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SHORT PROOFS OF TWO CONVERGENCE THEOREMS FOR CONDITIONAL EXPECTATIONS

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In this paper there are given new proofs of two convergence theorems for conditional expectations, concerning convergence in measure and convergence almost everywhere of a sequence of conditional expectations $P_n^{\mathscr{F}_0}f$ of a bounded function f, given a σ-field $\mathscr{F}_0$, with respect to varying probability measures P_n.

We shall give new proofs of two convergence theorems for conditional expectations which were proven in [1]. These new proofs are shorter and more transparent.

If $\mu | \mathscr{F}$ is a measure, σ-finite on the σ-field $\mathscr{F}_0 \subset \mathscr{F}$, and $f: X \to \mathbb{R}$ is μ-integrable, then $\mu^{\mathscr{F}_0}f$ denotes the $\mu | \mathscr{F}_0$-equivalence class of all conditional expectations of f, relative $\mu | \mathscr{F}$, given $\mathscr{F}_0$. A sequence of equivalence classes of conditional expectations converges a.e. or in measure if every choice of versions converges a.e. or in measure. A sequence $P_n | \mathscr{F}$, $n \in \mathbb{N}$, of probability measures converges uniformly to $P_0 | \mathscr{F}$ if $\sup \{|P_n(A) - P_0(A)| : A \in \mathscr{F}\} \to 0$.

THEOREM 1. *Let $\mu | \mathscr{F}$ be a measure which is σ-finite on the σ-field $\mathscr{F}_0 \subset \mathscr{F}$ and dominates the probability measures $P_n | \mathscr{F}$, $n \in \mathbb{N} \cup \{0\}$. Let h_n be a density of $P_n | \mathscr{F}$ with respect to $\mu | \mathscr{F}$ and $h_{0,n}$ a density of $P_n | \mathscr{F}_0$ with respect to $\mu | \mathscr{F}_0$, $n \in \mathbb{N} \cup \{0\}$. Assume that*

(i) $h_0 \leqq \liminf_{n \in \mathbb{N}} h_n \quad \mu$-a.e.,

(ii) $h_{0,0} = \lim_{n \in \mathbb{N}} h_{0,n} \quad \mu$-a.e.

Then $(P_n^{\mathscr{F}_0}f)_{n \in \mathbb{N}}$ converges P_0-a.e. to $P_0^{\mathscr{F}_0}f$ for each $\mathscr{F}$-measurable bounded function f.

PROOF. We have $(*)$: $P_n^{\mathscr{F}_0}f\mu^{\mathscr{F}_0}h_n = \mu^{\mathscr{F}_0}fh_n$ for all $n \in \mathbb{N} \cup \{0\}$. As $h_{0,n} \in \mu^{\mathscr{F}_0}h_n$, (ii) implies $\mu^{\mathscr{F}_0}(h_n - h_0) \to 0$ μ-a.e. Since $(h_n - h_0)^- \leqq h_0$, (i) implies $\mu^{\mathscr{F}_0}(h_n - h_0)^- \to 0$ μ-a.e. Therefore we obtain $\mu^{\mathscr{F}_0}|h_n - h_0| = \mu^{\mathscr{F}_0}(h_n - h_0) + 2\mu^{\mathscr{F}_0}(h_n - h_0)^- \to 0$ μ-a.e. As f is bounded this implies $\mu^{\mathscr{F}_0}fh_n \to \mu^{\mathscr{F}_0}fh_0$ μ-a.e. Hence by $(*)P_n^{\mathscr{F}_0}f\mu^{\mathscr{F}_0}h_n \to P_0^{\mathscr{F}_0}f\mu^{\mathscr{F}_0}h_0$ μ-a.e. whence $P_0\{x: (\mu^{\mathscr{F}_0}h_0)(x) > 0\} = 1$ and (ii) imply the assertion.

We remark that condition (i) is slightly weaker than condition (i) of Theorem 1 of [1].

Using $\mu^{\mathscr{F}_0}|h_n - h_0| \to 0$ μ-a.e. we can also obtain the following result of [1]: If $\mu | \mathscr{F}$ is a probability measure admitting a regular conditional probability,

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given $\mathscr{F}_0$, then there exist regular conditional probabilities $p_n \mid \mathscr{F} \times X$, relative P_n, given $\mathscr{F}_0$, $n \in \mathbb{N} \cup \{0\}$, such that $\sup \{|p_n(A, x) - p_0(A, x)| : A \in \mathscr{F}\} \to 0$ P_0-a.e.

THEOREM 2. *Let $P_n \mid \mathscr{F}$, $n \in \mathbb{N}$, be probability measures, converging uniformly to the probability measure $P_0 \mid \mathscr{F}$ and let $\mathscr{F}_0 \subset \mathscr{F}$ be a σ-field. Then $(P_n^{\mathscr{F}_0} f)_{n \in \mathbb{N}}$ converges to $P_0^{\mathscr{F}_0} f$ in P_0-measure for every $\mathscr{F}$-measurable bounded function f.*

PROOF. Let $\mu := \sum \{2^{-n} P_n : n \in \mathbb{N}\}$ and h_n be a density of $P_n \mid \mathscr{F}$ with respect to $\mu \mid \mathscr{F}$, $n \in \mathbb{N} \cup \{0\}$. Since $P_n \mid \mathscr{F}$ converges uniformly to $P_0 \mid \mathscr{F}$, we obtain $\mu(\mu^{\mathscr{F}_0} |h_n - h_0|) = \mu(|h_n - h_0|) \to 0$; hence $\mu^{\mathscr{F}_0} |h_n - h_0| \to 0$ in μ-measure, whence $\mu^{\mathscr{F}_0} h_n \to \mu^{\mathscr{F}_0} h_0$ and $\mu^{\mathscr{F}_0} f h_n \to \mu^{\mathscr{F}_0} f h_0$ in μ-measure. According to relation (∗) of Theorem 1 this implies $P_n^{\mathscr{F}_0} f \mu^{\mathscr{F}_0} h_n \to P_0^{\mathscr{F}_0} f \mu^{\mathscr{F}_0} h_0$, in μ-measure. As $P_0\{x : (\mu^{\mathscr{F}_0} h_0)(x) > 0\} = 1$ and $\mu^{\mathscr{F}_0} h_n \to \mu^{\mathscr{F}_0} h_0$ in μ-measure, we obtain the assertion.

In Lemma 2 of [1] it is proved that $\mu^{\mathscr{F}_0} h_n f \to \mu^{\mathscr{F}_0} h_0 f$ P_0-a.e. and $\mu^{\mathscr{F}_0} h_n \to \mu^{\mathscr{F}_0} h_0$ P_0-a.e. imply $P_n^{\mathscr{F}_0} f \to P_0^{\mathscr{F}_0} f$ P_0-a.e. This follows directly from relation (∗) of Theorem 1.

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