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CONDITIONS FOR METRIC TRANSITIVITY FOR STATIONARY GAUSSIAN PROCESSES ON GROUPS

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Grenander and Maruyama independently proved that a stationary Gaussian process on the line or the integers with continuous covariance function is ergodic if and only if its spectral measure has no atoms. In this paper this theorem is extended to processes parameterized by locally compact Abelian groups.

Introduction. Grenander (1950) and Maruyama (1949) independently proved that a stationary Gaussian process on the line or the integers with continuous covariance function is ergodic if and only if its spectral measure has no atoms. More recently [1], [6] there have been studies of stationary Gaussian processes parameterized by R^n and Z^n, the so-called homogeneous random fields, and processed parameterized by more general groups. Many results for processes on the line generalize easily to results for processes on groups. But results about ergodicity depend on order properties of the real line and are not as easily extended to the group context.

In [2] it is shown that if a stationary Gaussian process $X = (x_g, P, \Omega, F)$ on a locally compact Abelian group G is metrically transitive, i.e., has no non-trivial sets A in F such that $T_g A = A$ for all $g \in G$ then its spectral measure must have no atoms. Classically this result depended on ergodic theorems for such processes, theorems nonexistent in the general case.

In this paper the converse question is attacked. It is shown that if the spectral measure of X has a density with respect to Haar measure on the dual group $\hat{G}$ of G then X is metrically transitive. It is also shown that if G is of the form $R^n \times Z^m \times K$ where K is any Abelian compact group then if the spectral measure of X has no atoms, X is metrically transitive.

1. A covariance condition for metric transitivity. Since the theorem in this section follows directly from Grenander's work for processes on the line, the proof will only be sketched. The difficulty comes in applying the theorem when the group involved is not the real numbers.

Throughout the paper let X be a mean zero real-valued Gaussian process with continuous covariance $R(g) = E(X_h X_{hg})$, where $R(g)$ is a continuous function on G.

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THEOREM 1. *If* $\forall \varepsilon > 0$ *and all finite sets* $\{g_i\}$ *in* G *there exists some* g *in* G *such that* $|R(gg_i)| < \varepsilon$ *then* X *is metrically transitive.*

PROOF. If B is a set of the form $\{x_h \in b_h, h \in G\}$ denote by $T_g B$ or B_g the set of the form $\{x_{hg} \in b_h, h \in G\}$. Then T_g extends to a transformation on sets A in F. Let A be a set in F such that $A = T_g A$ modulo a null set for all g in G. Then for any δ there exists a set I depending on finitely many x_{g_i}, $i = 1, \cdots, m$, such that $P(A \triangle I) \leqq \delta/8$. Since T_g is measure preserving, $|P(AA_g) - P(II_g)| \leqq \delta/2$ for all g in G. The characteristic function C_I of I depends only on $x_{g_1}, \cdots, x_{g_m}$ and the characteristic function C_{I_g} of I_g depends only on $x_{g_1 g}, \cdots, x_{g_m g}$. The covariances $E(x_{g_i} x_{g_j g})$ equal $R(gg_j g_i^{-1})$ so that if for all ε there exists a g such that $|R(gg_j g_i^{-1})| < \varepsilon$ then by the continuity of matrix inversion there exists a g such that the joint density $f(x_{g_1}, \cdots, x_{g_m}, \cdots, x_{gg_m})$ is arbitrarily close to a product density $\tilde{f}(x_{g_1}, \cdots, x_{g_m})\tilde{f}(x_{gg_1}, \cdots)$. It follows that for the proper g, $|P(II_g) - P(I)^2| < \delta/8$.

Now using the fact that $P(AA_g) = P(A_g)$,

$$|P(A) - P^2(A)| \leqq |P(AA_g) - P(II_g)| + |P(II_g) - P^2(I)| + |P^2(I) - P^2(A)| \leqq \delta/2 + \delta/8 + \delta/4\,.$$

But δ is arbitrary. Hence $P(A) = P(A)^2 = 0$ or 1. Thus X is metrically transitive. □

If X is a process on the line with continuous spectral function, it follows that for any finite set t_i,

$$\lim_T \frac{1}{T} \int_0^T \sum_i |R(t_i + t)|^2\, dt = 0$$

and hence $\liminf_t \sum_i |R(t_i + t)|^2 = 0$. This part of Grenander's argument does not generalize as easily.

2. Spectral conditions for metric transitivity. If G is compact it is known [2] that no non-zero stationary Gaussian process on G is metrically transitive. Therefore, in the rest of this section assume G is locally compact, but not compact.

THEOREM 2. *If* X *has spectral measure* dF *which has a density with respect to Haar measure on* $\hat{G}$ *then* X *is metrically transitive.*

PROOF. By the Riemann-Lebesgue lemma $\forall \varepsilon > 0$, there is a compact K such that for $h \in K^c$, $|R(h)| < \varepsilon$. Choose any ε and any finite set $\{g_i\}$. Now consider the set $S = \{g \mid gg_i \in K \text{ for some } i\}$. Then $S = \bigcup_i g_i^{-1}K$ which is compact.

Since G is not compact there exists some $g \in S^c$ and for this g, $gg_i \in K^c$, for all i. For this g, $|R(gg_i)| < \varepsilon$ all i and by Theorem 1, X is metrically transitive. □

For the next theorem we use the following well-known facts:

$$\frac{1}{T} \int_0^T e^{ist}\, dt \to 1 \quad \text{if } s = 0$$
$$\to 0 \quad \text{if } s \neq 0$$
$$\frac{1}{N} \sum_0^{N-1} e^{ins} \to 1 \quad \text{if } s = 0$$
$$\to 0 \quad \text{if } s \neq 0\,.$$

If x is a character on a compact group K, then

$$\int_K x(g)\, d\mu(g) = 1 \quad \text{if} \quad x = 1$$
$$= 0 \quad \text{if} \quad x \neq 1$$

where μ is Haar measure.

THEOREM 3. *Let $G = R^n \times Z^m \times K$, where R is the real line, Z is the set of integers, and K is any compact group. If X is a process on G with continuous spectral measure then X is metrically transitive.*

LEMMA. *Let $G = R^n \times Z^m \times K$, then*

$$\frac{1}{\mu(E_N)} \int_{E_N} x(g)\, d\mu(g) \to 1 \quad \textit{if} \quad x = 1$$
$$\to 0 \quad \textit{if} \quad x \neq 1\,,$$

where $E_N = [0, N]^n \times \{1, 2, \cdots, N\}^m \times K$.

PROOF. By Fubini's theorem

$$\frac{1}{\mu(E_N)} \int_{E_N} x(g)\, d\mu(g)$$
$$= \Big(\prod_{j=1}^{n} \frac{1}{N} \int_0^N e^{its_j}\, dt\Big)\Big(\prod_{p=1}^{m} \frac{1}{N} \sum_{k=1}^{N} e^{ipr_k}\Big)\Big(\int_K x'(g)\, d\mu(g)\Big)\,.$$

If $x = 1$ then all factors are 1. If $x \not\equiv 1$ then for some j, $s_j \neq 0$ or for some k, $r_k \neq 0$ or $x' \not\equiv 1$. In that case one of the factors in the above expression goes to zero and the rest are bounded by 1 as $N \to \infty$. Hence

$$\frac{1}{\mu(E_N)} \int_{E_N} x(g)\, d\mu(g) \to 0\,. \qquad \square$$

PROOF OF THEOREM 3.

$$|R(h)|^2 = \int \int_{x \neq y} h(xy^{-1})\, dF(x)\, dF(y) + \int \int_{x=y} dF(x)\, dF(y)\,.$$

Since dF has no atoms

$$\int \int C_{x=y}(x, y)\, dF(x)\, dF(y) = \int 0\, dF(y) = 0\,.$$

Choosing E_N as in the lemma it is seen

$$\frac{1}{\mu(E_N)} \int_{E_N} |R(gg_i)|^2\, d\mu(g)$$
$$= \int \int_{x \neq y} \Big(\frac{1}{\mu(E_N)} \int_{E_N} g_i(xy^{-1}) g(xy^{-1})\, d\mu(g)\Big) dF\, dF\,.$$

But by Lemma 3, as $N \to \infty$,

$$\frac{1}{\mu(E_N)} \int_{E_N} g(xy^{-1})\, d\mu(g) \to 0\,, \qquad \text{for each} \quad x \neq y\,.$$

Hence for any finite set $\{g_i\}$

$$\frac{1}{\mu(E_N)} \int_{E_N} \sum_i |R(gg_i)|^2 \, d\mu(g) \to 0 \qquad \text{as} \quad N \to \infty .$$

It follows that for any $\varepsilon > 0$, there is a g such that

$$\sum_i |R(gg_i)|^2 < \varepsilon .$$

Hence by Theorem 1, X is metrically transitive. □

It is seen in the above argument that the crucial reason a continuous spectral function implies metric transitivity is that there exists a sequence of sets E_N on the group such that for any character $x \neq 1$ on the group

$$\frac{1}{\mu(E_N)} \int_{E_N} x(g) \, d\mu(g) \to 0 .$$

It is an open question as to whether there exists such a sequence on an arbitrary group. As a step toward answering this question we offer the following:

PROPOSITION 1. *Given any sequence of compact sets E_n in a locally compact Abelian group G with $\mu(E_n) \to \infty$, where μ is Haar measure, there exists a subsequence E_{n_k} such that*

$$\frac{1}{\mu(E_{n_k})} \int_{E_{n_k}} x(g) \, d\mu(g) \to 0$$

for almost every x in G.

PROOF. Let $f_n(x) = (1/\mu(E_n)) \int_{E_n} x(g) \, d\mu(g)$. Then $f_n(x)$ is the Fourier transform of $C_{E_n}(g)/\mu(E_n)$, where C_{E_n} is the characteristic function of E_n. By the Plancherel theorem,

$$\int_{\hat{G}} |f_n(x)|^2 \, dx = \int_G \frac{C_{E_N}(g)}{\mu(E_n)^2} \, d\mu(g) = \frac{1}{\mu(E_n)} \to 0 .$$

Hence there is a subsequence $f_{n_k}(x) \to 0$ a.e. dx. □

A curious consequence of this proposition is the following:

COROLLARY. *Let E_n be any sequence of sets of integers with $\# E_n$ increasing. Then there exists a subsequence E_{n_k}, where*

$$\frac{1}{\# E_{n_k}} \sum_{p \in E_{n_k}} e^{ipt} \to 0 \quad \text{a.e.}$$

Added Note. Since this paper was first written this open question has been completely answered. Namely, there is a sequence of sets of the above type existing on a locally compact Abelian group if and only if the group is sigma-compact. It follows that Theorem 3 holds for all sigma-compact groups. This result along with some new ergodic theorems will appear in a paper entitled "Mean Ergodic Theorems for Unitary Groups."

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