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ON EMBEDDING RIGHT CONTINUOUS MARTINGALES IN BROWNIAN MOTION

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A stopping time T for the Wiener process $W(t)$ is called minimal if there is no stopping time $S \leqq T$ such that $W(S)$ and $W(T)$ have the same distribution. In the first section, it is shown that if $E\{W(T)\} = 0$, then T is minimal if and only if the process $W(t \wedge T)$ is uniformly integrable. Also, if T is minimal and $E\{W(T)\} = 0$ then $E\{T\} = E\{W(T)^2\}$.

In the second section, these ideas are used to show that for any right continuous martingale $M(t)$, there is a right continuous family of minimal stopping times $T(t)$ such that $W(T(t))$ has the same finite joint distributions as $M(t)$.

In the last section it is shown that if T is defined in the manner proposed by Skorokhod (and therefore minimal) such that $W(T)$ has a stable distribution of index $\alpha > 1$ then T is in the domain of attraction of a stable distribution of index $\alpha/2$.

0. The Skorokhod embedding theorem has proved to be a very powerful tool for extending results from Brownian motion to random walks. However, for the most part, the embedding theorem has only been discussed for random variables with finite variances. This is clearly unnecessary as the stopping times that Skorokhod defines require only that the random variables have finite means which are then assumed to be zero. However, if the random variable that is being embedded in Brownian motion does not have a finite variance, it is no longer clear how one selects "good" stopping times; that is, one can not simply ask that the expectation of the stopping time be finite and therefore equal to the variance of the random variable.

In this paper, a class of "good" stopping times will be explored. These "minimal" stopping times were first singled out by Doob. In the first section, a characterization of these stopping times will be given and a bound on the stopping time in terms of the distribution of the random variable being embedded will be obtained. In the second section it will be shown that every right continuous martingale can be embedded in Brownian motion by means of a family of minimal stopping times $\{T_t\}$ which as functions on t are right continuous and non-decreasing. The stopping times in general require larger σ-fields than those generated by the Brownian motion. In the third section, the particular case where the martingale is a process with independent increments is discussed.

The notation used will be that of Blumenthal and Getoor [1]. However, Brownian motion on the real line or the Wiener process will be denoted by W_t or more precisely $(\Omega, \mathscr{F}, \mathscr{F}_t, W_t, \theta_t, P^x)$. It will usually be assumed that the process starts at zero and P^0 will often be written simply P. Also $\mathscr{L}(X) =$

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$\mathscr{L}(Y)$ will mean that the random variables X and Y defined on Ω have the same distribution with respect to P.

1.

DEFINITION 1. A stopping time T will be said to be minimal if for any stopping time $S \leqq T$, $\mathscr{L}(W(T)) = \mathscr{L}(W(S))$ implies $S = T$ a.s.

All hitting times are minimal. All stopping times with finite expectation are minimal. Indeed, if $E\{T\} < \infty$, then $E\{W_T\} = 0$ and $E\{W_T^2\} = E\{T\}$. If $S \leqq T$ and $\mathscr{L}(W_S) = \mathscr{L}(W_T)$ then

$$E\{S\} = E\{W_S^2\} = E\{W_T^2\} = E\{T\}$$

so $S = T$ a.s.

PROPOSITION 2. *For any stopping time S there is a minimal stopping time $T \leqq S$ such that $\mathscr{L}(W_T) = \mathscr{L}(W_S)$.*

PROOF. The set of all stopping times $T \leqq S$ such that $\mathscr{L}(W_T) = \mathscr{L}(W_S)$ is partially ordered by $\leqq$. Choose a maximal chain $\{T_\alpha\}$ and let $\beta_\alpha = E\{e^{-T_\alpha}\}$. Extract a sequence $\{T_n\}$ such that

$$\lim E\{e^{-T_n}\} = \sup \beta_\alpha .$$

The sequence T_n is decreasing so it converges to some stopping time T and therefore $W_{T_n} \to W_T$. This means that $\mathscr{L}(W(T)) = \mathscr{L}(W(T_n)) = \mathscr{L}(W(S))$. Clearly T is minimal.

THEOREM 3. *Let S be a stopping time such that $E\{W(S)\} = 0$. Then S is minimal if and only if the process $W(t \wedge S)$ is uniformly integrable.*

PROOF. First suppose that $W(t \wedge S)$ is uniformly integrable. Let T be a stopping time such that $T \leqq S$ and $\mathscr{L}(W(T)) = \mathscr{L}(W(S))$. Then for all $a > 0$

$$E\{W_S : W_S \geqq a\} = E\{W_T : W_T \geqq a\} = E\{W_S : W_T \geqq a\} .$$

Likewise if $a < 0$ $E\{W_S : W_S \leqq a\} = E\{W_S : W_T \leqq a\}$. These imply that $W_S = W_T$ a.s. If R is any stopping time such that $T \leqq R \leqq S$ then

$$W_R = E\{W_S \mid \mathscr{F}_R\} = E\{W_T \mid \mathscr{F}_R\} = W_T = W_S \text{ a.s.}$$

But the paths of W_t are continuous so W_t is constant on the interval $T \leqq t \leqq S$ which is impossible unless $T = S$ a.s.

Before we prove the converse, we will prove a number of lemmas in which some results of Root [7] will be used. These are now summarized.

A *barrier* is a subset B of $[0, +\infty] \times [-\infty, +\infty]$ satisfying

(i) B is closed
(ii) $(+\infty, x) \in B$ for all $x \in [0, +\infty]$
(iii) $(t, \pm\infty) \in B$ for all $t \in [0, +\infty]$
(iv) if $(t, x) \in B$ then $(s, x) \in B$ whenever $s > t$.

The set $\mathscr{B}$ of all barriers admits a metric which makes B into a compact separable metric space. (If B_n is a sequence in $\mathscr{B}$ which converges to B then $B = \bigcap_k (\overline{\bigcup_{n=k}^{\infty} B_n})$.) For each barrier $B \in \mathscr{B}$ one defines the stopping time τ_B by

$$\tau_B = \inf\{t : (t, W(t)) \in B\} .$$

Root has shown that if B_n is a sequence of barriers and $E\{\tau_{B_n}\} < M$ for all n then there is a subsequence $B_{n'}$ of B_n and a barrier B_0 such that $B_n \to B_0$ in $\mathscr{B}$ and $\tau_{B_n} \to \tau_{B_0}$ a.s.

LEMMA 4. *If S is any stopping time such that $E\{W(S)\} = 0$ then there is a barrier $B \in \mathscr{B}$ such that $S \wedge \tau_B$ is minimal and $\mathscr{L}(W(S)) = \mathscr{L}(W(S \wedge \tau_B))$.*

PROOF. Suppose first that there is a finite set $A = \{a_1, a_2, \cdots, a_n\}$ such that $P\{W(S) \in A\} = 1$. For convenience, we suppose that $a_1 = \min A$ and $a_n = \max A$. Let $\mathscr{B}_S$ be the set of all barriers B which

(a) are subsets of the set

$$\{(t, x); t \geqq 0, x \in A\}$$

(b) contain $\{(t, x); t \geqq 0, x = a_1\}$ and $\{(t, x); t \geqq 0, x = a_n\}$ and
(c) for all $i \neq 1$, $i \neq n$

$$P\{W(\tau_B \wedge S) = a_i\} \leqq P\{W(S) = a_i\} .$$

$\mathscr{B}_S$ is not empty since it contains the barrier

$$[(t, x) : t \geqq 0, x \in \{a_1, a_n\}] .$$

The elements of $\mathscr{B}_S$ are partially ordered by inclusion. Select a maximal chain $\mathscr{C}$ and denote the closure of the union of $\mathscr{C}$ by B_0. Clearly $B_0 \in \mathscr{B}$. Moreover one can select an increasing sequence $\{B_n\}$ of barriers in $\mathscr{B}_S$ such that $B_0 = \overline{\bigcup\{B_n\}}$. It follows that

$$\tau_{B_n} \to \tau_{B_0} \quad \text{a.s.}$$

so that for $i \neq 1$, $i \neq n$

$$P\{W(\tau_{B_0} \wedge S) = a_i\} = \lim P\{W(\tau_{B_n} \wedge S) = a_i\} \leqq P\{W_S = a_i\}$$

so $B_0 \in \mathscr{B}_S$.

Suppose that for some a_i, $i \neq 1$, $i \neq n$,

$$P\{W(\tau_{B_0} \wedge S) = a_i\} < P\{W_S = a_i\} .$$

Let $t_0 = \inf\{t : (t, a_i) \in B_0\}$. Since

$$P\{\exists t, t_0 - 1/n \leqq t \leqq t_0, W(t) = a_i\} \to 0$$

there is an n such that

$$P\{\exists t, W(t) = a_i, t_0 - 1/n \leqq t \leqq t_0\} + P\{W(\tau_{B_0} \wedge S) = a_i\} \leqq P\{W(S) = a_i\}$$

so that B_0 is not maximal. Thus for all $i \neq 1$, $i \neq n$,

$$P\{W(\tau_{B_0} \wedge S) = a_i\} = P\{W(S) = a_i\} .$$

But $E\{W(\tau_{B_0} \wedge S)\} = 0 = E\{W(S)\}$ and $P\{W(S) \in A\} = 1 = P\{W(\tau_{B_0} \wedge S) \in A\}$ so it must follow that $\mathscr{L}(W(S)) = \mathscr{L}(W(\tau_{B_0} \wedge S))$. Finally, note that $\tau_{B_0} \leqq T_{\{a_1, a_n\}}$ so $E\{\tau_{B_0} \wedge S\} < \infty$ so that $\tau_{B_0} \wedge S$ is minimal. This proves the lemma when $W(S)$ takes on only a finite number of values.

Now suppose that there is a compact interval $[a, b]$ such that $P\{W(S) \in [a, b]\} = 1$. Let

$$A_n = \{k/n; k = 0, \pm 1, \pm 2, \cdots, a \leqq k/n \leqq b\}$$

and $S_n = \inf\{t: t \geqq S, W_t \in A_n\}$.

S_n is a stopping time and $W(S_n)$ takes on only a finite number of values so there is a barrier B_n such that $W(\tau_{B_n} \wedge S_n)$ has the same distribution as $W(S_n)$ and $\tau_{B_n} \leqq T_{\{a,b\}}$. This means that $E\{\tau_{B_n}\} \leqq E\{T_{\{a,b\}}\} < \infty$. Thus by Root's result, there is a subsequence $B_{n'}$ and a barrier B_0 such that

$$\tau_{B_{n'}} \to \tau_{B_0} \quad \text{a.s.}$$

It follows that $S_{n'} \wedge \tau_{B_{n'}} \to S \wedge \tau_{B_0}$ a.s. and

$$\begin{aligned}\mathscr{L}(W(S \wedge \tau_{B_0})) &= \lim \mathscr{L}(W(S_{n'} \wedge \tau_{B_{n'}})) \\ &= \lim \mathscr{L}(W(S_{n'})) = \mathscr{L}(W(S)) .\end{aligned}$$

Since $\tau_{B_n} \leqq T_{\{a,b\}}$ for all n, $E\{\tau_{B_0} \wedge S\} < \infty$ so $\tau_{B_0} \wedge S$ is minimal. Also note that $\tau_{B_0} \leqq T_{\{a,b\}}$ a.s.

Now we turn to the general case. For each n, select $a_n < 0$, $b_n > 0$ such that $\max\{|a_n|, b_n\} = n$ and

$$E\{W(S_n)\} = 0$$

where now

$$S_n = \inf\{t > S;\ W_t \in [a_n, b_n]\} .$$

Since $P\{W(S_n) \in [-n, n]\} = 1$, there is a barrier B_n (which we can assume contains $\{(t, x);\ x \leqq a_n \text{ or } x \geqq b_n\}$) such that $W(\tau_{B_n} \wedge S_n)$ has the same distribution as $W(S_n)$.

We now show that if $n < m$ then we can assume that $B_m \subset B_n$. The techniques used are due to Root.

Let $\mathscr{B}_n = \{B \in \mathscr{B};\ \{(t, x): x \leqq a_n \text{ or } x \geqq b_n\} \subset B,\ \mathscr{L}(W(\tau_B \wedge S_n)) = \mathscr{L}(W(S_n))\}$. We show that if $n < m$, $B_n \in \mathscr{B}_n$ and $B_m \in \mathscr{B}_m$ then $B_n \cup B_m \in \mathscr{B}_n$. Certainly $\{(t, x); x \leqq a_n \text{ or } x \geqq b_n\} \subset B_n \cup B_m$. Let $A_n = \{x; \inf\{t: (t, x) \in B_n\} \leqq \inf\{t: (t, x) \in B_m\}\}$ and $A_m = R \backslash A_n = (a_n, b_n) \backslash A_n$. If $A \subset A_n \cap [a_n, b_n]$ then, since $\tau_{B_n} \wedge S_n = \tau_{B_n} \wedge S$, $\tau_{B_n \cup B_m} \wedge S = \tau_{B_n \cup B_m} \wedge S_n$, and $W(S \wedge \tau_{B_n}) \in A$ if $W(S \wedge \tau_{B_n \cup B_m}) \in A$, $(A \subset A_n)$, we have

$$\begin{aligned}P\{W(S_n) \in A\} &= P\{W(S_n \wedge \tau_{B_n}) \in A\} = P\{W(S \wedge \tau_{B_n}) \in A\} \\ &\geqq P\{W(S \wedge \tau_{B_n \cup B_m}) \in A\} = P\{W(S_n \wedge \tau_{B_n \cup B_m}) \in A\} .\end{aligned}$$

Likewise if $A \subset A_m$, then

$$\begin{aligned}P\{W(S_n) \in A\} = P\{W(S_m) \in A\} &= P\{W(S_m \wedge \tau_{B_m}) \in A\} = P\{W(S \wedge \tau_{B_m}) \in A\} \\ &\geqq P\{W(S \wedge \tau_{B_m \cup B_n}) \in A\} = P\{W(S_n \wedge \tau_{B_n \cup B_m}) \in A\} .\end{aligned}$$

Thus for every set $A \subset [a_n, b_n]$,

$$P\{W(S_n) \in A\} \geqq P\{W(S_n \wedge \tau_{B_n \cup B_m}) \in A\}.$$

But $P\{W(S_n \wedge \tau_{B_n \cup B_m}) \in [a_n, b_n]\} = 1$ so equality holds and $B_n \cup B_m \in \mathscr{B}_n$.

Now fix a sequence B_n chosen from the sets $\mathscr{B}_n$. For each n and $k > n$ let

$$B_n{}^k = \bigcup_{i=n}^k B_i .$$

We know that $B_n{}^k \in B_n$ for each k. Moreover $E\{\tau_{B_n{}^k}\} \leqq E\{_{\{a_n, b_n\}}\}$ where $T_{\{a_n, b_n\}}$ is the hitting time of the set $\{a_n, b_n\}$ so there is a subsequence $B_n{}^{k'}$ and a barrier $B_n{}^0$ such that $B_n{}^{k'}$ converges to $B_n{}^0$, $\tau_{B_n{}^{k'}} \to \tau_{B_n{}^0}$ a.s. and $B_n{}^0$ is the closure of $\bigcap_k \bigcup_{i=k}^\infty B_n{}^i = \bigcup_{i=n}^\infty B_i$. Since $\tau_{B_n{}^{k'}} \to \tau_{B_n{}^0}$, $B_n{}^0 \in \mathscr{B}_n$. It follows that for $m > n$ $B_m{}^0 \subset B_n{}^0$. We will simply assume that if $m > n$, $B_m \subset B_n$.

Define $B_0 = \bigcap B_n$. Then $\tau_{B_n} \uparrow \tau_{B_0}$. Also $S_n = S$ on $\{S_n < \tau_{B_n}\}$ so $\tau_{B_n} \wedge S_n \uparrow \tau_{B_0} \wedge S$. It follows that

$$\mathscr{L}(W(\tau_{B_0} \wedge S)) = \lim \mathscr{L}(W(\tau_{B_n} \wedge S_n)) = \lim \mathscr{L}(W(S_n)) = \mathscr{L}(W(S)).$$

We will show that the process $W(t \wedge \tau_{B_0} \wedge S)$ is uniformly integrable and conclude that $\tau_{B_0} \wedge S$ is minimal.

Since $P\{|W(t \wedge \tau_{B_n} \wedge S_n)| > n\} = 0$, this process is uniformly integrable. Thus

$$E\{|W(t \wedge \tau_{B_n} \wedge S_n)|\} \leqq E\{|W(\tau_{B_n} \wedge S_n)|\} \leqq E\{|W(S)|\} < \infty$$

so $P\{|W(t \wedge \tau_{B_n} \wedge S_n)| \geqq a\} \leqq E\{|W(S)|\}/a$ for all a. Choose ε and δ small enough that if $P\{\Delta\} < \delta$, $\Delta \subset \Omega$, then

$$E\{|W(S)|; \Delta\} < \varepsilon .$$

This implies in particular that if $P\{\Delta\} < \delta$ then

$$E\{|W(\tau_{B_n} \wedge S_n)|; \Delta\} < \varepsilon$$

because of the relationship between the $\mathscr{L}(W(\tau_{B_n} \wedge S_n)) = \mathscr{L}(W(S_n))$ and $\mathscr{L}(S)$. Choose a large enough that $E\{|W(S)|\}/a < \delta$. Then letting $\Delta = \{|W(t \wedge \tau_{B_n} \wedge S_n)| > a\}$, we have

$$E\{|W(t \wedge \tau_{B_n} \wedge S_n)|; \Delta\} \leqq E\{|W(\tau_{B_n} \wedge S_n)|; \Delta\} < \varepsilon .$$

Now as $n \to \infty$, $t \wedge \tau_{B_n} \wedge S_n \to t \wedge \tau_{B_0} \wedge S$ so

$$E\{|W(t \wedge S \wedge \tau_{B_0})|; |W(t \wedge S \wedge \tau_{B_0})| > a\} < \varepsilon$$

by the continuity of paths and Fatou's lemma. Thus the process $W(t \wedge \tau_{B_0} \wedge S)$ is uniformly integrable so the proof of the lemma is complete.

Note that the a chosen above does not depend on S but rather on an ε and $\mathscr{L}(W(S))$.

Returning to the proof of Theorem 3, suppose that S is minimal. By the proof to the lemma, there is a barrier B such that the process $W(t \wedge \tau_B \wedge S)$ is uniformly integrable and $\mathscr{L}(W(\tau_B \wedge S) = \mathscr{L}(S))$. It follows that $\tau_B \wedge S = S$ so the process $W(t \wedge S)$ is uniformly integrable.

THEOREM 5. *If T is minimal and $E\{W(T)\} = 0$, then $E\{T\} = E\{W(T)^2\}$.*

PROOF. If $E\{T\} < \infty$, the result is well known. Suppose $E\{W(T)^2\} < \infty$. As in Lemma 4, for each n choose a_n, b_n such that $\max\{|a_n|, b_n\} = n$ and $E\{W(T_n)\} = 0$ where

$$T_n = \inf\{t: t \geqq T, W_t \in [a_n, b_n]\}.$$

Then $E\{W(T_n)^2\} \leqq E\{W(T)^2\}$. In Lemma 4, it was shown that a sequence of barriers B_n could be chosen such that

$$\tau_{B_n} \wedge T_n \to T$$

and

$$E\{\tau_{B_n} \wedge T_n\} = E\{W(\tau_{B_n} \wedge T_n)^2\} = E\{W(T_n)^2\} \leqq E\{W(T)^2\}.$$

It follows by Fatou's lemma that $E\{T\} \leqq E\{W(T)^2\} < \infty$ so the proof is complete.

In the proof of the lemma, we also proved the following.

PROPOSITION 6. *If S is minimal, $E\{W(S)\} = 0$ and $P\{W(S) \in [a, b]\} = 1$ then $S \leqq T_{\{a,b\}}$ a.s.*

Indeed we showed that there was a barrier B containing $\{(t, x): t \geqq 0, x = a$ or $x = b\}$ such that $\tau_B \wedge S = S$.

PROPOSITION 7. *If T is minimal, $E\{W_T\} = 0$ and $E\{|W_T|\} = M$ then for all $\lambda > 0$*

$$P\{T \geqq \lambda\} \leqq (M^2 + 1)/\lambda^{\frac{1}{3}}$$

and

$$P\{T \geqq \lambda\} \leqq M^{\frac{2}{3}}(1 + \lambda^{-1}).$$

PROOF. Let $T_{KM} = \inf\{t: |W_t| \geqq KM\}$. Then as

$$KMP\{T > T_{KM}\} \leqq E\{|W_{T \wedge T_{KM}}|\} \leqq E\{|W_T|\} \leqq M$$

we have

$$P\{T > T_{KM}\} \leqq 1/K.$$

On the other hand $E\{T_{KM}\} = K^2M^2$ so

$$P\{T_{KM} > \lambda\} < K^2M^2/\lambda.$$

Therefore

$$P\{T > \lambda\} \leqq P\{T > T_{KM}\} + P\{T_{KM} > \lambda\} \leqq 1/K + \frac{K^2M^2}{\lambda}.$$

Letting $K = \lambda^{\frac{1}{3}}$, we obtain the first inequality. Letting $K = M^{-\frac{2}{3}}$ we obtain the second inequality.

2. In this section we prove that every right continuous martingale can be embedded in Brownian motion with minimal stopping times. There are several related results already known. Dubins and Schwartz [5] have shown that every continuous martingale is a time change of Brownian motion. See also [3]. Clark [2] has shown that every stochastic process which is a martingale with respect to the σ-field $\mathscr{F}_t$ generated by a Brownian motion process W_s $s \leqq t$ is a stochastic integral of this Brownian motion. Such processes are continuous. Finally Dubins

[4] has shown that every discrete time martingale can be embedded in Brownian motion and the stopping times that he uses are minimal as we now show.

LEMMA 8. *For every martingale M_n, there are minimal stopping times T_n such that $W(T_n)$ has the same finite joint distributions as M_n.*

PROOF. It is only necessary to show that the stopping times T_n are minimal. T_1 is the usual Dubins stopping time which embeds M_1 in W_t so it is minimal. Suppose that T_{n-1} is minimal. To see that T_n is minimal, let us examine the manner in which T_n is defined.

For simplicity, it is assumed that the martingale M_n is defined on Ω also. Let $\mathscr{G}_0$ be the σ-field generated by $M_1, M_2, \cdots, M_{n-1}$ and $Y_0 = M_{n-1}$. Inductively define $\mathscr{G}_i$ to be the σ-field generated by $\mathscr{G}_{i-1}$ and the set $\{M_n > Y_{i-1}\}$ and define Y_i to be $E\{M_n \mid \mathscr{G}_i\}$. Then $Y_i \to M_n$ almost surely. The Dubins stopping time T_n is the supremum of an increasing sequence of stopping times S_i where S_i is chosen such that the process $W(T_1), \cdots, W(T_{n-1}), W(S_i)$ has the same joint distributions as $M_1, M_2, \cdots, M_{n-1}, Y_i$.

It will be shown that the stopping times S_i are minimal and it will then follow from arguments like those in Lemma 4 that T_n is minimal.

The stopping times S_i are defined as follows. Let $F_i(\omega, x)$ be the conditional distribution of Y_i given $\mathscr{G}_{i-1}$. Then $F_i(\omega, x)$ is measurable in ω, non-decreasing in x and in fact has at most two points of increase as a function of x almost surely. Since $\int xF_i(\omega, dx) = Y_{i-1}$ a.s., one of the points of increase is larger than Y_{i-1} and the other is smaller. The larger point of increase is

$$\frac{\int_{Y_{i-1}(\omega)}^{\infty} xF_i(\omega, dx)}{1 - F_i(\omega, Y_{i-1}(\omega))}$$

which is clearly measurable with respect to $\mathscr{G}_{i-1}$. Denote this function by f_2. Similarly define f_1 to be the smaller point of increase. Now by assumption, the stopping times $T_1, T_2, \cdots, T_{n-1}, S_1, S_2, \cdots, S_{i-1}$ are already defined so we can assume that there is a σ-field $\mathscr{G}_{i-1}'$ generated by $W(T_1), W(T_2), \cdots, W(T_n)$, $W(S_1), \cdots, W(S_{i-1})$ which corresponds to $\mathscr{G}_{i-1}$ and $\mathscr{G}_{i-1}'$ measurable functions f_1', f_2' which correspond to f_1 and f_2. The stopping time S_i is

$$S_i = \inf\{t > S_{i-1} \colon W(t) = f_1' \text{ or. } W(t) = f_2'\}\,.$$

Now $E\{|W(S_i)|\} = E\{|Y_i|\} \leqq E\{|M_n|\} < \infty$ and $E\{W(S_i)\} = F\{Y_i\} = 0$. Thus it is enough to prove the following lemma.

LEMMA 9. *Suppose that S is a minimal stopping time, $E\{W(S)\} = 0$ and $E\{|W(S)|\} < \infty$. Suppose moreover that there are functions g_1 and g_2 which are measurable with respect to $\mathscr{G}_S$ such that $g_1(\omega) \leqq W(S(\omega)) \leqq g_2(\omega)$. If*

$$T = \inf\{t > S \colon W(t) = g_1 \textit{ or } W(t) = g_2\}$$

and $E\{W(T)\} = 0$, then T is minimal.

PROOF. To prove the lemma, it is enough to show that the process $W(t \wedge T)$

is uniformly integrable. Since the process $W(t \wedge S)$ is uniformly integrable, for any $\varepsilon > 0$ there is an $a > 0$ such that

$$E\{|W(t \wedge S)|;\ |W(t \wedge S)| > a\} < \varepsilon/2\,.$$

Since

$$\begin{aligned} E\{|W(t \wedge T)|;\ |W(t \wedge T)| > a\} &\leqq E\{|W(t \wedge S)|;\ |W(t \wedge S)| > a\} \\ &\quad + E\{|W((t \vee S) \wedge T)|;\ |W((t \vee S) \wedge T)| > a\} \end{aligned}$$

it is enough to show that for a large enough

$$E\{|W((t \vee S) \wedge T)|;\ W((t \vee S) \wedge T)| > a\} < \varepsilon/2\,.$$

The argument is the usual one. We first show that for any $\Delta \in \mathscr{F}_{(t \vee S) \wedge T}$,

$$E\{|W((t \vee S) \wedge T)| : \Delta\} \leqq E\{|W(\dot{T})|;\ \Delta\}\,.$$

Then if δ is chosen small enough that $P\{\Delta\} < \delta$ implies that $E\{W(T)|;\ \Delta\} < \varepsilon/2$ and a is chosen such that

$$(E\{|W(T)|\}/a) < \delta \qquad (\text{so that } (E\{|W((t \vee S) \wedge T)|\}/a) < \delta)$$

then $P\{|W((t \vee S) \wedge T)| > a\} < \delta$ and

$$E\{|W((t \vee S) \wedge T)|;\ |W((t \vee S) \wedge T)| > a\} < \varepsilon/2\,.$$

To show that

$$E\{|W((t \vee S) \wedge T)|;\ \Delta\} \leqq E\{|W(t)|;\ \Delta\}$$

for $\Delta \in \mathscr{F}_{(t \vee S) \wedge T}$, write $\Omega = \cup \Delta_n$ where

$$\Delta_n = \{\sup_{t<S} |W(t)| \leqq n\,; \quad |f_1| \leqq n\,; \quad |f_2| \leqq n\}\,.$$

Then $\Delta_n \in \mathscr{F}_S$ and $W(t)$, $t \leqq T$, is bounded on Δ_n. But this implies that if I_n is the indicator function of Δ_n then the process

$$I_n W((t \vee S) \wedge T)$$

is a uniformly integrable martingale. Thus

$$E\{|W((t \vee S) \wedge T);\ \Delta_n \cap \Delta\} \leqq E\{|W(T)|\colon \Delta_n \cap \Delta\}\,.$$

It follows that

$$E\{|W((t \vee S) \vee T)|;\ \Delta\} \leqq E\{|W(T)|;\ \Delta\}$$

so the proof is complete.

The proof given above can be modified to prove the following corollary which will be used in Section 3.

COROLLARY 10. *Let S and T be minimal stopping times. Suppose that $T(\omega_1) = T(\omega_2)$ if $W(\omega_1, t) = W(\omega_2, t) + a$ for all t and some constant a. Then the stopping time $S + T \circ \theta_S$ is minimal.*

THEOREM 11. *Let M_s, $s \geqq 0$ be a right continuous martingale, Then there is a Wiener process $(\Omega, \mathscr{G}_t, W_t)$ and a family $T(s)$ of $\mathscr{G}_s$ stopping times such that the* *process $Y(s) = W(T(s))$ has the same finite joint distributions as $M(s)$. The family $T(s)$ is right continuous, increasing, and for each s, $T(s)$ minimal. $T(s)$ is right continuous, increasing, and for each s, $T(s)$ minimal. Moreover, if $M(s)$ has stationary independent increments, so does $T(s)$.*

The σ-field $\mathscr{G}_t$ is in general larger than $\mathscr{F}_t$, the smallest σ-field making W_s $s \leqq t$ measurable.

PROOF. Let $(\Omega', \mathscr{F}', \mathscr{F}_t', X_t, P)$ be a Wiener process, and let $\{T_k^n\}$ be a family of minimal $\mathscr{F}_t$ stopping times such that the process $X(T_k^n)$ has the same finite joint distributions as $M_{k/n}$, $k = 1, 2, \cdots, n$.

Let C be the set of all continuous maps from $[0, \infty)$ into R. The set C admits a metric which makes C into a complete separable metric space. See Whitt [9]. Let T be the set of all non-decreasing, right continuous integrable functions from $[0, 1]$ into $[0, \infty]$. The $\mathscr{L}_1$ metric makes T also into a complete separable metric space. Note that any set of functions in T bounded by some constant is compact.

Let $\Omega = C \times T$. Then with the product topology, Ω is a complete separable metric space. Define $f_n : \Omega' \to \Omega$ by

$$f_n(\omega') = (x(s), t(s))$$

where

$$x(s) = X_s(\omega')$$

and

$$t(s) = T_k^n(\omega'), \qquad \text{where } k = [ns].$$

The functions f_n are measurable. Let μ_n be the measure induced on Ω by the random variable f_n. It will be shown that the measures μ_n are tight and if μ is an accumulation point of μ_n then Ω, $W_s(x, t) = x(s)$, $T_s(x, t) = t(s)$ and μ satisfy the theorem.

To show that the μ_n are tight, it is enough to show that the projections onto C and T are tight. That the projections on C are tight is obvious since all of the measures coincide there. On the other hand T_n^n is minimal so

$$P\{T_n^n \geqq \lambda\} \leqq \lambda^{-\frac{1}{3}}\{E\{|M_1|\}^2 + 1\}.$$

Let A_λ be the set of functions in T which are bounded by λ. Then if π is the projection of $C \times T$ onto T then

$$\mu_n(\pi^{-1}(A_\lambda)) = P\{T_n^n \leqq \lambda\} \geqq 1 - \lambda^{-\frac{1}{3}}\{E[|M_1|]^2 + 1\}.$$

Since A_λ is compact for all λ, the projections of μ_n on T are tight. Thus the sequence $\{\mu_n\}$ has an accumulation point μ and we assume that μ_n converges to μ.

Since for any open sets $U_1, U_2 \subset R$

$$\begin{aligned}\mu\{W(s_1) \in U_1, W(s_2) \in U_2\} &= \mu\{x(s_1) \in U_1, x(s_2) \in U_2\} \\ &= P\{X(s_1) \in U_1, X(s_2) \in U_2\}\end{aligned}$$

it is clear that (Ω, W_s, μ) is a Wiener process.

We now show that the process $W(T(s))$ has the same finite joint distributions as M_s. It is enough to show that for any continuous function with compact support $f\colon R^k \to R$, $i = 1, 2, \cdots, k$

$$\int f(x(t(s_1)), x(t(s_2)), \cdots, x(t(s_k)))\, d\mu = E\{f(M(s_1), M(s_2), \cdots, M(s_k))\} \quad \text{for } s_1 < s_2 < \cdots < s_k < 1 .$$

Define $g_m : \Omega \to R^k$ for $m^{-1} < 1 - s_k$ by

$$g_m(x, t) = (m \textstyle\int_0^{m^{-1}} x \circ t(s_1 + z)\, dz, \cdots, m \int_0^{m^{-1}} x \circ t(s_k + z)\, dz) .$$

Then g_m is continuous.

Now clearly

$$\lim_{m\to\infty} \int f \circ g_m\, d\mu = \int f(x \circ t(s_1), x \circ t(s_2), \cdots, x \circ t(s_k))\, d\mu$$

and

$$\lim_{n\to\infty} \int f \circ g_m\, d\mu_n = \int f \circ g_m\, d\mu .$$

Thus we must show that

$$\lim_{m\to\infty} \lim_{n\to\infty} \int f \circ g_m\, d\mu_n = E\{f(M(s_1), M(s_2), \cdots, M(s_k))\} .$$

Now

$$\int f \circ g_m\, d\mu_n = E\{f(Q_1, Q_2, \cdots, Q_k)\}$$

where

$$Q_i = m \left\{ \left(\frac{[ns_i] + 1}{n} - s_i \right) M_{[ns_i]/n} + n^{-1}[M_{([ns_i]+1)/n} + M_{([ns_i]+2)/n} + \cdots + M_{([n(s_i+m^{-1})]-1)/n}] + \left(s_i + m^{-1} - \frac{[n(s_i + m^{-1})]}{n} \right) M_{([n(s_i+m^{-1})])/n} \right\} .$$

Since the paths of M are almost surely right continuous and bounded on $[0, 1]$,

$$\lim_{n\to\infty} Q_i = m \int_0^{m^{-1}} M(s_i + z)\, dz .$$

Of course $\lim_{m\to\infty} m \int_0^{m^{-1}} M(s_i + z)\, dz = M(s_i)$ so

$$\lim_{m\to\infty} \lim_{n\to\infty} E\{f(Q_1, Q_2, \cdots, Q_k)\} = E\{f(M(s_1), M(s_2), \cdots, M(s_k))\} .$$

We have not yet discussed σ-fields. The σ-field we have implicitly been using on Ω is the one of Borel sets generated by the product topology on Ω. Denote this field by $\mathscr{F}$. If $\mathscr{F}_t$ is the field generated by the functions W_v, $v \leqq t$, then clearly $\mathscr{F}_t \subset \mathscr{F}$ since the projections are measurable functions. Moreover W_t is Markovian with respect to $\mathscr{F}_t$. But $T(s)$ (for fixed s) is not in general a stopping time with respect to $\mathscr{F}_t$. Let $\mathscr{G}_t$ be the σ-field generated by $\mathscr{F}_t$ and the sets of the form $\{T(s) \leqq v; v \leqq t, s \in R^+\}$. We will show that W_t is still Markovian with respect to $\mathscr{G}_t$.

It must be shown that the σ-field generated by the functions $W_{t_1} - W_{t_0}$, $t_1 > t_0$ is independent of the sets $\{T(s) \leqq v\}$, $v \leqq t_0$. It is enough, however, to show that, for each fixed v, there is a sequence $v_i \downarrow v$, such that the σ-field generated by the functions $W_{t_2} - W_{t_1}$, $t_2 > t_1 > \max(v_i, t_0)$ is independent of the set $\{T(s) < v_i\}$. In turn, it is enough to show that for each fixed s, there is a sequence $s_i \downarrow s$ such that the σ-field generated by the function $W_{t_2} - W_{t_1}$, $t_2 > t_1 > \max(v_i, t_0)$

is independent of the set $\{T(s_j) < v_i\}$. The point of all this is that it is possible to choose the s_j and v_i such that the set $\{T(s_j) < v_i\}$ is a continuity set of μ. Indeed the boundary of the set $\{T(s) < v\}$ is the set

$$\{T(s-) < v;\ T(s) \geqq v\}$$

and since (for fixed v) these sets are all disjoint, only a countable number can have positive measure. The assertion follows.

The proof now goes quickly. Let Λ be a set in the σ-field generated by the functions $W_{t_2} - W_{t_1}$, $t_2 - t_1 > \max(v_i, t_0)$. Then $\mu(\Lambda) = \mu_n(\Lambda)$ for all n so

$$\begin{aligned}
\mu\{T(s_j) < v_i;\ \Lambda\} &= \lim_{n\to\infty} \mu_n\{T(s_j) < v_i;\ \Lambda\} \\
&= \lim_{n\to\infty} P\{T_k{}^n < v_i; f_n{}^{-1}(\Lambda)\} \qquad k = [ns_j] \\
&= \lim_{n\to\infty} P\{T_k{}^n < v_i\}P\{f_n{}^{-1}(\Lambda)\} \qquad k = [ns_j] \\
&= \lim_{n\to\infty} \mu_n\{T(s_j) < v_i\}\mu_n\{\Lambda\} \\
&= \mu\{T(s_j) < v_i\}\mu\{\Lambda\}
\end{aligned}$$

since $\{T_n{}^k < v_j\} \in F_{v_j}$ and $f_n{}^{-1}(\Lambda)$ is in the σ-field generated by $X_{t_2} - X_{t_1}$, $t_2 > t_1 > \max(v_i, t_0)$, and the process has independent increments.

Finally, we must show that the stopping times $\{T(s)\}$ are minimal. It is enough to show that the process $W(t \wedge T(s_0))$, $s_0 < 1$, is uniformly integrable.

For $\varepsilon > 0$, choose $\delta > 0$ such that if $P\{\Delta\} < \delta$ then $E\{|M(1)|;\ \Delta\} < \varepsilon$. Also let $\lambda = E\{|M(1)|\}/\delta$. If $s < 1$ and h_λ is the function defined by

$$\begin{aligned}
h_\lambda &= |x| \qquad \text{if} \quad |x| \geqq \lambda \\
&= 0 \qquad \text{if} \quad |x| < \lambda\,,
\end{aligned}$$

then since $T_n{}^n$ is minimal

$$\begin{aligned}
\int h_\lambda \circ x(t(s_0) \wedge t_0)\, d\mu_n &= E\{h_\lambda \circ X(T_k{}^n \wedge t_0)\} \qquad k = [ns_0] \\
&\leqq E\{|X(T_n{}^n)|;\ |X(T_k{}^n \wedge t_0)| > \lambda\}\,.
\end{aligned}$$

But $E\{|X(t_k{}^n \wedge t_0)|\} \leqq E\{|X(T_n{}^n)|\}$ so

$$P\{|X(T_k{}^n \wedge t_0)| > \lambda\} < \delta\,.$$

Thus

$$\int h_\lambda \circ x(t(s_0) \wedge t_0)\, d\mu_n < \varepsilon\,.$$

This bound is independent of s_0 and t_0. Therefore if $0 \leqq f \leqq h_\lambda$ is a continuous function, then

$$\begin{aligned}
E\{f \circ W(T(s_0) \wedge t_0)\} &\leqq \lim_{\alpha\to 0} (1/\alpha) E \int_0^\alpha f \circ W(T(s_0 + z) \wedge t_0)\, dz \\
&\leqq \lim_{\alpha\to 0} (1/\alpha) \int\int_0^\alpha f \circ x(t(s_0 + z) \wedge t_0)\, dz\, d\mu \\
&\leqq \lim_{\alpha\to 0} (1/\alpha)\{\lim_{n\to\infty} \int\int_0^\alpha f \circ x(t(s_0 + z) \wedge t_0)\, dz\, d\mu_n\} \\
&\leqq \lim_{\alpha\to 0} (1/\alpha)\{\lim_{n\to\infty} \int_0^\alpha \varepsilon\, dz\} = \varepsilon\,.
\end{aligned}$$

This clearly implies that

$$E\{|W(T(s_0) \wedge t_0)|;\ |W(T(s_0) \wedge t_0| > \lambda\} < \varepsilon$$

for all t_0 so the proof is completed.

3. The theorem of the preceding section shows that any process with independent identically distributed increments with mean zero can be embedded in Brownian motion with minimal stopping times. The stopping times used to embed the discrete time process need not be the ones defined by Dubins. In fact any method for defining minimal stopping times will serve. (See Corollary 10.) This brings up the following question. What is the relation between the stopping times $\{T_t\}$ and the process X_t being embedded? This question will not be answered here but we will show that if the Skorokhod stopping times are used to embed a stable process of index $\alpha < 2$ in Brownian motion, then the stopping times $\{T_t\}$ form a stable process of index $\alpha/2$. This is very reminiscent of subordination but note that $\{T_t\}$ is, at least in most cases, *not* independent of W_t. The theorem we prove is somewhat more general than suggested above. It shows in particular that the invariance principle can be proved using an embedding approach even when the random variables are only assumed to be in the domain of attraction of a normal distribution and might not have finite variances.

The definition of Skorokhod's stopping times can be found in [8].

THEOREM 12. *Let $W(t)$ be the Wiener process, let X be a random variable with mean zero in the domain of attraction of a stable distribution of index $\alpha > 1$, and let T be a stopping time defined in the manner proposed by Skorokhod such that $W(T)$ has the same distribution as X. Then if $1 < \alpha < 2$, T is in the domain of attraction of the one-sided stable distribution of index $\alpha/2$. If $\alpha = 2$, then there is a sequence $\{a_n\}$ such that $a_n^{-2} \sum_{i=1}^n T_i$ converges to 1 in probability. Here the T_i are independent copies of T.*

PROOF. For simplicity, it will be assumed that the distribution function F of X is continuous and strictly increasing but this is not necessary. Skorokhod defines the function $G(x)$ by

$$\int_x^{G(x)} tF(dt) = 0 \qquad x < 0$$

$$\int_{G(x)}^x tF(dt) = 0 \qquad x > 0 .$$

Under the above assumptions, $G(G(x)) = x$. Also $xF(dx) = G(x)F(dG(x))$. An examination of the manner in which Skorokhod defines T shows that the Laplace transform of T is

$$\varphi(\lambda) = \int_{-\infty}^{\infty} \frac{\sinh x(2\lambda)^{\frac{1}{2}} - \sinh G(x)(2\lambda)^{\frac{1}{2}}}{\sinh (x - G(x))(2\lambda)^{\frac{1}{2}}} F(dx) .$$

We first consider the case $1 < \alpha < 2$. For such α, there is a sequence a_n such that $a_n/a_{n+1} \to 1$,

$$nx^2F_n(dx) \to M(dx)$$

where F_n is the distribution function of X/a_n and

$$M\{[0, x]\} = Cpx^{2-\alpha} \qquad x > 0 , \quad p \geqq 0$$

$$M\{[x, 0]\} = Cqx^{2-\alpha} \qquad x < 0 , \quad q \geqq 0$$

and $p + q = 1$. (See for instance [6] page 303.) Moreover

$$\frac{1 - F(x)}{1 - F(x) + F(-x)} \to p\,, \qquad \frac{F(-x)}{1 - F(x) + F(-x)} \to q\,,$$

and

$$1 - F(x) + F(-x) \sim \frac{2 - \alpha}{\alpha} x^{-\alpha} L(x) \qquad x \to \infty$$

where $L(x)$ is a slowly varying function.

LEMMA. $\lim_{x\to\infty} |G(x)/x| = (p/q)^{1/(1-\alpha)}$ *and* $\lim_{x\to-\infty} |G(x)/x| = (q/p)^{1/(1-\alpha)}$.

PROOF OF LEMMA. For $x > 0$,

$$\begin{aligned}\int_x^\infty tF(dt) &= x[1 - F(x)] + \int_x^\infty 1 - F(t)\,dt \\ &\sim p\frac{2-\alpha}{\alpha} x^{1-\alpha}L(x) + \int_x^\infty p\frac{2-\alpha}{\alpha} t^{-\alpha}L(t)\,dt \\ &\sim p\frac{2-\alpha}{\alpha} x^{1-\alpha}L(x) + p\frac{2-\alpha}{\alpha}\cdot\frac{1}{1-\alpha}\cdot x^{1-\alpha}L(x)\end{aligned}$$

by 9.6 of Chapter VIII of [6]. Thus

$$\int_x^\infty tF(dt) \sim p\frac{(2-\alpha)^2}{\alpha(1-\alpha)}\cdot x^{1-\alpha}L(x)\,.$$

Likewise

$$\int_{-\infty}^{G(x)} F(dt) \sim q\frac{(2-\alpha)^2}{\alpha(1-\alpha)}(|G(x)|)^{1-\alpha}L(|G(x)|)\,.$$

As $\int_{-\infty}^{\infty} tF(dt) = 0$ we must have for $x > 0$

$$\int_x^\infty tF(dt) = -\int_{-\infty}^{G(x)} tF(dt)\,.$$

Thus

$$px^{1-\alpha}L(x) \sim q(|G(x)|)^{1-\alpha}L(|G(x)|)$$

or

$$L(x)/L(|G(x)|)\cdot(x/|G(x)|)^{1-\alpha} \sim q/p\,.$$

Now $L(x) = a(x)\exp\int_1^x(\varepsilon(y)/y)\,dy$ where $a(x) \to c < 0$ and $\varepsilon(x) \to 0$ as $x \to \infty$ (see 9.9 Chapter VIII of [6]). This shows that

$$Q(x)(x/|G(x)|)^{1-\alpha} \sim q/p$$

where

$$(x/|G(x)|)^{-\delta(x)} < Q(x) < (x/|G(x)|)^{\delta(x)}$$

and $\delta(x) \to 0$ as $x \to \infty$. We then obtain

$$x/|G(x)| \sim (q/p)^{\alpha-1}\,.$$

(The case $p = 0$ or $q = 0$ requires special attention but it is straightforward.)

To show that T is in the domain of attraction of the one-sided stable distribution of index $\alpha/2$, it will be shown that $\lim_{n\to\infty}\varphi(\lambda/a_n^2)^n = \exp[-K\lambda^{\alpha/2}]$. As usual, this is equivalent to

$$\lim_{n\to\infty} n\ln\varphi(\lambda/a_n^2) = -K\lambda^{\alpha/2}$$

or

$$\lim_{n\to\infty} n[1 - \varphi(\lambda/a_n^2)] = K\lambda^{\alpha/2}.$$

Now

$$n[1 - \varphi(\lambda/a_n^2)] = n \int_{-\infty}^{\infty} - \frac{\sinh \dfrac{x(2\lambda)^{\frac{1}{2}}}{a_n} - \sinh \dfrac{G(x)(2\lambda)^{\frac{1}{2}}}{a_n}}{\sinh (x - G(x)) \cdot \dfrac{(2\lambda)^{\frac{1}{2}}}{a_n}} dF.$$

Consider the case $pq \neq 0$. We show first that the integral over the set $\{|x| < a_n\varepsilon\}$ can be made arbitrarily small by making ε small.

Since G is a monotone function and $G(x)/x$ has a finite limit as $x \to \infty$ or $-x \to \infty$, we can conclude that if n is large enough and $|x| < a_n\varepsilon$ then $|G(x)| < ka_n\varepsilon$ where k is some constant. One can show that

$$1 - \frac{\sinh x + \sinh y}{\sinh x + y} \sim \frac{xy}{2} \qquad x \to 0, \quad y \to 0, \quad x > 0, \quad y > 0.$$

Thus we have

$$n \int_{|x|<\varepsilon a_n} 1 - \frac{\sinh \dfrac{x(2\lambda)^{\frac{1}{2}}}{a_n} - \sinh \dfrac{G(x)(2\lambda)^{\frac{1}{2}}}{a_n}}{\sinh (x - G(x)) \dfrac{(2\lambda)^{\frac{1}{2}}}{a_n}} dF$$

$$< 2 \left\{ -\frac{n}{2} \int_{|x|<\varepsilon a_n} \frac{x(2\lambda)^{\frac{1}{2}}}{a_n} \cdot \frac{G(x)(2\lambda)^{\frac{1}{2}}}{a_n} dF \right\}$$

if ε is small enough and n is large enough. But

$$\begin{aligned} -n \int_{|x|<\varepsilon a_n} \frac{xG(x)}{a_n^2} dF(x) &= (-n/a_n^2) \int_{|x|<\varepsilon a_n} G(x) \cdot G(x)\, dF(G(x)) \\ &\leqq (n/a_n^2) \int_{|y|<\varepsilon k a_n} y^2\, dF(y) \\ &= n \int_{|y|<\varepsilon k} y^2\, dF_n(y) \to c(\varepsilon k)^{2-\alpha} \end{aligned}$$

which can be made small by making ε small.

The integral over the set $\{|x| > Ma_n\}$, M large, can be dealt with even more easily. One verifies that

$$\left| 1 - \frac{\sinh x + \sinh y}{\sinh x + y} \right| \leqq 1 \qquad x > 0, \quad y > 0$$

and obtains

$$n \int_{|x|>Ma_n} 1 - \frac{\sinh \dfrac{x(2\lambda)^{\frac{1}{2}}}{a_n} - \sinh \dfrac{G(x)(2\lambda)^{\frac{1}{2}}}{a_n}}{\sinh (x - G(x)) \dfrac{(2\lambda)^{\frac{1}{2}}}{a_n}} dF$$

$$\leqq n \int_{|x|>Ma_n} dF = n \int_{|y|>M} dF_n \to KM^{-\alpha}L(M)$$

which also can be made small. Thus we need only consider $\varepsilon a_n < |x| < Ma_n$.

Since $|G(a_n x)/a_n x| \to (p/q)^{1/(1-\alpha)}$ uniformly on $\varepsilon < x < M$ and $|G(a_n x)/a_n x| \to (q/p)^{1/(1-\alpha)}$ uniformly on $-M < x < -\varepsilon$ we have

$$n \int_{\varepsilon a_n < |x| < M a_n} 1 - \frac{\sinh \frac{x(2\lambda)^{\frac12}}{a_n} - \sinh \frac{G(x)(2\lambda)^{\frac12}}{a_n}}{\sinh (x - G(x)) \frac{(2\lambda)^{\frac12}}{a_n}} \, dF$$

$$= n \int_{\varepsilon < |x| < M} 1 - \frac{\sinh x(2\lambda)^{\frac12} - \sinh G(a_n x) \cdot \frac{(2\lambda)^{\frac12}}{a_n}}{\sinh \left(x - \frac{G(x a_n)}{a_n}\right) (2\lambda)^{\frac12}} \, dF_n$$

which converges to

$$qk \int_\varepsilon^M 1 - \frac{\sinh x(2\lambda)^{\frac12} + \sinh (q/p)^{1/(1-\alpha)} x(2\lambda)^{\frac12}}{\sinh (x + (q/p)^{1/(1-\alpha)} x)(2\lambda)^{\frac12}} \cdot x^{-1-\alpha} \, dx$$

$$+ pk \int_\varepsilon^M 1 - \frac{\sinh x(2\lambda)^{\frac12} + \sinh (p/q)^{1/(1-\alpha)} x(2\lambda)^{\frac12}}{\sinh (x + (p/q)^{1/(1-\alpha)} x)(2\lambda)^{\frac12}} \, x^{-1-\alpha} \, dx$$

$$= qk(2\lambda)^{\alpha/2} \int_\varepsilon^M 1 - \frac{\sinh z + \sinh \gamma^{-1} z}{\sinh z(1 + \gamma^{-1})} \, z^{-1-\alpha} \, dz$$

$$+ pk(2\lambda)^{\alpha/2} \int_\varepsilon^M 1 - \frac{\sinh z + \sinh \gamma z}{\sinh z(1 + \gamma)} \, z^{-1-\alpha} \, dz$$

$$= K\lambda^{\alpha/2}$$

where k and K are unspecified constants.

Now consider the case $pq = 0$ and assume that $p = 0$. Then we have for any $\varepsilon > 0$

$$n \int_{\varepsilon a_n}^\infty 1 - \frac{\sinh \frac{x(2\lambda)^{\frac12}}{a_n} - \sinh \frac{G(x)(2\lambda)^{\frac12}}{a_n}}{\sinh \frac{(x - G(x))(2\lambda)^{\frac12}}{a_n}} \, dF \leqq n \int_\varepsilon^\infty dF_n \to 0 \, .$$

We now use the fact that for $x, y > 0$

$$1 - \frac{\sinh x + \sinh y}{\sinh x + y} \sim x \left(\frac{\cosh y - 1}{\sinh y} \right) \qquad x \to 0$$

and the convergence is independent of y, $y > 0$. Thus we can write

$$n \int_0^{\varepsilon a_n} 1 - \frac{\sinh \frac{x(2\lambda)^{\frac12}}{a_n} - \sinh \frac{G(x)(2\lambda)^{\frac12}}{a_n}}{\sinh \frac{(x - G(x))(2\lambda)^{\frac12}}{a_n}} \, dF$$

$$\sim -n \int_0^{\varepsilon a_n} \frac{x(2\lambda)^{\frac12}}{a_n} \left(\frac{\cosh \frac{G(x)(2\lambda)^{\frac12}}{a_n} - 1}{\sinh \frac{G(x)(2\lambda)^{\frac12}}{a_n}} \right) dF(x)$$

which is (since $x\,dF(x) = G(x)\,dF(G(x))$)

$$-\frac{(2\lambda)^{\frac12}n}{a_n}\int_0^{\varepsilon a_n} G(x)\frac{\cosh\frac{G(x)(2\lambda)^{\frac12}}{a_n}-1}{\sinh\frac{G(x)(2\lambda)^{\frac12}}{a_n}}\,dF(G(x))$$

$$=\frac{(2\lambda)^{\frac12}n}{a_n}\int_{0\leqq G(y)\leqq\varepsilon a_n} y\frac{\cosh\frac{y(2\lambda)^{\frac12}}{a_n}-1}{\sinh\frac{y(2\lambda)^{\frac12}}{a_n}}\,dF(y)$$

$$=(2\lambda)^{\frac12}n\int_{0\leqq G(a_ny)\leqq\varepsilon a_n} y\frac{\cosh y(2\lambda)^{\frac12}-1}{\sinh y(2\lambda)^{\frac12}}\,dF_n(y)\,.$$

Since the sets $\{0\leqq G(a_n y)\leqq \varepsilon a_n\}\uparrow[-\infty, 0]$, this converges to

$$k(2\lambda)^{\frac12}\int_0^\infty y\frac{\cosh y(2\lambda)^{\frac12}-1}{\sinh y(2\lambda)^{\frac12}}y^{-1-\alpha}\,dy=\left(k\int_0^\infty\frac{(\cosh z)-1}{\sinh z}\cdot z^{-\alpha}\,dz\right)(2\lambda)^{\alpha/2}$$
$$=K\lambda^{\alpha/2}$$

where, again, k is some constant. Above we used the fact that for $y<0$

$$\lim_{n\to\infty}\frac{G(a_n y)}{a_n}=0\qquad\text{or}\qquad\lim_{n\to\infty}\frac{G(a_n y)}{a_n y}=0$$

which follows from the fact that $p=0$.

We use this again along with the fact that for y small, $xy>0$

$$1-\frac{\sinh x+\sinh y}{\sinh x+y}\sim y\left(\frac{\cosh x-1}{\sinh x}\right)$$

to write, for instance

$$n\int_{-Ma_n}^{-\varepsilon a_n} 1-\frac{\sinh\frac{x(2\lambda)^{\frac12}}{a_n}-\sinh\frac{G(x)(2\lambda)^{\frac12}}{a_n}}{\sinh\frac{(x-G(x))(2\lambda)^{\frac12}}{a_n}}\,dF$$

$$\sim -n\int_{-Ma_n}^{-\varepsilon a_n}\frac{G(x)(2\lambda)^{\frac12}}{a_n}\left(\frac{\cosh\frac{x(2\lambda)^{\frac12}}{a_n}-1}{\sinh\frac{x(2\lambda)^{\frac12}}{a_n}}\right)dF$$

$$\leqq n\int_{-Ma_n}^{-\varepsilon a_n}\frac{G(x)(2\lambda)^{\frac12}}{a_n}\,dF$$

$$=n\int_{-M}^{-\varepsilon}\frac{G(xa_n)}{a_n}(2\lambda)^{\frac12}\,dF_n\to 0\,.$$

Also

$$n\int_{-\infty}^{-Ma_n} 1-\frac{\sinh\frac{x(2\lambda)^{\frac12}}{a_n}-\sinh\frac{G(x)(2\lambda)^{\frac12}}{a_n}}{\sinh\frac{(x+G(x))(2\lambda)^{\frac12}}{a_n}}\,dF\leqq n\int_{-\infty}^{-M}dF_n$$

which can be made small by making M large. Finally

$$n \int_{-\varepsilon a_n}^{0} 1 - \frac{\sinh \frac{x(2\lambda)^{\frac{1}{2}}}{a_n} - \sinh \frac{G(x)(2\lambda)^{\frac{1}{2}}}{a_n}}{\sinh \frac{(x - G(x))(2\lambda)^{\frac{1}{2}}}{a_n}} dF$$

$$\sim -n \int_{-\varepsilon a_n}^{0} \frac{xG(x)}{2a_n^2} 2\lambda \, dF$$

$$\sim -\lambda n \int_{-\varepsilon a_n}^{0} (G(x))^2/a_n^2 \, dF(G(x))$$

$$= \lambda n \int_0^{\varepsilon} y^2 \, dF_n \to \lambda c \varepsilon^{2-\alpha}$$

which can be made small by making ε small. This concludes the proof of the theorem when $1 < \alpha < 2$.

Now suppose that $\alpha = 2$. In this case

$$nx^2 F_n(dx) \to k\varepsilon_0$$

where ε_0 is unit mass at the origin. Since

$$1 - \frac{\sinh \frac{x(2\lambda)^{\frac{1}{2}}}{a_n} + \sinh \frac{G(x)(2\lambda)^{\frac{1}{2}}}{a_n}}{\sinh \frac{(x + G(x))(2\lambda)^{\frac{1}{2}}}{a_n}} < 1$$

we have

$$\lim_{n\to\infty} n[1 - \varphi(\lambda/a_n^2)] = \lim_{n\to\infty} n \int_{-\varepsilon a_n}^{\varepsilon a_n} 1 - \frac{\sinh \frac{x(2\lambda)^{\frac{1}{2}}}{a_n} - \sinh \frac{G(x)(2\lambda)^{\frac{1}{2}}}{a_n}}{\sinh \frac{(x - G(x))(2\lambda)^{\frac{1}{2}}}{a_n}} F(dx)$$

for all $\varepsilon > 0$.

We will again use the fact that

$$1 - \frac{\sinh x + \sinh y}{\sinh x + y} \sim \frac{xy}{2} \qquad x \to 0\,, \quad y \to 0\,, \quad x > 0\,, \quad y > 0$$

and

$$1 - \frac{\sinh x + \sinh y}{\sinh x + y} < 2x \qquad x, y > 0\,.$$

Let $A_n = [-a_n\varepsilon, a_n\varepsilon]$ and $B_n = \{x\colon G(x) \in A_n\}$. Then

$$\lim_{n\to\infty} n \int_{A_n \setminus B_n} 1 - \frac{\sinh \frac{x(2\lambda)^{\frac{1}{2}}}{a_n} - \sinh \frac{G(x)(2\lambda)^{\frac{1}{2}}}{a_n}}{\sinh \frac{(x - G(x))(2\lambda)^{\frac{1}{2}}}{a_n}} F(dx)$$

$$\leqq \lim_{n\to\infty} (2n(2\lambda)^{\frac{1}{2}}/a_n) \int_{A_n \setminus B_n} |x| F(dx)$$

$$= \lim_{n\to\infty} (2n(2\lambda)^{\frac{1}{2}}/a_n) \int_{A_n \setminus B_n} |G(x)| \, dF(G(x))$$

$$\leqq \lim_{n\to\infty} 2(2\lambda)^{\frac{1}{2}}(n/a_n) \int_{|y| > \varepsilon a_n} |y| F(dy) = 0$$

as will be verified later.

Also we have

$$\lim_{n\to\infty} n \int_{A_n\cap B_n} 1 - \frac{\sinh \frac{x(2\lambda)^{\frac12}}{a_n} + \sinh \frac{G(x)(2\lambda)^{\frac12}}{a_n}}{\sinh \frac{(x+G(x))(2\lambda)^{\frac12}}{a_n}} F(dx)$$

$$\sim \lim_{n\to\infty} n \int_{A_n\cap B_n} \frac{2\lambda}{2a_n^{\ 2}} xG(x)F(dx)$$

$$\sim \lambda \lim_{n\to\infty} \frac{n}{a_n^{\ 2}} \int_{A_n\cap B_n} G(x)^2\, dF(G(x))$$

$$\sim \lambda \lim_{n\to\infty} \frac{n}{a_n^{\ 2}} \int_{A_n\cap B_n} y^2 F(dy)$$

since $G(G(x)) = x$. Now

$$\lim_{n\to\infty} \frac{n}{a_n^{\ 2}} \int_{A_n} y^2F(dy) = \lim_{n\to\infty} n \int_{-\varepsilon}^{\varepsilon} y^2F_n(dy) = k\,.$$

Also
$$\lim_{n\to\infty} \frac{n}{a_n^{\ 2}} \int_{A_n\setminus B_n} y^2\, F(dy) = \lim_{n\to\infty} \frac{n}{a_n^{\ 2}} \int_{A_n\setminus B_n} yG(y)\, dF(G(y))$$

$$\leqq \lim_{n\to\infty} \frac{n\varepsilon}{a_n} \int_{A_n\setminus B_n} |G(y)|\, dF(Gy))$$

$$\leqq \lim_{n\to\infty} \frac{n\varepsilon}{a_n} \int_{A_n^c} |z|\, F(dz) = 0\,.$$

Thus
$$\lim_{n\to\infty} \frac{n}{a_n^{\ 2}} \int_{A_n\cap B_n} y^2F(dy) = k\lambda$$

and concludes the proof except for the assertion that

$$\lim_{n\to\infty} \frac{n}{a_n} \int_{|x|>\varepsilon a_n} |x|F(dx) = 0\,.$$

We know that
$$\frac{n}{a_n^{\ 2}} \int_{\varepsilon a_n\leqq|x|\leqq 2a_n} x^2F(dx) \to 0 \qquad \text{for any } \varepsilon < 2$$

so we take $\varepsilon = 2$. We write

$$\frac{n}{a_n} \int_{|x|>2a_n} |x|\, F(dx) = \frac{n}{a_n} \sum_{i=1}^{\infty} \int_{2^ia_n\leqq|x|\leqq 2^{i+1}a_n} |x|F(dx)$$

$$\leqq \frac{n}{a_n} \sum_{i=1}^{\infty} \int_{a_{m(i,n)}\leqq|x|\leqq 3a_{m(i,n)}} |x|F(dx)$$

where $m(i, n)$ has been selected so that $a_{m(i,n)} \leqq 2^ia_n \leqq a_{m(i,n)+1}$. Since $a_n \to \infty$, and $a_n/a_{n+1} \to 1$, $3a_{m(i,n)} > 2^{i+1}a_n$ if n is large enough. Thus

$$\frac{n}{a_n} \int_{|x|>2a_n} |x|F(dx) \leqq \frac{n}{a_n} \sum_{i=1}^{\infty} \int_{a_{m(i,n)}<|x|\leqq 3a_{m(i,n)}} |x|F(dx)$$

$$\leqq \frac{n}{a_n} \sum_{i=1}^{\infty} \frac{a_{m(i,n)}}{m(i,n)} \frac{m(i,n)}{a_{m(i,n)}} \int_{a_{m(i,n)}\leqq|x|\leqq 3a_{m(i,n)}} |x|F(dx)$$

$$\leqq \frac{n}{a_n} \sum_{i=1}^{\infty} \frac{a_{m(i,n)}}{m(i,n)}\, \varepsilon$$

if n is large since $(k/a_k) \int_{a_k<|x|<3a_k} |x|F(dx) \to 0$. Now $a_{3n}/a_n \to 3^{\frac{1}{2}} < 2$ so we can assume $a_{3k}/a_k < 2$ for all $k > n$. It follows that

$$a_{3^i n} \leqq 2^i a_n$$

so that $m(i, n) \geqq 3^i n$. Also $a_{m(i,n)}/a_n \to 2^i$ so we can assume that $a_{m(i,n)}/a_n < (2.5)^i$. Then we have

$$\begin{aligned} \frac{n}{a_n} \int_{|x|>2a_n} |x|F(dx) &\leqq \varepsilon \sum_{i=1}^{\infty} \frac{n}{m(i, n)} \cdot \frac{a_{m(i,n)}}{a_n} \\ &\leqq \varepsilon \sum_{i=1}^{\infty} 3^{-i}(2.5)^i \\ &\leqq k\varepsilon \end{aligned}$$

for some k. As ε was arbitrary we have

$$\lim_{n\to\infty} \frac{n}{a_n} \int_{|x|>2a_n} |x|F(dx) = 0 .$$

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