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FURTHER REMARKS ON SEQUENTIAL ESTIMATION: THE EXPONENTIAL CASE[1]

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A sequential procedure for estimating the mean of an exponential distribution is proposed. It is shown to perform well for large values of the mean, and the results of a Monte Carlo study indicate that it also performs well for moderate values of the mean.

1. Introduction. Let $x_1, x_2, \cdots$ be independent random variables with common exponential density defined by

$$f(x) = \frac{1}{\mu} \exp(-x/\mu), \qquad x > 0,$$

where $\mu > 0$ is unknown. Given a sample $x_1, \cdots, x_k$ of size $k \geqq 1$, we shall estimate μ by $\bar{x}_k = (x_1 + \cdots + x_k)/k$, incurring the loss

$$L_k = A(\bar{x}_k - \mu)^2 + k. \tag{1}$$

Here $A > 0$ is chosen in advance of experimentation to express the weight that the experimenter assigns to estimation error relative to sampling costs.

The expected loss $E(L_k) = (1/k)A\mu^2 + k$ is minimized by taking a sample of size $k = c$, where by definition $c = A^{\frac{1}{2}}\mu$ and we have treated k as a continuous variable. The minimum expected loss is then

$$\beta_c = A^{\frac{1}{2}}\mu + A^{\frac{1}{2}}\mu = c + c = 2c.$$

Thus, the minimum expected loss is equally divided between losses assignable to estimation error and the cost of sampling.

Of course, since it is the parameter μ which we wish to estimate, the optimal sample size c is *unknown*. However, as a possible measure of the efficiency of any procedure for estimating μ, we may compare the expected loss $\bar{\beta}_c$ arising from the particular procedure with β_c. We shall give a sequential procedure which determines the sample size as a random variable in such a manner that the regret R_c, defined by $R_c = \bar{\beta}_c - \beta_c$, is small for all $c > 0$.

Procedures similar to the one which we shall propose have been discussed in [4], [5], and [7] for normal observations. There the sample size n was determined by estimating the variance of x_i at each stage, so that, as a consequence of normality, the estimate $\bar{x}_k$ and the event that $n = k$ were independent for every k. Similarly, we shall also determine the sample size by estimating the variance of x_i at each stage, but since the variance is now μ^2, $\bar{x}_k$ and the event

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$\{n = k\}$ will be highly dependent in our case. This presents several additional difficulties in the analysis of the regret.

2. The Procedure. We define our sample size n to be the least integer $k \geqq m$ for which $k \geqq A^{\frac{1}{2}} \cdot \bar{x}_k$ where the starting sample size $m \geqq 2$ is at the disposal of the experimenter. In the sequel, we shall see that the choice of m plays a crucial role in determining the efficiency of the procedure.

When sampling is terminated, we will have observed $x_1, \cdots, x_n$ and will estimate μ by $\bar{x}_n$, incurring the loss L_n defined by (1). The expected loss is then

$$\bar{\beta}_c = AE[(\bar{x}_n - \mu)^2] + E(n) ,$$

which is easily seen to be a function of c alone.

Although it follows from the results of [6] that $E(\bar{x}_n) < \mu$, there is still reason to believe that our procedure should be efficient. To see why this is so, consider the transformation $y_j = x_j/\mu$, $j = 1, 2, \cdots$. The y_j are then independent, exponentially distributed random variables with common expectation one. Moreover, $A^{\frac{1}{2}} \cdot \bar{x}_k = c\bar{y}_k$, so that

$$n = \text{least } k \geqq m \quad \text{for which} \quad k \geqq c\bar{y}_k , \tag{2}$$

and the resulting loss becomes $L_n = c^2(\bar{y}_n - 1)^2 + n$. Therefore,

$$\begin{aligned} \Pr(L_n < \beta_c) &= \Pr(c^2(\bar{y}_n - 1)^2 + n < 2c) \\ &= \Pr(c^{\frac{1}{2}}\,|\bar{y}_n - 1| < (2 - n/c)^{\frac{1}{2}}) . \end{aligned} \tag{3}$$

Now, as $c \to \infty$, $n/c \to 1$ w.p. 1 ([3]), and $c^{\frac{1}{2}}(\bar{y}_n - 1)$ has a limiting standard normal distribution ([1]). Therefore, from (3) we have

$$\lim \Pr(L_n < \beta_c) = 2\Phi(1) - 1 = .683$$

as $c \to \infty$. Here Φ defines the standard normal distribution function. Therefore, approximately 68% of experiments in which the sequential procedure is used will result in losses less than β_c.

Concerning the regret, we shall show in the next section that R_c is bounded as $c \to \infty$, and in Section 4 we shall present the results of a Monte Carlo simulation for several moderate values of c. We shall see there that the magnitude of R_c for moderate c depends crucially on the choice of the starting sample size m. Indeed, we have already excluded the choice $m = 1$ from our sampling plan, for in this case $R_c \to \infty$ as $c \to \infty$, as we shall also show in Section 3. It appears from the Monte Carlo simulation that in the absence of prior information about c, $m = 4$ or 5 is a reasonable choice of the starting sample size.

3. The Regret for large c.

THEOREM. *Let n be defined by* (2). *Then,* $R_c \leqq 0(1)$ *as* $c \to \infty$.

The proof of the theorem depends on several lemmas.

LEMMA 1. $E(n) - c \leqq O(1)$ *as* $c \to \infty$.

The proof of the lemma is given in [6]. Indeed, it is shown in [6] that $E(n) - c \leqq 1 + m \Pr(c\bar{y}_k \leqq k, \text{ for all } k \geqq m)$, so that $\limsup (E(n) - c) \leqq 1$ as $c \to \infty$.

LEMMA 2. $E(n^2) \leqq (c + m)E(n)$.

PROOF. From the definition of n and the fact that the y_j are nonnegative, we have $(n-1)^2 \leqq c(y_1 + \cdots + y_n)$ on $\{n > m\}$. Moreover, for any $m \geqq 2$, $n^2 - nm \leqq (n-1)^2$, so that $n^2 - nm \leqq c(y_1 + \cdots + y_n)$ w.p. 1. The lemma now follows from Wald's lemma.

LEMMA 3. *Let $p \geqq m$ be an integer. Then,* $\Pr(p \leqq n \leqq c/2) = O(c^{-p})$ *as* $c \to \infty$.

PROOF. The proof may be developed along lines similar to those of [5]. Sketching it, we have

$$\begin{aligned}
\Pr(p \leqq n \leqq c/2) &= \textstyle\sum_{k=p}^{c/2} \Pr(n = k) \\
&\leqq \textstyle\sum_{k=p}^{c/2} \Pr(c\bar{y}_k \leqq k) \\
&\leqq \textstyle\sum_{k=p}^{c/2} \int_0^{k^2/c} \frac{1}{\Gamma(k)} x^{k-1} e^{-x}\, dx \\
&\leqq e^p c^{-p} \textstyle\sum_{k=p}^{c/2} k^{p+1} \delta_k^{\,k-p} ,
\end{aligned} \tag{4}$$

where $\delta_k = (k/c)e^{(1-k/c)}$. Since $\delta_k \leqq (1/2)e^{\frac{1}{2}} < 1$ for $k \leqq c/2$, the last summation in (4) converges to a finite limit as $c \to \infty$, and the lemma follows.

LEMMA 4. *For* $k \geqq 1$, $E[(n-c)^{2k}] = O(c^k)$ *as* $c \to \infty$.

The lemma may be proved by a martingale argument similar to that given in [7] for a related problem.

COROLLARY 1. $E(n) \geqq c + O(1)$ *as* $c \to \infty$.

PROOF. From (2) and Wald's Lemma, we have $E(n^2) \geqq cE(n)$. Therefore, by Lemma 4, $O(c) = E[(n-c)^2] = E(n^2) - 2cE(n) + c^2 \geqq c^2 - cE(n)$, as asserted. Lemma 4 and Corollary 1 yield stronger conclusions: see Section 3.

Let $z_k = k(\bar{y}_k - 1) = y_1 + \cdots + y_k - k$, $k \geqq 1$. Then, we have

LEMMA 5. (i) $E(z_n^{\,2}) = c + O(1)$; (ii) $E(z_n^{\,3}) \geqq O(c)$; *and* (iii) $E(z_n^{\,4}) \leqq O(c^2)$ *as* $c \to \infty$.

PROOF. We rely heavily on the results of [2], which imply

$$E(z_n^{\,2}) = E(n) \quad \text{and} \quad E(z_n^{\,3}) = 3E(nz_n) + \gamma E(n)$$

where $\gamma > 0$. Thus, (i) follows from Lemma 1 and Corollary 1, and (ii) will follow if we can show that $E(nz_n) \geqq 0$. To see this observe first that since

$$\begin{aligned}
|k \textstyle\int_{n>k} z_k\, dP| &\leqq k(\textstyle\int z_k^{\,2}\, dP)^{\frac{1}{2}} \Pr(n > k)^{\frac{1}{2}} \\
&\leqq k^{\frac{3}{2}} \Pr(n > k)^{\frac{1}{2}} \to 0 \qquad \text{as} \quad k \to \infty ,
\end{aligned}$$

we may safely write

$$E(nz_n) = \sum_{k=m}^{\infty} [(k-1) \int_{n>k-1} z_{k-1} dP - k \int_{n>k} z_k \, dP] + \sum_{k=m-1}^{\infty} \int_{n>k} z_k \, dP .$$

The first sum obviously telescopes to zero, while the second is nonnegative by the results of [6].

To prove (iii), observe that from [2], $E(z_n^4) \leqq 6E(nz_n^2) + 4\gamma E(nz_n) + \delta E(n)$ where γ and δ are positive. Therefore, by the Schwarz Inequality and Lemmas 1 and 2, we have

$$\begin{aligned} E(z_n^4) &\leqq 6E(n^2)^{\frac{1}{2}} E(z_n^4)^{\frac{1}{2}} + 4\gamma E(n^2)^{\frac{1}{2}} E(z_n^2)^{\frac{1}{2}} + O(c) \\ &\leqq O(c) E(z_n^4)^{\frac{1}{2}} + O(c^{\frac{3}{2}}) . \end{aligned}$$

A contradiction now follows easily from the assumption that $\limsup c^{-2}E(z_n^4) = \infty$, thus completing the proof of the lemma.

Returning to the proof of the theorem, we observe that the regret is

$$R_c = c^2 E[(\bar{y}_n - 1)^2] - c + E(n) - c .$$

By Lemma 1, $E(n) - c = O(1)$, so it will suffice to show $E(S_c) \leqq c + O(1)$, where $S_c = c^2(\bar{y}_n - 1)^2$. With z_k as defined above, we have

$$S_c = c^2 n^{-2} z_n^2 = z_n^2 + (c^2 n^{-2} - 1) z_n^2 .$$

Therefore, by Lemma 5, it will suffice to show that $E[(c^2 n^{-2} - 1) z_n^2] \leqq O(1)$. Expanding $c^2 n^{-2} - 1$ as a function of n about $n = c$, yields

$$(c^2 n^{-2} - 1) z_n^2 = -\left(\frac{2}{c}\right)(n - c) z_n^2 + 3c^2 d^{-4} (n - c)^2 z_n^2 , \tag{5}$$

where d lies between c and n. Let

$$U_c = -c^{-1}(n - c) z_n^2 \quad \text{and} \quad V_c = c^2 d^{-4} (n - c)^2 z_n^2 .$$

Then, it will suffice to show that $E(U_c) \leqq O(1)$ and $E(V_c) = O(1)$.

To see that $E(U_c) \leqq O(1)$, observe that $n - c \geqq c\bar{y}_n - c = cz_n/n$ by definition of n and z_n. Therefore,

$$U_c \leqq -\left(\frac{1}{n}\right) z_n^3 = -\left(\frac{1}{c}\right) z_n^3 + \left(\frac{n - c}{nc}\right) z_n^3 .$$

The expectation of $-c^{-1} z_n^3$ is at most $O(1)$ by Lemma 5 (ii). To analyze the expectation of $(n - c) z_n^3 / nc$ write

$$E\left[\left(\frac{n - c}{nc}\right) z_n^3\right] = \int_{n \leqq c/2} \left(\frac{n - c}{nc}\right) z_n^3 \, dP + \int_{n > c/2} \left(\frac{n - c}{nc}\right) z_n^3 \, dP . \tag{6}$$

Now, on $n \leqq c/2$ we have $\bar{y}_n < 1$ by (2), so that $0 < -z_n \leqq n$. Therefore, the first integral on the right side of (6) is bounded in absolute value by

$$\int_{n\leqq c/2} n^2\, dP \leqq c^2 \Pr(n \leqq c/2) = O(1)\,.$$

Moreover, by Hölder's Inequality, the absolute value of the second integral on the right side of (6) is at most

$$2c^{-2}E[(n-c)^4]^{\frac{1}{4}}E(z_n^{\,4})^{\frac{3}{4}}\,,$$

which is $O(1)$ by Lemmas 4 and 5.

To see that $E(V_c) = O(1)$, again write

$$E(V_c) = \int_{n\leqq c/2} c^2 d^{-4}(n-c)^2 z_n^{\,2}\, dP + \int_{n>c/2} c^2 d^{-4}(n-c)^2 z_n^{\,2}\, dP\,. \tag{7}$$

It follows easily from (5) that on $n \leqq c/2$, $d^4 \geqq (\frac{3}{2})m^2c^2 > c^2$, so that the first integral on the right side of (7) is at most

$$\begin{aligned}\int_{n\leqq c/2}(n-c)^2 z_n^{\,2}\, dP &= \int_{n\leqq 2m}(n-c)^2 z_n^{\,2}\, dP + \int_{2m<n\leqq c/2}(n-c)^2 z_n^{\,2}\, dP\\ &\leqq 4m^2c^2 \Pr(n\leqq 2m) + c^4 \Pr(2m < n \leqq c/2)\,,\end{aligned}$$

which is $O(1)$ by Lemma 3. Here we used the fact that for $k \leqq c/2$, $n \leqq k$ implies $z_k^{\,2} \leqq k^2$. Finally, since $d > c/2$ on $n > c/2$, the second integral on the right side of (7) is at most

$$16c^{-2}E[(n-c)^2 z_n^{\,2}] \leqq 16c^{-2}E[(n-c)^4]^{\frac{1}{2}}E(z_n^{\,4})^{\frac{1}{2}}\,,$$

which is $O(1)$ by Lemma 4 and 5. This completes the proof of the theorem.

We will now show that if $m = 1$ in (2), then $R_c \to \infty$ as $c \to \infty$, in contradistinction to the result of our theorem. To see this write $R_c = c^2E[(\bar{y}_n - 1)^2] - c + E(n) - c$, as in the proof of the theorem. Moreover, when $m = 1$, it is still true that $E(n) = c + O(1)$, so it will suffice to show that when $m = 1$, $c^2E[(\bar{y}_n - 1)^2] - c \to \infty$ as $c \to \infty$. To see this write

$$c^2E[(\bar{y}_n - 1)^2] = c^2 \int_{n=1}(y_1 - 1)^2\, dP + c^2 \int_{n>1}(\bar{y}_n - 1)^2\, dP\,. \tag{8}$$

The first integral on the right side of (8) equals

$$-(e^{-1/c} + c^2(e^{-1/c} - 1)) = c + O(1)\,. \tag{9}$$

To determine the asymptotic behavior of the second, observe that $c(\bar{y}_n - 1)^2$ has limiting chi-square distribution with one degree of freedom as $c \to \infty$. Therefore,

$$\liminf \int_{n>1} c(\bar{y}_n - 1)^2\, dP \geqq 1 \tag{10}$$

as $c \to \infty$ by Fatou's Lemma. Equations (8), (9), and (10) combine to give the desired conclusion.

4. The Regret for moderate c. In order to study the procedure for moderate values of c, standard exponential deviates were generated on a computer and 5000 values of n, $\bar{y}_n$, and L_n computed for $m = 2, \cdots, 16$ and for various values of c. The results are tabled below. In cases where the computations for several consecutive values of m produced the same values of n, $\bar{y}_n$, and L_n, these values were given for the smallest of the several values of m.

It is noteworthy that when $m = 2$, the regret is always nearly twice as large as when $m = 4$. The reason for this is easily understood. For example, when $m = 2$ and $c = 100$, the probability that $n = m$ is

$$\Pr(n = m) = \Pr(y_1 + y_2 \leqq .04)$$
$$= \int_0^{.04} xe^{-x}\,dx \approx 0.0008 ,$$

so that we may expect the event $n = m$ to occur about 4 times in 5000 runs. But $n = m$ implies $\bar{y}_n \leqq 0.02$ in which case L_n exceeds $c^2(1 - \bar{y}_2)^2 \geqq 9604$. On the other hand, when $m = 4$ and $c = 100$,

$$\Pr(n = m) = \int_0^{.16} (\tfrac{1}{6})x^3e^{-x}\,dx < 0.000027 ,$$

so that we may expect the event $n = m$ to occur fewer than 3 times in every 100,000 repetitions of the procedure.

With the exception of the case $c = 25$, the choices $m = 4$ and $m = 5$ seem to do about as well as can be anticipated. In the absence of prior information, we recommend their use in practice.

	n		$\bar{y}_n$		R_c	
	mean	st. dev.	mean	st. dev.	mean	st. dev.
$c = 10$						
$m = 2$	9.19	3.76	.849	.37	5.36	19.8
$m = 3$	9.45	3.49	.873	.35	3.14	16.3
$m = 4$	9.61	3.30	.888	.33	1.88	14.3
$m = 5$	9.78	3.11	.901	.32	0.90	13.1
$m = 10$	11.25	1.79	.963	.27	−1.48	10.5
$m = 15$	15.07	0.38	.999	.25	1.37	9.1
$c = 25$						
$m = 2$	24.12	6.06	.936	.24	13.18	79.0
$m = 3$	24.31	5.69	.943	.23	8.47	61.5
$m = 4$	24.39	5.54	.946	.22	6.57	53.5
$m = 5$	24.42	5.47	.948	.22	5.76	49.9
$m = 10$	24.52	5.28	.951	.21	3.81	42.1
$m = 15$	24.66	5.00	.955	.20	1.70	35.6
$c = 50$						
$m = 2$	49.72	7.92	.979	.16	13.10	167
$m = 3$	49.87	7.45	.983	.15	5.98	108
$m = 4$	49.89	7.40	.983	.15	5.05	98

(cont.)

	n mean	n st. dev.	$\bar{y}_n$ mean	$\bar{y}_n$ st. dev.	R_c mean	R_c st. dev.
$c = 50$						
$m = 5$	49.92	7.32	.984	.15	3.82	84
$m = 16$	49.92	7.31	.984	.15	3.70	82
$c = 100$						
$m = 2$	99.50	10.6	.988	.10	12.40	359
$m = 3$	99.52	10.6	.988	.10	10.33	329
$m = 4$	99.61	10.0	.989	.10	1.05	145
$c = 150$						
$m = 2$	149.94	12.7	.995	.08	11.94	488
$m = 3$	149.97	12.5	.995	.08	7.61	380
$m = 4$	150.00	12.3	.995	.08	3.18	218
$c = 200$						
$m = 2$	199.53	14.6	.994	.07	13.70	630
$m = 3$	199.62	14.1	.994	.07	-1.98	284

5. Concluding remarks.

1. We do not know whether R_c is nonnegative for all $c > 0$, or not. The negative value -1.48 when $m = c = 10$ appears too large to be due to chance, while the negative value -1.98 when $m = 3$ and $c = 200$ does not. In any case $R_c \to m$ as $c \to 0$, and $\lim \bar{\beta}_c/\beta_c = 1$ as $c \to \infty$.

2. The methodology here developed applies also to estimating a normal variance. Let $w_1, w_2, \cdots$ be independent random variables having a common normal distribution with unknown mean θ and unknown variance σ^2, and suppose that by estimating σ^2 with $s_k^2 = (w_1^2 + \cdots + w_k^2 - k\bar{w}_k^2)/(k-1)$ we incur the loss

$$L_k = A(s_k^2 - \sigma^2)^2 + k\,.$$

The expected loss is minimized, among fixed sample size procedures, by taking $k = (2A)^{\frac{1}{2}} \cdot \sigma^2 + 1$ observations, in which case the expected loss is $\beta_c = 4c + 1$ with $c = \sigma^2(A/2)^{\frac{1}{2}}$. We determine a random sample size j by $j =$ least *odd* integer $k \geqq m$ for which $k \geqq (2A)^{\frac{1}{2}} \cdot s_k^2 + 1$. (We permit stopping only with an odd number of observations in order to expedite the analysis.) Write $\bar{y}_k = s^2_{2k+1}/\sigma^2$. Then, $\bar{y}_1, \bar{y}_2, \cdots$ has the same distribution as the sequence of successive averages of standard exponential random variables, and $j = 2n + 1$, where $n =$ least integer $k \geqq m'$ for which $k > c\bar{y}_k$ and $m' = (m-1)/2$. Moreover,

$$\begin{aligned} \bar{\beta}_c = E(L_j) &= AE(s^2_{2j+1} - \sigma^2)^2 + E(j) \\ &= 2[c^2E(\bar{y}_n^2 - 1)^2 + E(n)] + 1\,. \end{aligned}$$

Therefore, if $m' \geqq 2(m \geqq 5)$, then the regret $R_c = \bar{\beta}_c - \beta_c = 2[c^2E(\bar{y}_n^2 - 1)^2 - c + E(n-c)]$ is bounded above as $c \to \infty$ by the theorem of Section 3.

3. The referee has remarked that stronger conclusions are possible in Lemma 4 and Corollary 1 than were given, namely

$$E(n) \geqq c - 1 - 2c^{-\frac{1}{2}} + O(c^{-1}) \tag{11}$$

$$\operatorname{Var}(n) \leqq c + 2c^{\frac{1}{2}} + O(1) . \tag{12}$$

To see this, observe that since $n\bar{y}_n \geqq (n-1)\bar{y}_{n-1} \geqq (n-1)^2/c$ on $n > m$. Consequently, $n > m$ implies $n - c \geqq c(\bar{y}_n - 1) \geqq (n-1)^2/n - c \geqq n - c - 2$, so that

$$|n - c - 1| \leqq c|\bar{y}_n - 1| + 1 = S_c^{\frac{1}{2}} + 1 .$$

Therefore, $E[(n-c-1)^2] \leqq E(S_c) + 2E(S_c)^{\frac{1}{2}} + 1 + m \Pr(n = m) \leqq c + 2c^{\frac{1}{2}} + O(1)$. (11) now follows as in the proof of Corollary 1, and then (12) follows from $\operatorname{Var}(n) = E[(n - c + 1)^2] - (E(n) - c + 1)^2$.

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