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AN ANALOGUE, FOR SIGNED RANK STATISTICS, OF JUREČKOVÁ'S ASYMPTOTIC LINEARITY THEOREM FOR RANK STATISTICS[1]

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1. Introduction. The purpose of this note is to prove that if, for each $\nu = 1, 2, \cdots, X_{\nu,1}, \cdots, X_{\nu,n_\nu}$ are a random sample from a distribution symmetric around 0, then the signed-rank statistic

$$T_\nu(\theta) = \sum_{i=1}^{n_\nu} p_{\nu,i}\psi\left(\frac{R_{|X_{\nu,i}-q_{\nu,i}\theta|}}{n_\nu+1}\right)\operatorname{sgn}(X_{\nu,i} - q_{\nu,i}\theta),$$

where $R_{|X_{\nu,i}-q_{\nu,i}\theta|}$ is the rank of $|X_{\nu,i} - q_{\nu,i}\theta|$ among $|X_{\nu,1} - q_{\nu,1}\theta|, \cdots, |X_{\nu,n_\nu} - q_{\nu,n_\nu}\theta|$, is under certain conditions on the common distribution of the $X_{\nu,i}$, on the constants $p_{\nu,i}$, $q_{\nu,i}$ and on the function ψ, asymptotically approximately a linear function of θ in the sense that

$$\lim_{n_\nu\to\infty} P\{\sup_{|\theta|\leq C} |T_\nu(\theta) - T_\nu(0) + \theta K \textstyle\sum_{i=1}^{n_\nu} p_{\nu,i}q_{\nu,i}| \geqq \varepsilon\sigma(T_\nu(0))\} = 0, \tag{1.1}$$

for every $C > 0$ and every $\varepsilon > 0$, where K is a constant depending on the common distribution of the $X_{\nu,i}$ and on the function ψ.

This result is related to a result of Jurečková [3]; she proves (1.1) for the special case where $p_{\nu,i} \equiv 1$ and $\sum_{i=1}^{n_\nu} q_{\nu,i} \equiv 0$ under different conditions on the sequence of vectors $(q_{\nu,1}, \cdots, q_{\nu,n_\nu})$.

An analogous result was proved by Jurečková [2] for the statistic

$$S_\nu(\theta) = \sum_{i=1}^{n_\nu} C_{\nu,i}\varphi\left(\frac{R_{X_{\nu,i}-d_{\nu,i}\theta}}{n_\nu+1}\right),$$

where $R_{X_{\nu,i}-d_{\nu,i}\theta}$ is the rank of $X_{\nu,i} - d_{\nu,i}\theta$ among $X_{\nu,1} - d_{\nu,1}\theta, \cdots, X_{\nu,n_\nu} - d_{\nu,n_\nu}\theta$ and where, for each $\nu = 1, 2, \cdots$, the $X_{\nu,i}$ are independently and identically distributed.

For the proof of our result some lemmas are needed which are given in Section 2; one of these lemmas is a generalization of Theorem 5 of Lehmann [9]; two of the lemmas are analogous to Corollaries 1 and 2 of Lehmann [9]. The main result and their proofs are given in Section 3.

2. Some Lemmas. Let $i_1, \cdots, i_n$ and $j_1, \cdots, j_n$ each be a permutation of the numbers $1, \cdots, n$ and let $\varepsilon_1, \cdots, \varepsilon_n, \delta_1, \cdots, \delta_n$ each be $+1$ or -1 such that $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies

CONDITION A_n.1. $\quad \delta_k = 1 \Rightarrow \varepsilon_k = 1$

CONDITION A_n.2. $\quad \{l < k, \delta_k = 1, j_l < j_k\} \Rightarrow i_l < i_k$

CONDITION A_n.3. $\quad \{l < k, \varepsilon_k = -1, j_l > j_k\} \Rightarrow i_l > i_k$.

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For fixed $M(1 \leqq M \leqq n)$ define

$$a_{M,1} > a_{M,2} > \cdots > a_{M,K_M} \tag{2.1}$$

as the ordered values of those i_k among $i_{n-M+1}, i_{n-M+2}, \cdots, i_n$ for which $\varepsilon_k = +1$ and

$$b_{M,1} > b_{M,2} > \cdots > b_{M,L_M} \tag{2.2}$$

as the ordered values of those j_k among $j_{n-M+1}, j_{n-M+2}, \cdots, j_n$ for which $\delta_k = +1$. Obviously, by Condition $A_n.1$, $K_M \geqq L_M$; further $K_M \leqq M$. Further define

$$c_{M,1} > c_{M,2} > \cdots > c_{M,M-K_M} \tag{2.3}$$

as the ordered values of those i_k among $i_{n-M+1}, i_{n-M+2}, \cdots, i_n$ for which $\varepsilon_k = -1$ and

$$d_{M,1} > d_{M,2} > \cdots > d_{M,M-L_M} \tag{2.4}$$

as the ordered values of those j_k among $j_{n-M+1}, j_{n-M+2}, \cdots, j_n$ for which $\delta_k = -1$.

LEMMA 2.1. *If $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition A_n, then*

$$\begin{aligned} b_{M,l} &\leqq a_{M,l} \qquad l = 1, \cdots, L_M \\ c_{M,l} &\leqq d_{M,l} \qquad l = 1, \cdots, M - K_M \qquad M = 1, \cdots, n . \end{aligned} \tag{2.5}$$

PROOF. The proof will be given in four parts.

(i) The lemma is true for $M = 1$ and any $n \geqq 1$. To prove this, notice that by Condition $A_n.1$ it is sufficient to prove that

$$\begin{aligned} j_n &\leqq i_n \qquad \text{if} \quad \delta_n = 1 \\ j_n &\geqq i_n \qquad \text{if} \quad \varepsilon_n = -1 . \end{aligned} \tag{2.6}$$

This can be seen as follows.

$$\begin{aligned} j_n &= (\# \text{ of } j_k \leqq j_n) = n - (\# \text{ of } j_k > j_n) \\ i_n &= (\# \text{ of } i_k \leqq i_n) = n - (\# \text{ of } i_k > i_n) . \end{aligned} \tag{2.7}$$

By Condition $A_n.2$

$$(\# \text{ of } j_k \leqq j_n) \leqq (\# \text{ of } i_k \leqq i_n) \quad \text{if} \quad \delta_n = 1 \tag{2.8}$$

and by Condition $A_n.3$

$$(\# \text{ of } j_k > j_n) \leqq (\# \text{ of } i_k > i_n) \quad \text{if} \quad \varepsilon_n = -1 . \tag{2.9}$$

(ii) If the lemma is true for some (n, M) then the lemma is true for $(n + 1, M)$. To see this consider, for some $n \geqq 1$, $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^{n+1}$ satisfying Condition A_{n+1}. From $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^{n+1}$ derive $(i_k', \varepsilon_k, j_k', \delta_k)_{k=2}^{n+1}$, satisfying Condition A_n, as follows. Let

$$\begin{aligned} r_k &= \text{ rank of } i_k \text{ among } (i_1, i_k) \\ s_k &= \text{ rank of } j_k \text{ among } (j_1, j_k) \qquad k = 2, \cdots, n + 1 \end{aligned} \tag{2.10}$$

and let

$$\begin{aligned} i_k' &= i_k - (r_k - 1) \\ j_k' &= j_k - (s_k - 1) \qquad k = 2, \cdots, n + 1 . \end{aligned} \tag{2.11}$$

Then $i_2', \cdots, i_{n+1}'$ and $j_2', \cdots, j_{n+1}'$ are each permutations of the numbers $1, \cdots, n$ and from

$$i_k < i_l \Leftrightarrow i_k' < i_l' \qquad \qquad (2.12)$$
$$j_k < j_l \Leftrightarrow j_k' < j_l' \qquad k, l = 2, \cdots, n+1$$

it then follows that $\{i_k', \varepsilon_k, j_k', \delta_k\}_{k=2}^{n+1}$ satisfies condition A_n.

For fixed $M \leqq n$ let $a'_{M,l}$, $b'_{M,l}$, $c'_{M,l}$, $d'_{M,l}$ L_M' and K_M' be defined, as in (2.2) – (2.4), for $(i_k', \varepsilon_k, j_k', \delta_k)_{k=n+2-M}^{n+1}$ and let $a_{M,l}$, $b_{M,l}$, $c_{M,l}$, $d_{M,l}$, K_M, and L_M be so defined for $(i_k, \varepsilon_k, j_k, \delta_k)_{k=n+2-M}^{n+1}$, then $L_M = L_M'$ and $K_M = K_M'$. Assuming the lemma to be true for (n, M) we have

$$b'_{M,l} \leqq a'_{M,l} \qquad l = 1, \cdots, L_M \qquad (2.13)$$
$$c'_{M,l} \leqq d'_{M,l} \qquad l = 1, \cdots, M - K_M .$$

Now let l_0 be the number of $b_{M,l} > j_1$, then by (2.11)

$$b'_{M,l} = b_{M,l}^{-1} \qquad l = 1, \cdots, l_0 \qquad (2.14)$$
$$= b_{M,l} \qquad l = l_0 + 1, \cdots, L_M .$$

Let k_0 be the number of $a_{M,l} > i_1$, then by (2.11)

$$a'_{M,l} = a_{M,l}^{-1} \qquad l = 1, \cdots, k_0 \qquad (2.15)$$
$$= a_{M,l} \qquad l = k_0 + 1, \cdots, K_M .$$

Further, by Condition $A_{n+1}.2$, $l_0 \leqq k_0$. From (2.13) – (2.15) it then follows that

$$b_{M,l} \leqq a_{M,l} \qquad l = 1, \cdots, L_M . \qquad (2.16)$$

The proof that

$$c_{M,l} \leqq d_{M,l} \qquad l = 1, \cdots, M - K_M \qquad (2.17)$$

is analogous, using Condition $A_{n+1}.3$.

(iii) If the lemma is true for some $n \geqq 2$ with $M = n - 1$, then the lemma is true for the same n with $M = n$. This can be seen as follows. Assuming the lemma to be true for $M = n - 1$ we have

$$b_{n-1,l} \leqq a_{n-1,l} \qquad l = 1, \cdots, L_{n-1} \qquad (2.18)$$
$$c_{n-1,l} \leqq d_{n-1,l} \qquad l = 1, \cdots, n - 1 - K_{n-1}$$

and it will be proved that

$$b_{n,l} \leqq a_{n,l} \qquad l = 1, \cdots, L_n \qquad (2.19.i))$$

$$c_{n,l} \leqq d_{n,l} \qquad l = 1, \cdots, n - K_n . \qquad (2.19.ii))$$

The following three cases can be distinguished

(a) $\delta_1 = \varepsilon_1 = -1$. Then $L_n = L_{n-1}$, $K_n = K_{n-1}$, $b_{n,l} = b_{n-1,l}(l = 1, \cdots, L_n)$ and $a_{n,l} = a_{n-1,l}(l = 1, \cdots, K_n)$, so that (2.19.i)) is obvious. Further $(a_{n,l}, l = 1, \cdots, K_n, c_{n,l}, l = 1, \cdots, n - K_n)$ and $(b_{n,l}, l = 1, \cdots, L_n, d_{n,l}, l = 1, \cdots,$

$n - L_n$) are each permutations of the numbers $1, \cdots, n$ so that (2.19.ii)) follows from (2.19.i)).

(b) $\delta_1 = -1$, $\varepsilon_1 = 1$. Then $L_n = L_{n-1}$, $K_n = K_{n-1} + 1$, $b_{n,l} = b_{n-1,l}(l = 1, \cdots, L_n)$ and $c_{n,l} = c_{n-1,l}(l = 1, \cdots, n - K_n)$. To prove (2.19.i)) let k_0 be the number of $a_{n-1,l}(l = 1, \cdots, K_{n-1})$ larger than i_1, then

$$\begin{aligned} a_{n,l} &= a_{n-1,l} && l = 1, \cdots, k_0 \\ &= i_1 && l = k_0 + 1 \\ &= a_{n-1,l-1} && l = k_0 + 2, \cdots, K_n . \end{aligned} \tag{2.20}$$

If $L_n \leqq k_0 \leqq K_{n-1}$ then (2.19.i)) is immediate. If $0 \leqq k_0 < L_n = L_{n-1}$, then (2.19.i)) is immediate for $l = 1, \cdots, k_0$. Further

$$b_{n,k_0+1} = b_{n-1,k_0+1} \leqq a_{n-1,k_0+1} < i_1 = a_{n,k_0+1} \tag{2.21}$$

and for $l = k_0 + 2, \cdots, L_n$

$$b_{n,l} = b_{n-1,l} \leqq a_{n-1,l} = a_{n,l+1} \leqq a_{n,l} . \tag{2.22}$$

The proof of (2.19.ii)) is analogous.

(c) $\delta_1 = \varepsilon_1 = 1$. Then $L_n = L_{n-1} + 1$, $K_n = K_{n-1} + 1$, $c_{n,l} = c_{n-1,l}(l = 1, \cdots, n - K_n)$ and $d_{n,l} = d_{n-1,l}(l = 1, \cdots, n - L_n)$ so that (2.19.ii)) is obvious. Further (see (a)) (2.19.i)) follows from (2.19.ii)).

(iv) The lemma now follows by induction on M. According to part 1 of the proof, the lemma is true for $M = 1$ and any $n \geqq 1$. Let M_0 be an integer $\geqq 1$ and assume the lemma is true for $M = M_0$ and any $n \geqq M_0$, then it will be proved that the lemma is true for $M = M_0 + 1$ and any $n \geqq M_0 + 1$. This can be seen as follows. According to the induction hypothesis the lemma is true for $n = M_0 + 1$ and $M = M_0$; according to part 3 of the proof this implies the truth for $n = M_0 + 1$ and $M = M_0 + 1$; according to part 2 of the proof this implies the truth for $M = M_0 + 1$ and any $n \geqq M_0 + 1$. □

In Lemma 2.1 it was shown that Condition A_n is sufficient for (2.5) to hold for each $M = 1, \cdots, n$. For (2.5) to hold for a particular value of M it is obviously sufficient that $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies

$$\text{Condition } A_{n,M} \begin{cases} \text{For each } & k \geqq n - M + 1 \\ 1. & \delta_k = 1 \Rightarrow \varepsilon_k = 1 \\ 2. \text{ for each } & l \leqq k - 1 \qquad (\delta_k = 1, j_l < j_k) \Rightarrow i_l < i_k \\ 3. \text{ for each } & l \leqq k - 1 \qquad (\varepsilon_k = -1, j_l > j_k) \Rightarrow i_l > i_k . \end{cases}$$

Further, if $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for $M = M_0$ then $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for all $M \leqq M_0$, which proves the following lemma.

LEMMA 2.2. *If $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for $M = M_0$, then*

$$\begin{aligned} a_{M,l} &\leqq b_{M,l} && l = 1, \cdots, L_M \\ c_{M,l} &\leqq d_{M,l} && l = 1, \cdots, M - K_M \quad 1 \leqq M \leqq M_0 . \end{aligned} \tag{2.23}$$

LEMMA 2.3. *If h is nondecreasing and nonnegative and if $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for $M = M_0$, then*

$$\sum_{l=n+1-M;\varepsilon_l>0}^n h(i_l) \geqq \sum_{l=n+1-M;\delta_l>0}^n h(j_l)$$
$$\sum_{l=n+1-M;\varepsilon_l<0}^n h(i_l) \leqq \sum_{l=n+1-M;\delta_l<0}^n h(j_l) \qquad 1 \leqq M \leqq M_0 . \tag{2.24}$$

PROOF. Because h is nondecreasing, it follows from Lemma 2.2 that for $1 \leqq M \leqq M_0$

$$1. \quad h(b_{M,l}) \leqq h(a_{M,l}) \qquad l = 1, \cdots, L_M$$
$$2. \quad h(c_{M,l}) \leqq h(d_{M,l}) \qquad l = 1, \cdots, M - K_M . \tag{2.25}$$

From (2.25.1) and the fact that h is nonnegative it follows that, for $1 \leqq M \leqq M_0$,

$$\sum_{l=n+1-M;\delta_l>0}^n h(j_l) = \sum_{l=1}^{L_M} h(b_{M,l}) \leqq \sum_{l=1}^{L_M} h(a_{M,l}) \leqq \sum_{l=1}^{K_M} h(a_{M,l}) = \sum_{l=n+1-M;\varepsilon_l>0}^n h(i_l) . \tag{2.26}$$

From (2.25.2) and the fact that h is nonnegative it follows that for $1 \leqq M \leqq M_0$,

$$\sum_{l=n+1-M;\varepsilon<0}^n h(i_l) = \sum_{l=1}^{M-K_M} h(c_{M,l}) \leqq \sum_{l=1}^{M-K_M} h(d_{M,l}) \leqq \sum_{l=1}^{M-L_M} h(d_{M,l}) = \sum_{l=n+1-M;\delta<0}^n h(j_l) . \quad \square \tag{2.27}$$

REMARK. In the two special cases, where $\delta_k = 1$ for all k or $\varepsilon_k = -1$ for all k, Lemma 2.1 reduces to Theorem 5 of Lehmann [9]. Further, in each of these special cases, Lemma 2.3 is analogous to Corollary 1 of Lehmann [9].

LEMMA 2.4. *Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be n numbers satisfying*

$$0 \leqq \alpha_1 \leqq \cdots \leqq \alpha_n , \tag{2.28}$$

let h be nondecreasing and nonnegative and let $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfy

$$1. \quad (\delta_k = 1, \alpha_k > 0) \Rightarrow \varepsilon_k = 1$$
$$2. \quad (\delta_k = 1, \alpha_k > 0, l < k, j_l < j_k) \Rightarrow i_l < i_k$$
$$3. \quad (\varepsilon_k = -1, \alpha_k > 0, l < k, j_l > j_k) \Rightarrow i_l > i_k \tag{2.29}$$

then

$$\sum_{k=1}^n \alpha_k \varepsilon_k h(i_k) \geqq \sum_{k=1}^n \alpha_k \delta_k h(j_k) . \tag{2.30}$$

PROOF. The following proof is analogous to Lehmann's proof of his Corollary 2 in [9].

(2.30) is obviously true if $\alpha_k = 0$ for all $k = 1, \cdots, n$, so in the following it will be supposed that $\alpha_k > 0$ for at least one k. Further, since h is nonnegative,

$$\sum_{l=1}^n h(l) \geqq 0 \quad \text{and} \quad \sum_{l=1}^n h(l) = 0 \quad \text{if and only if} \quad h(l) = 0 \quad \text{for all} \quad l = 1, \cdots, n ,$$

in which case (2.30) is obvious. In the following it will be supposed that $\sum_{l=1}^n h(l) > 0$.

Let $0 \leqq \beta_1 < \beta_2 < \cdots < \beta_T$ be the different values of $\alpha_1, \cdots, \alpha_n$ and let

$n_t(t = 1, \cdots, T)$ be the number of α_k equal β_t. Further let $N_t = \sum_{s=1}^{t} n_s (t = 1, \cdots, T)$ and $N_0 = 0$. Consider the random variables X and Y each taking the values $(-\beta_T, -\beta_{T-1}, \cdots, -\beta_1, \beta_1, \cdots, \beta_{T-1}, \beta_T)$ with

$$(2.31)\qquad 1.\quad P(X \leqq -\beta_s) = \frac{\sum_{l=N_{s-1}+1;\varepsilon_l<0}^{N_T} h(i_l)}{\sum_{l=1}^{n} h(l)}$$

$$2.\quad P(X \leqq \beta_s) = 1 - \frac{\sum_{l=N_s+1;\varepsilon_l>0}^{N_T} h(i_l)}{\sum_{l=1}^{n} h(l)} \qquad s = 1, \cdots, T$$

and

$$(2.32)\qquad 1.\quad P(Y \leqq -\beta_s) = \frac{\sum_{l=N_{s-1}+1;\delta_l<0}^{N_T} h(j_l)}{\sum_{l=1}^{n} h(l)}$$

$$2.\quad P(Y \leqq \beta_s) = 1 - \frac{\sum_{l=N_s+1;\delta_l>0}^{N_T} h(j_l)}{\sum_{l=1}^{n} h(l)} \qquad s = 1, \cdots, T,$$

where, if $\beta_1 = 0$, $P(X \leqq 0)$ and $P(Y \leqq 0)$ are defined by (2.31.2) and (2.32.2) respectively.

If $\beta_1 > 0$, condition (2.29) reduces to Condition A_n and from Lemma 2.3 it then follows that

$$(2.33)\qquad P(X \leqq x) \leqq P(Y \leqq x) \qquad \text{for all } x.$$

If $\beta_1 = 0$, condition (2.29) is Condition $A_{n,M}$ for $M = N_T - N_1 = n - n_1$, so that in this case (2.24) holds for $M \leqq n - n_1$, which proves (2.33). From (2.33) it follows that

$$(2.34)\qquad \mathscr{E}X \geqq \mathscr{E}Y,$$

which is equivalent to

$$(2.35)\qquad \sum_{s=1}^{T} \beta_s \sum_{l=N_{s-1}+1}^{N_s} \varepsilon_l h(i_l) \geqq \sum_{s=1}^{T} \beta_s \sum_{l=N_{s-1}+1}^{N_s} \delta_l h(j_l),$$

which is equivalent to

$$(2.36)\qquad \sum_{l=1}^{n} \alpha_l \varepsilon_l h(i_l) \geqq \sum_{l=1}^{n} \alpha_l \delta_l h(j_l). \quad \square$$

3. Main Results. Let, for each $\nu = 1, 2, \cdots, X_{\nu,1}, \cdots, X_{\nu,n_\nu}$ be independently and identically distributed random variables with common distribution function $F(x)$ satisfying

(3.1) 1. $F(x)$ has an absolutely continuous density $f(x)$

2. $\int_0^1 \varphi_F^2(u)\, du < \infty$, where $\varphi_F(u) = -\dfrac{f'(F^{-1}(u))}{f(F^{-1}(u))}$ $(0 \leqq u \leqq 1)$

and where f' is the derivative of f

3. $f(x) = f(-x)$ for all x.

Let $\psi(u) (0 \leqq u \leqq 1)$ be a function satisfying

(3.2) 1. $\psi(u)$ can be written as the sum of two functions $\psi_1(u)$ and $\psi_2(u)$ where $\psi_1(u)$ is nondecreasing and nonnegative and $\psi_2(u)$ is nonincreasing and nonpositive.

2. $\int_0^1 \psi_i^2(u)\,du < \infty (i = 1, 2)$ and $\int_0^1 \psi^2(u)\,du > 0$.

Let $p_{\nu,1}, \cdots, p_{\nu,n_\nu}$ and $q_{\nu,1}, \cdots, q_{\nu,n_\nu}$ be vectors of constants satisfying

(3.3) 1. $\sum_{i=1}^{n_\nu} p_{\nu,i}^2 > 0$

2. $$\lim_{\nu\to\infty} \frac{\max_{1\leq i\leq n_\nu} p_{\nu,i}^2}{\sum_{i=1}^{n_\nu} p_{\nu,i}^2} = 0 ,$$

(3.4) 1. $\sum_{i=1}^{n_\nu} q_{\nu,i}^2 \leqq M$ for some positive number M independent of ν

2. $\lim_{\nu\to\infty} \max_{1\leqq i\leqq n_\nu} q_{\nu,i}^2 = 0$

and, for each $\nu = 1, 2, \cdots$, either

(3.5) 1. $p_{\nu,i}q_{\nu,i} \geqq 0$ for all $i = 1, \cdots, n_\nu$

2. $(|p_{\nu,i}| - |p_{\nu,i'}|)(|q_{\nu,i}| - |q_{\nu,i'}|) \geqq 0$ for all $i, i' = 1, \cdots, n_\nu$

or,

(3.6) 1. $p_{\nu,i}q_{\nu,i} \leqq 0$ for all $i = 1, \cdots, n_\nu$

2. $(|p_{\nu,i}| - |p_{\nu,i'}|)(|q_{\nu,i}| - |q_{\nu,i}|) \geqq 0$ for all $i, i' = 1, \cdots, n_\nu$.

Let $R_{|X_{\nu,i}-q_{\nu,i}\theta|}$ be the rank of $|X_{\nu,i} - q_{\nu,i}\theta|$ among $|X_{\nu,1} - q_{\nu,1}\theta|, \cdots, |X_{\nu,n_\nu} - q_{\nu,n_\nu}\theta|$, let

$$\text{(3.7)} \qquad \operatorname{sgn} u = 1 \quad \text{if } u > 0$$
$$= -1 \quad \text{if } u < 0$$

and let

$$\text{(3.8)} \qquad T_\nu(\theta) = \sum_{i=1}^{n_\nu} p_{\nu,i}\psi\left(\frac{R_{|X_{\nu,i}-q_{\nu,i}\theta|}}{n_\nu + 1}\right) \operatorname{sgn}(X_{\nu,i} - q_{\nu,i}\theta) .$$

THEOREM 3.1. *If $F(x)$ is continuous, if $\psi(u)$ is nondecreasing and nonnegative then, for each $\nu = 1, 2, \cdots$, $T_\nu(\theta)$ is with probability one a nonincreasing step function of θ if* (3.5) *holds and a nondecreasing step function of θ if* (3.6) *holds.*

PROOF. In the proof the index ν will be omitted. The proof will be given for the case that (3.5) holds. The result for the case that (3.6) holds is then obvious.

If $F(x)$ continuous, $T(\theta)$ is, with probability one, not well defined only for those values of θ satisfying $\theta = -(X_i/q_i)$ for some i with $q_i \neq 0$ and for those values of θ satisfying $|X_i - q_i\theta| = |X_{i'} - q_{i'}\theta|$ for some pair (i, i') with $q_i \neq 0$ or $q_{i'} \neq 0$. These values of θ where $T(\theta)$ is not well defined, define a finite number of intervals for θ within each of which $T(\theta)$ is independent of θ.

Now consider two values θ_1 and θ_2 of θ for which $T(\theta)$ is well defined and let $\theta_1 < \theta_2$. Then it will be proved that $T(\theta_1) \geqq T(\theta_2)$. Without loss of generality the X_i can be numbered in such a way that $|p_1| \leqq \cdots \leqq |p_n|$. Then, by (3.5.2),

$|q_1| \leqq \cdots \leqq |q_n|$. Write $T(\theta)$ as

$$T(\theta) = \sum_{k=1}^{n} |p_k| \, \psi\left(\frac{R_{|X_k - q_k\theta|}}{n+1}\right) \operatorname{sgn} p_k(X_k - q_k\theta) , \tag{3.9}$$

where, for $p_k = 0$, sgn $p_k(X_k - q_k\theta)$ is defined as 1.
Now apply Lemma 2.4 with, for $k = 1, \cdots, n$

$$\begin{aligned} \alpha_k &= |p_k| \\ \varepsilon_k &= \operatorname{sgn} p_k(X_k - q_k\theta_1) \qquad & \delta_k &= \operatorname{sgn} p_k(X_k - q_k\theta_2) \\ i_k &= R_{|X_k - q_k\theta_1|} & j_k &= R_{|X_k - q_k\theta_2|} \, . \end{aligned} \tag{3.10}$$

Then $T(\theta_1) \geqq T(\theta_2)$ if (2.29) is satisfied. That (2.29) is satisfied can be seen from the following steps (a), (b) and (c).

(a) (2.29.1) is identical with

$$\{p_k(X_k - q_k\theta_2) > 0, p_k \neq 0\} \Rightarrow p_k(X_k - q_k\theta_1) > 0$$

which follows immediately from (3.5.1) and

$$p_k(X_k - q_k\theta_1) = p_k(X_k - q_k\theta_2) + p_k q_k(\theta_2 - \theta_1) \, .$$

(b) (2.29.2) is identical with

$$\{p_k(X_k - q_k\theta_2) > 0, p_k \neq 0, l < k, |X_l - q_l\theta_2| < |X_k - q_k\theta_2|\} \\ \Rightarrow |X_l - q_l\theta_1| < |X_k - q_k\theta_1| \, .$$

This can be seen as follows. We have

$$-\frac{p_k}{|p_k|}(X_k - q_k\theta_2) < X_l - q_l\theta_2 < \frac{p_k}{|p_k|}(X_k - q_k\theta_2)$$

so that, using (3.5),

$$\begin{aligned} X_l - q_l\theta_1 &< \frac{p_k}{|p_k|}(X_k - q_k\theta_1) + (\theta_2 - \theta_1)\left(q_l - \frac{p_k}{|p_k|} q_k\right) \\ &= \frac{p_k}{|p_k|}(X_k - q_k\theta_1) + (\theta_2 - \theta_1)(q_l - |q_k|) \\ &\leqq \frac{p_k}{|p_k|}(X_k - q_k\theta_1) \, . \end{aligned}$$

Also

$$\begin{aligned} X_l - q_l\theta_1 &> -\frac{p_k}{|p_k|}(X_k - q_k\theta_1) + (\theta_2 - \theta_1)\left(q_l + \frac{p_k}{|p_k|} q_k\right) \\ &= -\frac{p_k}{|p_k|}(X_k - q_k\theta_1) + (\theta_2 - \theta_1)(q_l + |q_k|) \\ &\geqq -\frac{p_k}{|p_k|}(X_k - q_k\theta_1) \, , \end{aligned}$$

so that $|X_l - q_l\theta_1| \leqq |X_k - q_k\theta_1|$.

(c) (2.29.3) is identical with

$$\{p_k(X_k - q_k\theta_2) < 0, p_k \neq 0, l < k, |X_l - q_l\theta_2| > |X_k - q_k\theta_2|\} \Rightarrow |X_l - q_l\theta_1| > |X_k - q_k\theta_1| .$$

The proof of this is analogous to that for (2.29.2). □

A special case of Theorem 3.1 with $\psi(u) = u$ and $p_{\nu,i} = q_{\nu,i}(i = 1, \cdots, n_\nu)$ was proved by Koul ([5], Lemma 2.2).

THEOREM 3.2. *If* (3.1)–(3.4) *and* (3.5) *or* (3.6) *are satisfied then*

$$\lim_{\nu\to\infty} P\{\sup_{|\theta|\leq C} |T_\nu(\theta) - T_\nu(0) + \theta K \textstyle\sum_{i=1}^{n_\nu} p_{\nu,i}q_{\nu,i}| > \varepsilon\sigma(T_\nu(0))\} = 0 , \tag{3.11}$$

where $K = \int_0^1 \psi(u)\varphi_F((u+1)/2)\,du$.

PROOF. The index ν will be omitted in the proof. It is sufficient to prove the theorem for the case where $\psi_2(u) = 0$ for all u. Further the proof will be given for the case where (3.5) holds; the result for the case where (3.6) holds is then obvious.

The proof is analogous to the proof of Jurečková of her Theorem 3.1 in [2]. As in her case it can be supposed without loss of generality that $\sum_{i=1}^n p_i^2 = 1$ and it can be seen, using the result of Hájek and Šidák ([1], Theorem V. 1.7) that it is sufficient to prove

$$\lim_{\nu\to\infty} P\{\sup_{|\theta|\leq C} |T(\theta) - T(0) + \theta K \textstyle\sum_{i=1}^n p_i q_i| > \varepsilon\} = 0 .$$

As in Jurečková's proof and using the results of Hájek and Šidák ([1], section VI. 2.5) it can be proved that for any fixed set of points $\theta_1, \cdots, \theta_r$

$$\lim_{\nu\to\infty} P\{|T(\theta_i) - T(0) + \theta_i K \textstyle\sum_{j=1}^n p_j q_j| \leq \varepsilon \quad \text{for all} \quad i = 1, \cdots, r\} = 1 .$$

Further, for a fixed $C > 0$, choosing $\theta_1, \cdots, \theta_r$ with

$$-C = \theta_1 < \theta_2 < \cdots < \theta_{r-1} < \theta_r = C$$

and

$$|K|\,|\theta_{i+1} - \theta_i| \leq \tfrac{1}{2}\varepsilon M^{-\frac{1}{2}} ,$$

where M is the constant in (3.4), it can be seen, as in Jurečková's proof [2] and using Theorem 3.1 above, that

$$\{|T(\theta_i) - T(0) + \theta_i K \textstyle\sum_{j=1}^n p_j q_j| \leq \tfrac{1}{2}\varepsilon \quad \text{for all} \quad i = 1, \cdots, r\} \Rightarrow \sup_{|\theta|\leq C} |T(\theta) - T(0) + \theta K \textstyle\sum_{j=1}^n p_j q_j| \leq \varepsilon . \ \square$$

The conditions on the $p_{\nu,i}$ and $q_{\nu,i}$ in Theorem 3.2 can be weakened as follows (see also Jurečková [2], Remark, page 1897). First, it can be assumed, without loss of generality, that $q_{\nu,i} \geq 0$ for all $i = 1, \cdots, n_\nu$ or that $q_{\nu,i} \leq 0$ for all $i = 1, \cdots, n_\nu$. This can be seen as follows. Let $p_{\nu,i}$ and $q_{\nu,i}(i = 1, \cdots, n_\nu)$ satisfy (3.3) and (3.4) and suppose $q_{\nu,i} < 0$ for at least one i. Let A_ν be the set of values of i with $q_{\nu,i} < 0$ and define, for $i = 1, \cdots, n_\nu$,

$$\begin{aligned} p^*_{\nu,i} &= p_{\nu,i} \quad i \notin A_\nu & q^*_{\nu,i} &= q_{\nu,i} \quad i \notin A_\nu & Y_{\nu,i} &= X_{\nu,i} \quad i \notin A_\nu \\ &= -p_{\nu,i} \quad i \in A_\nu & &= -q_{\nu,i} \quad i \in A_\nu & &= -X_{\nu,i} \quad i \in A_\nu \end{aligned} \tag{3.12}$$

then

$$T_\nu(\theta) = \sum_{i=1}^{n_\nu} p^*_{\nu,i} \psi\left(\frac{R_{|Y_{\nu,i} - q^*_{\nu,i}\theta|}}{n_\nu + 1}\right) \operatorname{sgn}(Y_{\nu,i} - q^*_{\nu,i}\theta) ,$$

where $Y_{\nu,1}, \cdots, Y_{\nu,n_\nu}$ are independent random variables with common distribution function $F(x)$ satisfying (3.1), where the $p^*_{\nu,i}$ and $q^*_{\nu,i}$ satisfy (3.3) and (3.4) and where $q^*_{\nu,i} \geqq 0$ for all $i = 1, \cdots, n_\nu$.

Further, if $q_{\nu,i}$ has the same sign for all $i = 1, \cdots, n_\nu$, it is possible to find a sequence of pairs of vectors $(p^{(l)}_{\nu,1}, \cdots, p^{(l)}_{\nu,n_\nu})$ $(l = 1, 2)$ such that

(3.13)

1. $p_{\nu,i} = \sum_{l=1}^{2} p^{(l)}_{\nu,i}$ $\qquad i = 1, \cdots, n_\nu$
2. $p^{(1)}_{\nu,i} q_{\nu,i} \geqq 0$ $\qquad i = 1, \cdots, n_\nu$
   $p^{(2)}_{\nu,i} q_{\nu,i} \leqq 0$ $\qquad i = 1, \cdots, n_\nu$
3. $(|p^{(l)}_{\nu,i}| - |p^{(l)}_{\nu,i'}|)(|q_{\nu,i}| - |q_{\nu,i'}|) \geqq 0$
   $l = 1, 2$ and $i, i' = 1, \cdots, n_\nu$.

That this is possible can be seen as follows. Assume $q_{\nu,i} \geqq 0$ for all $i = 1, \cdots, n_\nu$. For every pair of vectors $(p_{\nu,1}, \cdots, p_{\nu,n_\nu})$, $(q_{\nu,1}, \cdots, q_{\nu,n_\nu})$ one can find $(\alpha_{\nu,1}, \cdots, \alpha_{\nu,n_\nu})$ and $(\beta_{\nu,1}, \cdots, \beta_{\nu,n_\nu})$ such that $p_{\nu,i} = \alpha_{\nu,i} + \beta_{\nu,i}$ and

$$(\alpha_{\nu,i} - \alpha_{\nu,i'})(|q_{\nu,i}| - |q_{\nu,i'}|) \geqq 0$$
$$(\beta_{\nu,i} - \beta_{\nu,i'})(|q_{\nu,i}| - |q_{\nu,i'}|) \leqq 0 \qquad i, i' = 1, \cdots, n_\nu .$$

Further one can find $\gamma_\nu \geqq 0$ such that $\alpha_{\nu,i} + \gamma_\nu \geqq 0$ and $\beta_{\nu,i} - \gamma_\nu \leqq 0$ for all $i = 1, \cdots, n_\nu$. By taking $p^{(1)}_{\nu,i} = \alpha_{\nu,i} + \gamma_\nu$, $p^{(2)}_{\nu,i} = \beta_{\nu,i} - \gamma_\nu$ one has found $(p^{(l)}_{\nu,1}, \cdots, p^{(l)}_{\nu,n_\nu})$, $l = 1, 2$ such that (3.13) is satisfied.

Further, if $p_{\nu,1}, \cdots, p_{\nu,n_\nu}$ satisfies $\sum_{i=1}^{n_\nu} p^2_{\nu,i} > 0$ for each ν (Condition 3.3.1) then, for each ν, there exists an $l(l = 1, 2)$ such that $\sum_{i=1}^{n_\nu} \{p^{(l)}_{\nu,i}\}^2 > 0$. Also, if $p_{\nu,i}$ is written as $\sum_{l=1}^{2} p^{(l)}_{\nu,i}$, $T_\nu(\theta)$ can be written as the sum of two statistics and (3.11) remains true if it is true for each of these two statistics and

$$\sum_{l=1}^{2} [\sum_{i=1}^{n_\nu} \{p^{(l)}_{\nu,i}\}^2]^{\frac{1}{2}} \leqq M_1[\sum_{i=1}^{n_\nu} p^2_{\nu,i}]^{\frac{1}{2}} \tag{3.14}$$

for some positive constant M_1 independent of ν. Further (3.11) is true for each of these two statistics if (3.1), (3.2) and (3.4) are satisfied and $p^{(l)}_{\nu,i} (l = 1, 2)$ satisfy (3.13) and

(3.15)

1. for at least one l
$$\sum_{i=1}^{n_\nu} \{p^{(l)}_{\nu,i}\}^2 > 0 \qquad \text{for each } \nu$$
2. for an l for which 1. is not satisfied
$$\sum_{i=1}^{n_\nu} \{p^{(l)}_{\nu,i}\}^2 = 0 \qquad \text{for each } \nu$$
3. for each l for which 1. is satisfied
$$\lim_{\nu\to\infty} \frac{\max_{1\leqq i\leqq n_\nu} \{p^{(l)}_{\nu,i}\}^2}{\sum_{i=1}^{n_\nu} \{p^{(l)}_{\nu,i}\}^2} = 0 .$$

This proves the following theorem.

THEOREM 3.3. *If* (3.1), (3.2) *and* (3.4) *are satisfied, if there exist* $p_{\nu,1}^{(l)}, \cdots, p_{\nu,n_\nu}^{(l)}$ $(l = 1, 2)$ *such that* (3.13), (3.14) *and* (3.15) *are satisfied, then* (3.11) *holds.*

This theorem is related to a theorem of Jurečková [3]. She proves (3.11) for the case where $p_{\nu,i} = 1 (i = 1, \cdots n_\nu)$ and $\sum_{i=1}^{n_\nu} q_{\nu,i} = 0$ under the conditions (3.1), (3.2) and (3.4). Jurečková's result [3] is not a special case of Theorem 3.3, as can be seen from the following two examples. Let, for n_ν even, $p_{\nu,i} = 1$, $i = 1, \cdots, n_\nu$, $q_{\nu,i} = n_\nu^{-\frac{1}{2}}$, $i = 1, \cdots, \frac{1}{2}n_\nu$ and $q_{\nu,i} = -n_\nu^{-\frac{1}{2}}$, $i = \frac{1}{2}n_\nu + 1, \cdots, n_\nu$. Then the conditions of Jurečková [3] are satisfied. That the conditions of Theorem 3.3 are also satisfied can be seen as follows. By (3.12) $T_\nu(\theta)$ can be written as

$$T_\nu(\theta) = \sum_{i=1}^{n_\nu} p_{\nu,i}^* \psi\left(\frac{R_{|Y_{\nu,i} - q_{\nu,i}^*\theta|}}{n_\nu + 1}\right) \operatorname{sgn}(Y_{\nu,i} - q_{\nu,i}^*\theta),$$

where $p_{\nu,i}^* = 1$, $i = 1, \cdots, \frac{1}{2}n_\nu$, $p_{\nu,i}^* = -1$, $i = \frac{1}{2}n_\nu + 1, \cdots, n_\nu$, $q_{\nu,i}^* = n_\nu^{-\frac{1}{2}} (i = 1, \cdots, n_\nu)$. Further $p_{\nu,i}^*$ can be written as $\sum_{l=1}^{2} p_{\nu,i}^{(l)}$ satisfying (3.13) by choosing $p_{\nu,i}^{(1)} = 1$, $i = 1, \cdots, n_\nu$ and $p_{\nu,i}^{(2)} = 0$, $i = 1, \cdots, \frac{1}{2}n_\nu$, $p_{\nu,i}^{(2)} = -2$, $i = \frac{1}{2}n_\nu + 1, \cdots, n_\nu$. Then $\sum_{i=1}^{n_\nu} \{p_{\nu,i}^{(1)}\}^2 = n_\nu$, $\sum_{i=1}^{n_\nu} \{p_{\nu,i}^{(2)}\}^2 = 2n_\nu$ and $\sum_{i=1}^{n_\nu} p_{\nu,i}^2 = n_\nu$, so that (3.14) and (3.15) are satisfied. However, if one takes e.g. $p_{\nu,i} = 1$, $i = 1, \cdots, n_\nu$ and $q_{\nu,i} = \{\frac{1}{2}(i + 1)(-1)^{i+1}\}/n_\nu^{\frac{3}{2}}$, $i = 1, \cdots, n_\nu$ then the conditions of Jurečková [3] are satisfied but those of Theorem 3.3 are not. This can be seen as follows. By (3.12), $p_{\nu,i}^* = (-1)^{i+1}$, $q_{\nu,i}^* = \frac{1}{2}(i + 1)/n_\nu^{\frac{3}{2}}$, $i = 1, \cdots, n_\nu$ and, for any $p_{\nu,i}^{(1)}$ and $p_{\nu,i}^{(2)}$ satisfying (3.13), $\sum_{i=1}^{n_\nu} \{p_{\nu,i}^{(1)}\}^2$ and $\sum_{i=1}^{n_\nu} \{p_{\nu,i}^{(2)}\}^2$ are of the order n_ν^3, whereas $\sum_{i=1}^{n_\nu} p_{\nu,i}^2 = n_\nu$, so that (3.14) is not satisfied.

A special case of Theorem 3.3 with $p_{\nu,i} = q_{\nu,i} = n_\nu^{-\frac{1}{2}}$ was used by Kraft and van Eeden ([6] and [7]) to find the asymptotic properties of linearized estimates based on signed ranks for the one-sample location problem.

An extension of Theorem 3.3 to the p-variate case, where $R_{|X_{\nu,i} - q_{\nu,i}\theta|}$ is replaced by $R_{|X_{\nu,i} - \Sigma_{j=1}^{p} q_{\nu,i,j}\theta_j|}$ with $p_{\nu,i} = q_{\nu,i,j}$ for some j and all $i = 1, \cdots, n_\nu$, is given in [8]; it is used there to find the asymptotic properties of linearized estimates based on signed ranks for the general linear hypothesis.

Koul [5] proves a theorem analogous to Theorem 3.2 for the p-variate case with $\psi(u) = u$ and conditions on F that are stronger than (3.1).

Jurečková also treated in [3] the p-variate case with $p_{\nu,i} = 1$, $i = 1, \cdots, n_\nu$ and $\sum_{i=1}^{n_\nu} q_{\nu,i,j} = 0$ for all $i = 1, \cdots, n_\nu$ and all $j = 1, \cdots, p$.

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