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A THRESHOLD FOR LOG-CONCAVITY FOR PROBABILITY GENERATING FUNCTIONS AND ASSOCIATED MOMENT INEQUALITIES

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Let $\{p_n\}_0^N$ be a discrete distribution on $0 \leqq n \leqq N$ and let $g(u) = \sum_0^\infty p_n u^n$ be its pgf. Then for $0 \leqq t < \infty$ $g_t(u) = g(u+t)/g(1+t) = \sum_0^N p_n(t) u^n$ is a family of pgf's indexed by t. It is shown that there is a unique value t^* such that $\{p_n(t)\}_0^N$ is log-concave (PF_2) for all $t \geqq t^*$ and is not log-concave for $0 < t < t^*$. As a consequence one finds the infinite set of moment inequalities $\{\mu_{[r]}/r!\}^{1/r} \geqq \{\mu_{[r+1]}/(r+1)!\}^{1/r+1}$ $r = 1, 2, 3, \cdots$ etc. where $\mu_{[r]}$ is the rth factorial moment of $\{p_n\}_0^N$ when the lattice distribution is log-concave. The known set of inequalities for the continuous analogue is shown to follow from the discrete inequalities.

0. Introduction and summary. A set of nonnegative masses $\{p_n\}_{-\infty}^{\infty}$ on the lattice of integers is PF_2 [5], and "log-concave" or "strongly unimodal" [8] if $p_n^2 \geqq p_{n+1}p_{n-1}$ and there are no gaps in the domain of positive support. Such strongly unimodal sequences play an important role in probability theory [8]. Let $P(z) = \sum_{k=0}^N p_k z^k$ be the generating function for a set of nonnegative masses on the lattice interval $[0, N]$, with $p_N > 0$. Then $P_t(z) = P(z+t) = \sum_0^N p_k(t) z^k$ is such a generating function for all $t \geqq 0$. It will be seen that the semi-infinite interval $[0, \infty)$ has precisely one value t^*, such that $\{p_k(t)\}_0^N$ is log-concave for $t \geqq t^*$, and is not log-concave for $t < t^*$. As one direct consequence, we provide new proofs of two important sets of inequalities, perhaps deserving of more attention than they have received. The first states that if $\{p_n\}_0^\infty$ is a log-concave probability distribution, then

$$\left\{\frac{\mu_{[r+1]}}{(r+1)!}\right\}^{1/(r+1)} \leqq \left\{\frac{\mu_{[r]}}{r!}\right\}^{1/r} \qquad r = 1, 2, \cdots. \tag{0.1}$$

Here $\mu_{[r]} = \sum_0^\infty p_k\{k(k-1)\cdots(k-r+1)\}$ is the factorial moment of order r [9]. In particular, one has that for all log-concave probability lattice distributions with nonnegative support,

$$\mu_2 \leqq 2\mu_1^2 + \mu_1 \tag{0.2}$$

where μ_1 and μ_2 are the ordinary first and second moments.

The inequality (0.2) has the continuous analogue

$$\mu_2 \leqq 2\mu_1^2 \tag{0.3}$$

where $\mu_k = \int_0^\infty x^k f(x)\,dx$ and $f(x)$ is any probability density function with purely positive support such that $\log f(x)$ is a concave function on its interval of support. Such probability density functions are also "strongly unimodal" and of interest

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to probability theory [4], [7]. Equation (0.3) is a special case of the analogue of (0.1) for such density functions, which takes the form

$$\left\{\frac{\mu_{k+1}}{(k+1)!}\right\}^{1/(k+1)} \leqq \left\{\frac{\mu_k}{k!}\right\}^{1/k}; \qquad k = 1, 2, \cdots . \tag{0.4}$$

The inequalities (0.1) are implicit in Theorem 2 of [6] by Karlin, Proschan and Barlow. The inequalities (0.4) for continuous time have been given explicitly in [6], and in Barlow, Marshall and Proschan [2] under weaker conditions.

The Pólya frequency sequences of order two and Pólya frequency functions of order two that have nonnegative support and total mass one are equivalent to the log-concave sequences and log-concave density functions above. They are described at length by Karlin [5] in his book on total positivity. The properties of PF_2 sequences needed for this paper are presented in a simple self-contained form in [8] oriented towards probability theory and unimodality.

1. Real polynomials and log-concavity. It is widely known that if $P(z) = \sum_0^N p_k z^k$ is a polynomial of degree N with real coefficients having all real zeros, then (cf. [1] Section 2.22), with the convention $p_{-1} = p_{N+1} = 0$,

$$p_k^2 \geqq p_{k+1}p_{k-1}, \qquad 0 \leqq k \leqq N. \tag{1.1}$$

The inequalities of (1.1) permit one to speak of the coefficients p_k as being "log-concave" when the coefficients are positive in that the sequence $\{\log p_k\}_0^N$ has nonpositive second differences. The log-concavity is always latent for polynomials with real coefficients, whether or not the zeros are real, in the sense of the following definition and theorem.

DEFINITION. A polynomial $P(z) = \sum_0^N p_k z^k$ for which $p_k \geqq 0$, $p_N > 0$, will be said to be of type $\mathscr{P}_N$ if its coefficients $\{p_k\}_0^N$ are a log-concave sequence (cf. [8]), i.e. if the coefficients satisfy (1.1) and the set $\{k : p_k > 0\}$ is connected. All such polynomials will be said to be of type $\mathscr{P}$.

THEOREM 1. *Let $P_0(z) = \sum_0^N p_{0k} z^k$ be a polynomial of degree N with $p_{0N} > 0$ and all coefficients p_{0k} real. Let the sequence $\{p_r(t)\}_0^N$ be defined for all $t \geqq 0$ by*

$$P_t(z) = P_0(z+t) = \textstyle\sum_0^N p_k(t) z^k , \tag{1.2}$$

so that

$$p_r(t) = \textstyle\sum_{k=0}^N \binom{k}{r} t^{k-r} p_{0k} .$$

Then the set of nonnegative numbers t has some smallest finite value t^* such that $P_0(z + t^*)$ is of type $\mathscr{P}_N$. Moreover, $P_0(z + t)$ is of type $\mathscr{P}_N$ for all $t \geqq t^*$.

The proof of the theorem is based on the following lemma of some interest in its own right. (cf. Chapter 8, Section 7 of [5].)

LEMMA. *Let $P_0(z)$ be defined as in Theorem 1. Then there exists an $A \geqq 0$ such that*

$$P_0(z+t) = p_{0N} \textstyle\prod_i \{z + \alpha_i(t)\} \prod_i \{z^2 + \beta_i(t) z + \gamma_i(t)\} \tag{1.3}$$

where for all $t \geqq A$,

$$\alpha_i(t) \geqq 0\,; \qquad \beta_i(t) \geqq 0\,; \qquad \gamma_i(t) \geqq 0\,; \qquad \beta_i^2(t) \geqq \gamma_i(t)\,.$$

As proof we note that we may write

$$P_0(z) = p_{0N} \prod_i (z - r_i) \prod_j \{(z - w_j)(z - \bar{w}_j)\} \tag{1.4}$$

where the r_i are real zeros of $P_0(z)$ and the w_j and $\bar{w}_j$ are complex zeros taken in conjugate pairs. Hence

$$P_0(z + t) = p_{0N} \prod_i \{z + (t - r_i)\} \prod_j \{(z + \zeta_j)(z + \bar{\zeta}_j)\} \tag{1.5}$$

where for $w_j = x_j + iy_j$, $\bar{\zeta}_j = (t - x_j) + iy_j$. When $A_1 = \max_{i,j}\{r_i, x_j\}$ and $t \geqq A_1$, coefficients in the monomial and quadratic factors of $P_0(z + t)$ will all be nonnegative, so that $P_0(z + t)$ will be a polynomial in z with nonnegative coefficients. Consider the quadratic term

$$\begin{aligned} Q_j(z) &= z^2 + (\zeta_j + \bar{\zeta}_j)z + |\zeta_j|^2 \\ &= z^2 + 2(t - x_j)z + (t - x_j)^2 + y_j^2 \\ &= z^2 + c_{1j}\,z + c_{0j}\,. \end{aligned} \tag{1.6}$$

We note that

$$c_{1j}^2/c_{0j} = 4[1 + y_j^2(t - x_j)^{-2}]^{-1} \geqq 1 \tag{1.7}$$

when $y_j^2/(t - x_j)^2 \leqq 3$. It follows that each quadratic factor will be a polynomial of type $\mathscr{P}$ whenever

$$\max_j \{y_j^2/(t - x_j)^2\} \leqq 3 \tag{1.8}$$

i.e., whenever

$$t \geqq A_2 = \max_j \{x_j + |y_j|/(3)^{\frac{1}{2}}\}\,. \tag{1.9}$$

The validity of the lemma then follows for $A = \max(A_1, A_2)$. □

It is known [5], [8] that the product of polynomials of type $\mathscr{P}$ is itself a polynomial of type $\mathscr{P}$. It then follows from Theorem 2 that $P_0(z + t)$ is such a polynomial in z for all $t \geqq A$. A polynomial of type $\mathscr{P}$ need not have factors of type $\mathscr{P}$. (To verify this the reader need only examine the case $P(z) = (z^2 + \alpha z + 1)^2$ to find that $P(z)$ is in the set $\mathscr{P}_4$ when $\alpha^2 \geqq \frac{2}{3}$. The factor $z^2 + \alpha z + 1$ is not of type $\mathscr{P}$ unless $\alpha^2 \geqq 1$.) To prove Theorem 1, we must show that if $P_0(z)$ is of type $\mathscr{P}_N$ then $P_0(z + t)$ will also be for all $t \geqq 0$.

PROOF OF THEOREM 1. Let $P_0(z) = \sum_0^N p_{0k} z^k$ be of type $\mathscr{P}_N$ with $p_{0N} > 0$; $\Delta_{0k} = p_k^2 - p_{k+1}p_{k-1} \geqq 0$, $0 \leqq k \leqq N$. We note that, for $p_k(t)$ defined by (1.2), we have for all $t \geqq 0$ $p_N(t) = p_{0N}$, and $\Delta_N(t) = p_{0N}^2$. Further,

$$\begin{aligned} P_t(z) = P_0(z + t) &= \textstyle\sum_0^N p_{0k}(z + t)^k = \sum_0^N \sum_0^N p_{0k}\binom{k}{r} z^r t^{k-r} \\ &= \textstyle\sum_0^N p_r(t) z^r\,. \end{aligned} \tag{1.10}$$

Hence,

$$p_r(t) = \textstyle\sum_{k=0}^N \binom{k}{r} t^{k-r} p_{0k} = p_{0r} + (r + 1)p_{0r+1}t + \cdots \tag{1.11}$$

and

$$p_r'(t) = \sum_{k=0}^{N} (k-r)\binom{k}{r} t^{k-r-1} p_{0k} . \tag{1.12}$$

We see that for $0 \leqq r \leqq N-1$, $p_r'(t)$ is positive for $t \geqq 0$, so that $p_r(t)$ is positive and monotonic increasing for $t > 0$. We note in particular that $p_r'(0) = (r+1)p_{0,r+1}$. Clearly $P_t(z+\varepsilon) = P_0(z+t+\varepsilon)$. It follows that[1]

$$p_k'(t) = (k+1)p_{k+1}(t) , \qquad 0 \leqq k \leqq N . \tag{1.13}$$

From simple algebra we then have

$$\begin{aligned} \frac{d}{dt}\Delta_k(t) &= \frac{d}{dt}\{p_k^2(t) - p_{k+1}(t)p_{k-1}(t)\} \\ &= (k+2)\{p_k(t)p_{k+1}(t) - p_{k+2}(t)p_{k-1}(t)\} , \end{aligned} \tag{1.14}$$

and this may be rewritten as

$$p_k(t)\Delta_k'(t) = (k+2)\{p_{k+1}(t)\Delta_k(t) + p_{k-1}(t)\Delta_{k+1}(t)\} . \tag{1.15}$$

Equation (1.15) may be written in vector matrix form as

$$\frac{d}{dt}\mathbf{\Delta}(t) = \mathbf{\Delta}(t)\mathbf{A}(t) . \tag{1.16}$$

When $p_k(0) > 0$, for $0 \leqq k \leqq N$, then $p_k(t) > 0$ for all $t \geqq 0$ and the matrix $\mathbf{A}(t)$ has finite nonnegative components for all $t \geqq 0$. If further $\mathbf{\Delta}(0)$ has nonnegative components (one always has $\Delta_N(0) = p_{0N}^2 > 0$), (1.16) will have a unique solution $\mathbf{\Delta}(t)$ and all of its components will be nonnegative as required for the log-concavity stipulated. We note from (1.11) that the vector $\mathbf{p}(t)$ is a continuous function of the vector $\mathbf{p}(0)$, and hence that $\mathbf{\Delta}(t)$ is a continuous function of the vector $\mathbf{p}(0)$. If some components of $\mathbf{p}(0)$ are zero, we may consider a sequence of vectors $\mathbf{p}^{(\alpha)}(0)$ with all positive components converging to $\mathbf{p}(0)$. By the above reasoning $\mathbf{\Delta}^{(\alpha)}(t)$ will be nonnegative and its limit $\mathbf{\Delta}(t)$ will be nonnegative.

We have established that the set $\mathscr{T}$ of nonnegative values t for which $P_0(z+t)$ is of type $\mathscr{P}_N$ is connected and unbounded. To complete the proof of the theorem we must show that the set $\mathscr{T}$ is closed on the left. The case $t^* = 0$ is trivial. Let t_1 be positive and let $P_0(z + t_1 + \varepsilon)$ be of type $\mathscr{P}_N$ for all $\varepsilon > 0$. Consider a sequence of positive ε_j converging to zero. It is known [5], [8] that convergent log-concave sequences have log-concave limits. This implies that $P_0(z+t_1)$ is of type $\mathscr{P}_N$ and $\mathscr{T}$ is closed on the left. □

2. A set of moment inequalities for log-concave lattice distributions. Many of the basic lattice distributions of importance to statistics and probability are log-concave sequences with connected support and $\Delta_n = p_n^2 - p_{n+1}p_{n-1} \geqq 0$ for all n. A discussion of this prevalence and its underlying origins may be found in [8].

[1] This may also be seen from $P_t(z) = P_0(z+t)$ so that $\partial/\partial t P_t(z) = \partial/\partial z P_t(z)$. Hence $\sum p_k'(t)z^k = \sum kp_k(t)z^{k-1} = \sum (k+1)p_{k+1}(t)z^k$.

The important moment inequalities of Karlin, Proschan and Barlow (Theorem 2 of [6]) may be obtained from Theorem 1.

THEOREM 2. *Let* $\{p_n\}_0^\infty$ *be a log-concave probability distribution on the lattice of nonnegative integers, and let* $\mu_{[r]} = \sum_{n=0}^{\infty} p_n\{n(n-1)\cdots(n-r+1)\}$ *be its* rth *factorial moment* [9]. *Then the sequence* $\{\mu_{[r]}/r!\}_1^\infty$ *is also* log-*concave. Moreover,*

$$\frac{\mu_{[1]}}{1!} \geqq \left\{\frac{\mu_{[2]}}{2!}\right\}^{\frac{1}{2}} \cdots . \tag{2.1}$$

Equality in (2.1) *holds for all r when* $\{p_n\}_0^\infty$ *is geometric, i.e., when* $p_n = (1-\theta)\theta^n$, $0 \leqq \theta < 1$.

PROOF. It is known [5], [8] that all moments of integral order $\mu_K = \sum_{-\infty}^{\infty} n^K p_n$ are finite for such lattice distributions. Consequently all factorial moments will be finite as well. Suppose now that $\{p_n\}_0^N$ is log-concave, i.e., is such a distribution with all support on $0 \leqq n \leqq N$. Then $P(z) = \sum_0^N p_n z^n$, the probability generating function for $\{p_n\}_0^N$ is a polynomial of the type described in Theorem 1 for which the value $t^* = 0$. Hence by Theorem 1, $P(z+1) = P_1(z)$ also has log-concave coefficients. But it is known that $P(z+1)$ is the generating function for the factorial moments, i.e. one has [9]

$$P(z+1) = \sum_{r=0}^{N} \{\mu_{[r]}/r!\} z^r . \tag{2.2}$$

This relationship may also be seen directly from (1.10) for $t = 1$. Consequently

$$\left\{\frac{\mu_{[r]}}{r!}\right\}^2 \geqq \frac{\mu_{[r+1]}}{(r+1)!} \frac{\mu_{[r-1]}}{(r-1)!} ; \qquad 1 \leqq r . \tag{2.3}$$

This together with $p_0(1) = 1$ implies (2.1) in the classical way, for lattice distributions on $0 \leqq n \leqq N$. For a log-concave distribution $\{p_n\}_0^\infty$ one considers the sequence of distributions $\{p_n^*\}_0^N$ where $p_n^* = p_n/(p_0 + p_1 + \cdots + p_N)$ which converges to the given distribution. It is known [6] that log-concavity is unaffected by truncation and convergence so that (2.3) and (2.1) continue to hold.

That the inequality bounds are tight may be seen for the geometric case. Here $P(z) = (1-\theta)/(1-\theta z)$ and

$$P(z+1) = \frac{1-\theta}{1-\theta-\theta z} = \sum_0^N \left(\frac{\theta}{1-\theta}\right)^k z^k \tag{2.4}$$

so that

$$\frac{\mu_{[r]}}{r!} = \left(\frac{\theta}{1-\theta}\right)^r ; \qquad 1 \leqq r \tag{2.5}$$

and the inequalities in (2.1) become equalities. This completes the proof of Theorem 2. □

The most important inequality for statistics is the case $r = 2$. One has for the ordinary moments $\mu_r = \sum_0^\infty n^r p_n$ the following.

COROLLARY. *If* $\{p_n\}_0^\infty$ *is a* log-*concave distribution, then*

$$\mu_2 \leqq 2\mu_1^2 + \mu_1 . \tag{2.6}$$

REMARK. The set of inequalities (2.1) does not characterize log-concavity. One may have the full set of inequalities (2.1), but $\{p_n\}_0^\infty$ need not be log-concave. The simplest example is provided by $p_0 = (1 + \lambda)^{-1}$, $p_1 = 0$, $p_2 = \lambda(1 + \lambda)^{-1}$ for which $P(z) = (1 + \lambda z^2)/(1 + \lambda)$. Then $P(z + 1) = \{(1 + \lambda) + 2\lambda z + \lambda z^2\}/(1 + \lambda)$ and $P(z) \in \mathscr{P}_3$ if $\lambda \geqq \frac{1}{3}$.

3. Continuous distributions. The inequalities of (0.4) may be obtained from those of (0.1) with the aid of a simple lemma, and associated limiting argument.

LEMMA. *Let* $f(x)$ *be* log-*concave with purely positive support and let* $F(x) = \int_0^x f(y)\,dy$ *be its cumulative distribution function. Then the lattice distribution* $\{g_n(a)\}_0^\infty$ *where*

$$g_n(a) = F(na) - F(na - a)$$

is log-*concave for every* $a > 0$.

PROOF. Since $f(x)$ is a log-concave function [4], and since the convolution of two such functions is also such a function [4], then

$$g(a, x) = \int_0^a f(x - u)\,du \tag{3.1}$$

is such a function. Moreover $g(a, x)$ is continuous in x and its interval of positivity is connected. Consequently $g(a, x + a)/g(a, x)$ is monotonic decreasing in the interior of the support interval I as x increases and $\{g(a, na)\}^2 \geqq g(a, na + a)g(a, na - a)$. Hence $\{g(a, na)\}_0^\infty$ is log-concave. The lemma follows from the identification $g_n(a) = g(a, na)$. □

If we define $\mu_{[K]}(a)$ to be the factorial moment of the lattice distribution with masses $g_n(a)$, it is easy to establish that $a^K\mu_{[K]}(a) \to \mu_K = \int xf(x)\,dx$, as $a \to 0+$, and to infer (0.4) from (0.1) thereby. The weak convergence of $F_a(x)$ to $F(x)$ and the standard lemma (Feller, II, page 245) provide the desired result.

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