence P_n converges from above to P, and write $P_n \downarrow P$ if $P_1 \succcurlyeq_h P_2 \succcurlyeq_h \cdots$ and $\{\theta : P_n(\theta) \neq P(\theta)\} \downarrow \varnothing$, where $\neq$ means "is not identical to."

The monotone continuity condition announced, is the following

(MC1) If $P_n \downarrow P$ and $P_n \succcurlyeq_h \bar{P}$ for all n, then $P \succcurlyeq_h \bar{P}$.

Although axiom MC1 looks like a stronger version of axiom ACA in Fishburn [4], on the basis of our second theorem it can be proved that they are equivalent conditions if H1, H2, and C1, C2, C3, hold.

3. Theorems.

THEOREM 1. *Let the weak orders* $\preccurlyeq$ *on* $\mathscr{P}$ *and* $\preccurlyeq_h$ *on* $\mathscr{H}^*$ *satisfy* H1 *and* H2, *and let* C1, C2, C3 *and* MC1 *hold. Then there exist utilities* u *on* $\mathscr{P}$ *and* u_h *on* $\mathscr{H}^*$ *agreeing with* $\preccurlyeq$ *and* $\preccurlyeq_h$, *and a σ-additive subjective probability* π *on* $\mathscr{S}$, *such that* (3) *holds for all* $P \in \mathscr{H}$.

Furthermore, if for $p_0, p_1 \in \mathscr{P}$ *such that* $p_0 \prec p_1$ *we define* $P_0, P_1 \in \mathscr{H}$ *as* $P_0(\theta) = p_0$, $P_1(\theta) = p_1$ *for all* θ, *and take* $u_h(P_0) = u(p_0) = 0$, $u_h(P_1) = u(p_1) = 1$, *then* (3) *holds with* $\pi(S) = u_h(P_S)$, *where* S *is an arbitrary subset of* Θ *and*

$$P_S(\theta) = p_1 \quad \text{if} \quad \theta \in S \qquad\qquad (4)$$
$$= p_0 \quad \text{if} \quad \theta \notin S .$$

In addition, every horse lottery is bounded for π *and* u, *and if there exists a denumerable partition of* Θ *such that each event in this partition has positive probability under* π, *then* u *is bounded.*

PROOF. If $p \sim q$ for all $p, q \in \mathscr{P}$, then by C1 all $P, Q \in \mathscr{H}$ satisfy $P \sim_h Q$ and taking constant utilities, the conclusions follow immediately.

Now let $p_0, p_1 \in \mathscr{P}$ be such that $p_0 \prec p_1$, define P_0, P_1 as above and take $u(p_0) = u_h(P_0) = 0$, $u(p_1) = u_h(P_1) = 1$.

For all $P, Q \in \mathscr{H}$, C1 implies that if $u(P(\theta)) = u(Q(\theta))$, then $P \sim_h Q$; hence the equivalence class of functions $\{r : r(\theta) = u(Q(\theta)),\ Q \sim_h P\}$ determines the class of horse lotteries equivalent to P. This allows us to abuse our notation and write r in place of P where $r(\theta) = u(P(\theta))$.

We shall now divide the proof in several steps.

(a) Additivity. The set function $\pi(S) = u_h(P_S)$, with P_S defined as in (4), is an additive probability on $\mathscr{S}$.

PROOF. Since $u_h(P_0) = 0$ and $u_h(P_1) = 1$ it follows immediately that $\pi(\varnothing) = 0$ and $\pi(\Theta) = 1$; and, in addition, C1 implies $0 \leqq \pi(S) \leqq 1$ for all $S \subseteq \Theta$.

Let A, B be two disjoint subsets of Θ with $C = A \cup B$. Also let $P \in \mathscr{H}$ be defined by $P(\theta) = \frac{1}{2}p_0 + \frac{1}{2}p_1$ if $\theta \in A \cup B$, and $P(\theta) = p_0$ otherwise. Then by C3, $(\frac{1}{2}P_A, \frac{1}{2}P_B) \sim_h P \sim_h (\frac{1}{2}P_C, \frac{1}{2}P_0)$, and by linearity it follows that $\frac{1}{2}u_h(P_A) + \frac{1}{2}u_h(P_B) = u_h[(\frac{1}{2}P_A, \frac{1}{2}P_B)] = u_h(P) = u_h[(\frac{1}{2}P_C, \frac{1}{2}P_0)] = \frac{1}{2}u_h(P_C) + \frac{1}{2}u_h(P_0)$. The definition of π then gives $\frac{1}{2}\pi(A) + \frac{1}{2}\pi(B) = \frac{1}{2}\pi(C) + \frac{1}{2}\pi(\varnothing) = \frac{1}{2}\pi(C)$ and the additivity is proved.

(b) σ-additivity. $\pi(S) = u_h(P_S)$ is a σ-additive probability on $\mathscr{S}$.

PROOF. In order to prove the σ-additivity, it is sufficient to show that the additive probability π is continuous from above at $\emptyset$. That is, if $S_n \subseteq \Theta$, $n = 1, 2, \cdots$ and $S_n \downarrow \emptyset$, then $\pi(S_n) \downarrow 0$.

If we set $P_n = P_{S_n}$, then $\pi(S_n) = u_h(P_n)$ is a monotone decreasing sequence. Let us call $c = \lim u_h(P_n)$ and consider $\bar{P} \in \mathscr{H}$ such that $\bar{P}(\theta) = cp_1 + (1-c)p_0$ for all θ. Hence $P_n \downarrow P_0$, $P_n \succ_h \bar{P}$ and therefore by MC1 $P_0 \succ_h \bar{P}$. From this last relation we conclude that $c \leqq 0$, but on the other hand $c = \lim u_h(P_n) \geqq 0$, thus $c = \lim \pi(S_n) = 0$.

(c) If $P \in \mathscr{H}$ satisfies $p_0 \prec P(\theta) \prec p_1$ for all $\theta \in \Theta$, then (3) holds.

PROOF. Let us call $r(\theta) = u(P(\theta))$ and for every $S \subseteq \Theta$, define $\mu(S) = u_h(\chi_S r)$. By similar arguments as those given above, it can be proved that μ is a σ-additive measure on $\mathscr{S}$. Obviously μ is absolutely continuous with respect to π ($\mu \ll \pi$) and hence by the Radon-Nikodym Theorem there exists a real function $r'(\theta)$ uniquely determined modulo π such that $\mu(S) = \int_S r'(\theta)\, d\pi$ for all $S \subseteq \Theta$. It remains to prove that $r'(\theta) = r(\theta)$ $[\pi]$.

Let S be an arbitrary subset of Θ, then by C1

$$\pi(S) \inf_{\theta \in S} r(\theta) \leqq \int_S r'(\theta)\, d\pi \leqq \pi(S) \sup_{\theta \in S} r(\theta)\,. \tag{5}$$

Now, for every positive integer n, let us call $A_{n,i} = \{\theta : r(\theta) \in ((i-1)/2^n, i/2^n)\}$, $i = 1, 2, \cdots, 2^n$ and $A_{n,0} = \{\theta : r(\theta) = 0\}$, and let us consider the functions $r_n(\theta)$, $r^n(\theta)$, constant on each $A_{n,i}$ defined by $r_n(\theta) = \inf\{r(\theta') : \theta' \in A_{n,i}\}$, $r^n(\theta) = \sup\{r(\theta') : \theta' \in A_{n,i}\}$, for $\theta \in A_{n,i}$, $i = 0, 1, 2, \cdots, 2^n$. Then $r_n(\theta) \uparrow r(\theta)$, $r^n(\theta) \downarrow r(\theta)$ and by the Lebesgue Dominated Convergence Theorem we have that for every A

$$\int_A r_n(\theta)\, d\pi \uparrow \int_A r(\theta)\, d\pi\,, \qquad \int_A r^n(\theta)\, d\pi \downarrow \int_A r(\theta)\, d\pi\,. \tag{6}$$

But

$$\begin{aligned}\int_A r_n(\theta)\, d\pi = \sum_{i=0}^{2n} \int_{A \cap A_{n,i}} r_n(\theta)\, d\pi &\leqq \sum_{i=0}^{2n} \pi(A \cap A_{n,i}) \inf\{r(\theta) : \theta \in A \cap A_{n,i}\} \\ &\leqq \sum_{i=0}^{2n} \int_{A \cap A_{n,i}} r'(\theta)\, d\pi = \int_A r'(\theta)\, d\pi\end{aligned}$$

where the last inequality follows from (5). By similar arguments on $r^n(\theta)$ it can be shown that

$$\int_A r_n(\theta)\, d\pi \leqq \int_A r'(\theta)\, d\pi \leqq \int_A r^n(\theta)\, d\pi$$

and then from (6) it follows that for every A

$$\int_A r(\theta)\, d\pi = \int_A r'(\theta)\, d\pi\,.$$

Hence $r(\theta) = r'(\theta)$ $[\pi]$ and (3) is proved.

(d) Until now, we have proved (3) provided that $p_0 \prec P(\theta) \prec p_1$ for all θ. Consider the set $\mathscr{H}_0$ of all $P \in \mathscr{H}$ such that there exists an interval $[p_0', p_1']$ depending on P, that contains $P(\theta)$ for all θ. Then by similar arguments as those given in Ferguson [2], it can be shown that (3) remains true for all $P \in \mathscr{H}_0$ with $\pi(S) = u_h(P_S)$ as above.

(e) Let $\mathscr{H}_1$ be the set of all $P \in \mathscr{H}$ such that there exists an interval $[a, b]$, depending on P, that contains $u(P(\theta))$ for all θ. We shall show that (3) holds for all $P \in \mathscr{H}_1$.

PROOF. Let us call $I = \inf\{u(P(\theta)) : \theta \in \Theta\}$ and $S = \sup\{u(P(\theta)) : \theta \in \Theta\}$. Assume for definiteness, that $u(P(\theta)) > I$ for all θ and that there exists some θ_0 such that $u(P(\theta_0)) = S$.

If we set $S_n = \{\theta : I < u(P(\theta)) \leqq I + 1/n\}$, then $S_n \downarrow \emptyset$ and $\pi(S_n) \downarrow 0$. Let us call $p = P(\theta_1)$ for an arbitrary $\theta_1 \in \Theta$ and define $\bar{P}_n(\theta) = p$ for $\theta \in S_n$ and $\bar{P}_n(\theta) = P(\theta)$ elsewhere. Since $S_n \neq \emptyset$ and for a given $\theta' \in S_n$, $u(\bar{P}_n(\theta)) \geqq \inf\{u(p), u(P(\theta'))\}$ for all θ, then $\bar{P}_n$ satisfy the assumption of boundedness in (d) and it follows that

$$u_h(\bar{P}_n) = \int_{\bar{S}_n} u(P(\theta))\, d\pi + u(p)\pi(S_n) \to \int u(P(\theta))\, d\pi\,. \tag{7}$$

Notice that if $n > N$, then $\theta_1 \notin S_n$ and C1 leads to

$$0 \leqq u_h(\bar{P}_n) - u_h(P)\,. \tag{8}$$

Obviously $u(\bar{P}_n(\theta)) - u(P(\theta)) \leqq (u(p) - I)\chi_{S_n}(\theta)$ and then $\frac{1}{2}\bar{P}_n + \frac{1}{2}P_0 \prec_h \frac{1}{2}P + \frac{1}{2}P^*_{S_n}$, where $u(P^*_{S_n}(\theta)) = (u(p) - I)\chi_{S_n}(\theta)$.

Thus

$$u_h(\bar{P}_n) - u_h(P) \leqq (u(p) - I)\pi(S_n) \to 0\,. \tag{9}$$

From (7), (8) and (9), (3) follows immediately. Analogous extensions can be proved for the other cases.

(f) We shall show in this last step that (3) holds for every $P \in \mathscr{H}$ with $\pi(S) = u_h(P_S)$. This extension is based on the fact that every horse lottery is bounded in the sense of (2). The proof of this property, as that of the boundedness of u when there exists a denumerable partition of Θ such that each event has positive probability, can be achieved in a similar way as in Fishburn [3].

PROOF. The definition of boundedness of a horse lottery P says that there exists a pair of real numbers a, b, depending on P, such that $\pi\{\theta : a \leqq u(P(\theta)) \leqq b\} = 1$.

If we set $\Theta_n = \{\theta : -2^n \leqq u(P(\theta)) \leqq 2^n\}$, then there exists a positive integer N such that for all $n \geqq N$, $[a, b] \subset [-2^n, 2^n]$ and then $\pi(\Theta_n) = 1$. Let us define

$$P^n(\theta) = P(\theta) \quad \text{if} \quad \theta \in \Theta_n \qquad \text{and} \qquad P_n(\theta) = p_0 \quad \text{if} \quad \theta \in \Theta_n$$
$$ = p_0 \quad \text{if} \quad \theta \notin \Theta_n \qquad\qquad\qquad = P(\theta) \quad \text{if} \quad \theta \notin \Theta_n\,,$$

hence, $\frac{1}{2}P + \frac{1}{2}P_0 = \frac{1}{2}P^n + \frac{1}{2}P_n$ and then

$$u_h(P) = u_h(P^n) + u_h(P_n) \qquad \text{for all} \quad n\,. \tag{10}$$

Now $P^n \in \mathscr{H}_1$ and then by (e)

$$u_h(P^n) = \int u(P^n(\theta))\, d\pi = \int_{\Theta_n} u(P(\theta))\, d\pi\,.$$

But if $n \geqq N$, then $\pi(\Theta_n) = 1$ and then

$$u_h(P^n) = \int_{\Theta_n} u(P(\theta))\, d\pi = \int_{\Theta} u(P(\theta))\, d\pi\,. \tag{11}$$

From (10) and (11) it follows that $u_h(P_n)$ does not depend on n if $n \geqq N$, and therefore it only remains to prove that $u_h(P_n) = 0$ if $n \geqq N$.

Now obviously, for all $n \geqq N$, $P^n \downarrow P$ and $P^n \succ_h P^N$, so that, by MC1

$$u_h(P) \geqq u_h(P^N) .$$

From (10) and this last relation, it follows immediately that $u_h(P_n) \geqq 0$.

Let us assume that $u_h(P_n) = c > 0$ for all $n \geqq N$; then it is easy to see that there exists $p_c \in \mathscr{P}$ such that $u(p_c) = c$ so if we take $P_c(\theta) = p_c$ for all θ, it follows that $P_n \succ_h P_c$ for all $n \geqq N$. Since $P_n \downarrow P_0$ and $P_n \succ_h P_c$ for all $n \geqq N$, then by MC1 $P_0 \succ_h P_c$ and this contradicts the assumption that $u_h(P_n) = c > 0$.

THEOREM 2. *Let the weak orders $\prec$ on $\mathscr{P}$ and $\prec_h$ on $\mathscr{H}^*$ satisfy* H1 *and* H2, *and let* C1, C2 *and* C3 *hold. If there exist utilities u on $\mathscr{P}$ and u_h on $\mathscr{H}^*$ agreeing with $\prec$ and $\prec_h$ and a σ-additive probability π on $\mathscr{S}$ such that* (3) *holds for all $P \in \mathscr{H}$, then* MC1 *holds.*

PROOF. Let us assume that $P_n \downarrow P$ and $P_n \succ_h \bar{P}$. If we set $A_n = \{\theta : P_n(\theta) \neq P(\theta)\}$, then by (3)

$$u_h(P_n) - u_h(P) = \int_{A_n} \{u(P_n(\theta)) - u(P(\theta))\}\, d\pi$$

where the right hand member converges to zero because $A_n \downarrow \emptyset$. If we notice that $u_h(P_n) \geqq u_h(\bar{P})$, it obviously follows from $u_h(P) - u_h(\bar{P}) = (u_h(P) - u_h(P_n)) + (u_h(P_n) - u_h(\bar{P}))$ that

$$u_h(P) - u_h(\bar{P}) \geqq 0$$

and MC1 follows immediately.

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