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BOOK REVIEW

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ALFRED RÉNYI. *Foundations of Probability*. Holden-Day Inc., San Francisco, 1970. xvi + 366 pp. $18.50.

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Those who have grown to love Professor Rényi's fresh and lively style will be pleased to find in this book an offspring which is unmistakably his. It was born in the classroom and embodies the changes in content and approach which he introduced in courses during his many years of teaching introductory probability. Much of the book is an interrelated account of his many research interests and results, together with his philosophical views of probability theory.

The fingerprints of his philosophy are most discernable in Chapters one and two. Chapter one is concerned with the mathematical notion of an experiment—a measurable space in the terminology of set theory. The term experiment, whose use may seem patronizing or otherwise annoying to some, is not introduced to aid the student with a new concept. Indeed, a knowledge of measure theory "is taken for granted throughout the book." Instead, it seems to us, the author is rejecting a perspective which *only* sees events as measurable sets and random variables as measurable functions and then concludes that probability theory is *only* a piece of mathematics.[1] The extent of his rejection may be seen in Chapter two where he appeals to a *real world example* to conclude that "*every probability is in reality a conditionl probability*." And later he states, "This example shows that *the basic notion of probability theory should be the notion of the conditional probabilty of* A *under the condition* B . . . ." (The italics in the quotes are his.) His mathematical notion of probability theory is inexorably tied to his intuitive notion of chance.[2]

Chapter one has an interesting section discussing the structure of σ-algebras associated with finite and denumerable basic spaces. The section on polynomials of events and the one introducing the concept of qualitative independence (set-theoretic independence) provide the reader with his first side trip—this time into

[1] This rejection is more thorough than the mere refusal by K. L. Chung and others to view probability theory as a "chapter of measure theory."

[2] The author has expressed his philosophical view more extensively in the form of fictitious correspondence between B. Pascal and P. Fermat. (*Briefe über die Wahrscheinlichkeit*, VEB Deutscher Verlag der Wissenschaften, (1969).)

the land of combinatorial mathematics. Even the "world traveler" will find in this book pleasant sights which he never saw before.

Chapter two eventually arrives at the classical notion of a probability but only after a long journey through a conditional probability space—a concept due to Rényi himself. He demonstrates that his model includes several situations which have an intuitive and natural explanation that are not expressible in Kolmogorov's model.

Without disparaging the author's concept, we must admit that the prospect of proving all the basic facts of probability with conditional probability spaces is not an appealing one. The concept is developed in the first five sections of Chapter two and is integrated into the remainder of the book. Nevertheless, we feel that—with some adaptations in the classroom—this book can provide the beginning student with a stimulating and successful introduction to the theory of probability.

Section 2.6 introduces the reader to an interesting result concerning inequalities of the type $\sum_{k=1}^{N} c_k P(F_k) \geqq 0$ where each c_k is a real constant and each F_k is a polynomial in the events $A_1, \cdots, A_n$, say. The result states that the inequality is valid for arbitrary events and probability space whenever it is valid in the special cases where each A_i is either the whole space Ω or the null set $\varnothing$. The remainder of Chapter two treats random variables in probability and conditional probability spaces, expectations, and inequalities involving random variables.

Chapter three is devoted to the notion of independence. Besides the usual subjects considered he relates independence to qualitative independence, orthogonality, ergodic theory and information theory. Several theorems are proven about the existence of probability measures under which qualitatively independent events become independent. The section on independence and orthogonality sheds light on the structure of stochastic independence while introducing concepts such as spanning system, saturation with respect to independence, and separating system. A brief discussion of Markov chains—defined in information theoretic terms—appears at the end of the chapter.

In Chapter four, the author chooses breadth rather than depth as he seeks to convey the essence of the basic laws of chance—sometimes with stronger assumptions than necessary—without burdening the student with difficult technical details. All the standard limit laws appear: The weak and strong laws of large numbers, the Lindeberg version of the central limit theorem (proved with linear contraction operators), the law of the iterated logarithm and the arcsine law.

Convergence for probability distributions is treated mainly on the real line, but the concepts of weak convergence and tightness are introduced in the more general framework of separable metric spaces. An elementary proof of the Vitali-Hahn-Saks theorem is presented which should interest teachers of probability.

Chapter five, on dependence, begins with the topics of conditional expectations and martingales. The selection of material is mainly dictated by the applications to sums of independent random variables which follow. The remainder of the chapter provides a good account of stable events, mixing and exchangeability. Many of the results are the product of the author's own research.

Each chapter is followed by ten exercises and ten problems many of which contain extensions of the material in the text, or introduce new, but related topics. The student will certainly profit by working through them. An extensive bibliography is provided with each chapter. The author's scholarship and linguistic versatility will be evident to anyone who examines the geographical distribution of their entries.

With so many delicious morsels to be savored in this tastefully written book, one is reluctant to examine whether the diet is balanced. But some may wish that fewer problems were slanted toward specialized topics in combinatorial mathematics and that there were more material on characteristic functions and certain other tools of the probabilist's trade. Such matters should be considered before it is chosen as a course textbook. On the other hand, this is no ordinary textbook; it is unorthodox and highly innovative. It may cause us to change our views about what a probability course should contain.