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RATES OF CONVERGENCE FOR WEIGHTED SUMS OF RANDOM VARIABLES[1]

BY F. T. WRIGHT

The University of Iowa

For $N = 1, 2, \cdots$ let $S_N = \sum_k a_{N,k} X_k$ where $a_{N,k}$ is a real number for $N, k = 1, 2, \cdots$ and $\{X_k\}$ is a sequence of not necessarily independent random variables. For the case $0 < t < 1$, with assumptions closely related to $E|X_k|^t < \infty$ it is shown that the rate of convergence of $P(|S_N| > \varepsilon)$ to zero is related to $\sum_k |a_{N,k}|^t$. The theorems presented here extend some of the results in the literature to not necessarily independent sequences $\{X_k\}$.

1. Introduction and summary. Let X_k for $k = 1, 2, \cdots$ be a sequence of random variables (not necessarily independent), let $a_{N,k}$ for $N, k = 1, 2, \cdots$ be real numbers, let $0 < t < 1$ and ρ_N be a sequence of positive numbers such that $\sum_k |a_{N,k}|^t \leqq \rho_N$, and let $S_{N,M} = \sum_{k=1}^M a_{N,k} X_k$ for $N, M = 1, 2, \cdots$. In Section 2 with assumptions closely related to $E|X_k|^t < \infty$, we show that for each N, $S_{N,M}$ has an almost sure limit S_N as $M \to \infty$ and that the rate at which $P(|S_N| > \varepsilon)$ converges to zero is related to ρ_N. We conclude with some remarks about the case $t = 1$.

The results of this paper are similar to those of [2], [3], [4], and [7]. In the references cited above the random variables were assumed to be independent. However in [4] it was observed that Theorems 1a and 2a of that paper were valid if the assumption of independence was omitted. Since Theorems 1a and 2a of [4] were generalizations of Theorems 1 and 2 of [2], the question is raised as to whether Theorems 3 and 4 of [2] can be generalized to include dependent sequences $\{X_k\}$ for $0 < t < 1$. In [7] Theorem 4 of [2] was generalized to the case $0 < t < 1$ but the sequence $\{X_k\}$ was still assumed to be independent. The above question is answered in the affirmative by Theorems 3 and 4 of this paper.

2. Results. Using the notation of Section 1, define for $y \geqq 0$,

$$F_k(y) = P(|X_k| \geqq y) \quad \text{and} \quad F(y) = \sup_k F_k(y) .$$

Throughout this paper C will denote various positive constants whose exact values do not matter. Where appropriate, summations will be taken over those values of k for which $a_{N,k} \neq 0$ and integrals will be Lebesgue-Stieltjes integrals.

We now prove the following

LEMMA. *If $y^t F(y) \leqq B < \infty$ for all $y > 0$, then for each N as $M \to \infty$ $S_{N,M}$ has an* a.s. *limit which we will denote by S_N.*

PROOF. We define $Y_{N,k} = X_k I_{[|a_{N,k} X_k| < 1]}$ and observe that for each N and for

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each $\varepsilon > 0$

$$(1)\qquad P(\sup_{j\geq 1}|S_{N,M+j} - S_{N,M}| > \varepsilon) \\ \leqq \sum_{k=M+1}^{\infty} P(|a_{N,k}X_k| \geqq 1) + P(\sup_{j\geq 1}|\sum_{k=M+1}^{M+j} a_{N,k} Y_{N,k}| > \varepsilon) .$$

Since $y^tF(y)$ is bounded for all $y > 0$, we see that the second expression in (1) is bounded by $C\sum_{k=M+1}^{\infty}|a_{N,k}|^t$. Using the inequality of Theorem 1 of [5] with $c_i \equiv 1$ and $r = 1$, the third expression in (1) is bounded by

$$C\sum_{k=M+1}^{\infty}|a_{N,k}| \int_{[0,|a_{N,k}|^{-1})} x\,|dF_k(x)|$$

(where the last integral is taken with respect to the Lebesgue-Stieltjes measure corresponding to $-F_k$). Integrating by parts we see the last expression is bounded by

$$C\sum_{k=M+1}^{\infty}|a_{N,k}| \int_0^{|a_{N,k}|^{-1}} F(x)\,dx \\ \leqq C\sum_{k=M+1}^{\infty}|a_{N,k}| \int_0^{|a_{N,k}|^{-1}} x^{-t}\,dt \\ = C\sum_{k=M+1}^{\infty}|a_{N,k}|^t .$$

So expression (1) tends to zero as $M \to \infty$ and hence $S_{N,M}$ has an almost sure limit (cf. page 115 of [6]).

The following theorems give rates of convergence for $P(|S_N| > \varepsilon)$.

THEOREM 1. *If $y^tF(y) \leqq B < \infty$ for all $y > 0$, then for every $\varepsilon > 0$*

$$P(|S_N| > \varepsilon) = O(\rho_N) .$$

THEOREM 2. *If $y^tF(y) \to 0$ as $y \to \infty$ and if $\max_k |a_{N,k}| \to 0$ as $N \to \infty$, then for every $\varepsilon > 0$*

$$P(|S_N| > \varepsilon) = o(\rho_N) .$$

For Theorems 3 and 4 we assume that ρ_N is of the form $CN^{-\rho}$ and hence there exists a constant β such that

$$(2)\qquad \max_k |a_{N,k}| \leqq CN^{-\beta} .$$

For Theorem 3 let s be a constant such that $0 < s < t$ and let α be a constant such that $\sum_k |a_{N,k}|^s \leqq CN^{\alpha}$. As in [4] it can be shown that we may assume $\beta \geqq \rho/t$, $\beta \geqq -\alpha/s$, and $\rho \geqq \beta(t-s) - \alpha$.

THEOREM 3. *If $\beta > 0$ and if F satisfies*

$$(3)\qquad \lim_{y\to\infty} F(y) = 0 \quad \textit{and} \quad \int_0^{\infty} y^t\,|dF(y)| < \infty ,$$

then for every $\varepsilon > 0$

$$\sum_N N^{\beta(t-s)-\alpha-1}P(|S_N| > \varepsilon) < \infty .$$

THEOREM 4. *If $\beta > 0$ and if there exists a non-increasing real valued function $G(x)$ satisfying (3) and such that $G(x) \geqq F(x)$ for all $x > 0$ and*

$$(4)\qquad \sup_{x\geq 1}\sup_{y\geq x} y^tF(y)/(x^tG(x)) < \infty ,$$

then for every $\varepsilon > 0$

$$\sum_N N^{\rho-1} P(|S_N| > \varepsilon) < \infty .$$

Note. Theorem 1 was proved in [4] and has been included here for completeness. Theorem 2 was proved in [4] under the assumption that $\rho_N \to 0$ as $N \to \infty$; however, examining that proof we see that the weaker assumption $\max_k |a_{N,k}| \to 0$ as $N \to \infty$ would suffice. Rohatgi in [7] extended Theorem 4 of [2] to the case $0 < t < 1$, but did not give an extension of Theorem 3 of [2]. Theorem 4 of this paper extends the above work to dependent sequences $\{X_k\}$ and removes assumption (6) of [7]. Theorem 3 of this paper extends Theorem 3 of [2] to the case $0 < t < 1$ and to not necessarily independent sequences $\{X_k\}$.

We now prove Theorems 3 and 4.

PROOFS. First we observe that

$$\begin{aligned} &P(|S_N| > \varepsilon) \\ &\quad \leqq \sum_k F(|a_{N,k}|^{-1}) + P(|\textstyle\sum_k a_{N,k} Y_{N,k}| > \varepsilon) \end{aligned} \tag{5}$$

where $Y_{N,k}$ is defined as in the proof of the lemma. The proofs of Theorems 3 and 4 are completed by showing that the last two expressions in (5) behave as specified in the theorems.

To show that the second expression in (5) behaves as specified in Theorem 3 one only needs to mimic the proof given for Theorem 3 of [3] found on pages 446 and 447. For Theorem 4 the argument on pages 351 and 352 of [2] suffices. It should be noted that the two arguments cited do not require $\beta(t-s) - \alpha > 0$ or $\rho > 0$ but only that $\beta > 0$.

In considering the last expression in (5), we define $\delta_{N,M} = \text{card.}\{k : M^{-1} \leqq |a_{N,k}|\}$ for $N, M = 1, 2, \cdots$. Using the Markov Inequality we see that

$$\begin{aligned} (6)\quad & \sum_N N^{\beta(t-s)-\alpha-1} P(|\textstyle\sum_k a_{N,k} Y_{N,k}| > \varepsilon) \\ & \leqq C \sum_N N^{\beta(t-s)-\alpha-1} E |\textstyle\sum_k a_{N,k} Y_{N,k}| \\ & \leqq C \sum_N N^{\beta(t-s)-\alpha-1} \sum_k |a_{N,k}| \int_{[0,|a_{N,k}|^{-1})} x\, |dF_k(x)| \\ & \leqq C \sum_N N^{\beta(t-s)-\alpha-1} \sum_k |a_{N,k}| \int_0^{|a_{N,k}|^{-1}} F(x)\, dx \\ (7)\quad & \leqq C \sum_N N^{\beta(t-s)-\alpha-1} \textstyle\sum' |a_{N,k}| \int_0^{|a_{N,k}|^{-1}} F(x)\, dx \\ (8)\quad & \qquad + C \sum_N N^{\beta(t-s)-\alpha-1} \sum_{M=2}^\infty (\delta_{N,M} - \delta_{N,M-1})(M-1)^{-1} \int_0^M F(x)\, dx \end{aligned}$$

where the prime on the summation in expression (7) indicates it is to be taken over those values of k for which $|a_{N,k}| \geqq 1$. Since $\beta > 0$ expression (7) is finite. Expression (8) is bounded by

$$\begin{aligned} & C \sum_N N^{\beta(t-s)-\alpha-1} \sum_{M=2}^\infty (\delta_{N,M} - \delta_{N,M-1}) M^{-1} \sum_{j=1}^M F(j-1) \\ & \quad \leqq C \sum_{j=1}^\infty F(j-1) \sum_{M=j}^\infty M^{-2} \sum_{N=1}^\infty N^{\beta(t-s)-\alpha-1} \delta_{N,M} . \end{aligned} \tag{9}$$

We now obtain estimates for $\delta_{N,M}$. Since $\sum_k |a_{N,k}|^s \leqq CN^\alpha$ and $\max_k |a_{N,k}| \leqq CN^{-\beta}$, we see that $\delta_{N,M} = 0$ unless $N \leqq CM^{1/\beta}$ and $\delta_{N,M} \leqq CN^\alpha M^s$. Therefore

(9) is bounded by

$$\begin{aligned} &C \sum_{j=1}^{\infty} F(j-1) \sum_{M=j}^{\infty} M^{-(2-s)} \sum_{N=1}^{[CM^{1/\beta}]} N^{\beta(t-s)-1} \\ &\leqq C \sum_{j=1}^{\infty} j^{t-1} F(j-1) \leqq C + C \sum_{j=1}^{\infty} j^{t-1} F(j) \\ &\leqq C + C \int_0^{\infty} x^t \,|dF(x)| < \infty . \end{aligned}$$

For Theorem 4 an argument similar to the one beginning at (6) shows that it is sufficient to consider

$$\sum_N N^{\rho-1} \sum_k |a_{N,k}| \int_0^{|a_{N,k}|^{-1}} F(x)\, dx . \tag{10}$$

From (2) wee see that there exists a positive constant A such that $|a_{N,k}|^{-1} \geqq AN^{\beta}$ for $N, k = 1, 2, \cdots$. Expression (10) is equal to

$$\sum_N N^{\rho-1} \sum_k |a_{N,k}| \int_0^{AN^{\beta}} F(x)\, dx + \sum_N N^{\rho-1} \sum_k |a_{N,k}| \int_{AN^{\beta}}^{|a_{N,k}|^{-1}} F(x)\, dx . \tag{11}$$

The first expression in (11) is bounded by

$$C \sum_N N^{-1-\beta(1-t)} \sum_{k=1}^{N} F(A(k-1)^{\beta})[k^{\beta} - (k-1)^{\beta}] . \tag{12}$$

Applying the Mean Value Theorem to the function $1 - (1-x)^{\beta}$, one can show that there exists a constant C depending only on β such that $k^{\beta} - (k-1)^{\beta} \leqq CK^{\beta-1}$. Hence expression (12) is bounded by

$$\begin{aligned} &C \sum_{k=1}^{\infty} k^{\beta-1} F(A(k-1)^{\beta}) \sum_{N=k}^{\infty} N^{-1-\beta(1-t)} \leqq C \sum_{k=1}^{\infty} k^{\beta t-1} F(A(k-1)^{\beta}) \\ &\quad \leqq C + C \sum_{k=1}^{\infty} k^{\beta t}[F(Ak^{\beta}) - F(A(k+1)^{\beta})] \leqq C + C \int_0^{\infty} x^t \,|dF(x)| < \infty . \end{aligned}$$

Choose N_0 so that $AN_0^{\beta} \geqq 1$. Using (4) we see that

$$\begin{aligned} &\sum_{N=N_0}^{\infty} N^{\rho-1} \sum_k |a_{N,k}| \int_{AN^{\beta}}^{|a_{N,k}|^{-1}} F(x)\, dx \\ &\leqq C \sum_{N=N_0}^{\infty} N^{\rho+\beta t-1} G(AN^{\beta}) \sum_k |a_{N,k}| \int_{AN^{\beta}}^{|a_{N,k}|^{-1}} x^{-t} dx \\ &\leqq C \sum_{N=1}^{\infty} N^{\beta t-1} G(AN^{\beta}) \leqq C \int_0^{\infty} x^t \,|dG(x)| < \infty . \end{aligned}$$

We have shown that the second expression in (11) is finite and the proofs are completed.

In [3] (see Theorem 6), it was shown that $\int_0^{\infty} x^t \log^+ x \,|dF(x)|$ finite implies the existence of the hypothesized G of Theorem 4. In [3] and [4] the sharpness of these theorems has been investigated for sequences of independent random variables.

For the case $t = 1$ it was shown in [4] that Theorems 1 and 2 are not valid for independent random variables even if it is assumed that $\rho_N \to 0$ as $N \to \infty$. However for $t = 1$ it was shown that with the additional hypotheses that $\rho_N \to 0$ as $N \to \infty$ and $\limsup_{T\to\infty} \sup_k |\int_{[-T,T]} x dP(X_k \leqq x)| < \infty$ the conclusions of Theorems 1 and 2 hold for independent variables. The following example shows that this is not the case for dependent variables.

Example. Let $Z, Y_1, Y_2, \cdots$ be independent random variables such that $P(Z = -1) = P(Z = 1) = \frac{1}{2}$, $Y_1, Y_2, \cdots$ are identically distributed, $P(Y_1 \geqq 0) = 1$, $yP(Y_1 \geqq y) \to 0$ as $y \to \infty$, and $EY_1 = \infty$. Set $X_k = ZY_k$ for $k = 1, 2, \cdots$. Clearly $yF(y) \to 0$ as $y \to \infty$ and $\int_{[-T,T]} x dP(X_k \leqq x) = 0$ for all T and $k =$

1, 2, $\cdots$. Now $N^{-1} \sum_{k=1}^{N} Y_k \rightarrow_{\text{a.s.}} \infty$ since $EY_1 = \infty$ and so there exists a sequence of positive numbers $\delta_N \rightarrow \infty$ such that

$$P(|N^{-1} \sum_{k=1}^{N} Y_k| > \delta_N) \geqq \tfrac{1}{2} \qquad \text{for} \quad N = 1, 2, \cdots .$$

Let $a_{N,k}$ be $(N\delta_N)^{-1}$ for $1 \leqq k \leqq N$ and zero for $k > N$. For this example $\rho_N = \delta_N^{-1}$ and $\rho_N^{-1} P(|S_N| > 1) \rightarrow \infty$.

In [1] an example of a stationary ergodic sequence X_k was given for which $EX_1 = 0$, $|X_1| = 1$, and

$$\sum_N N^{-1} P(|N^{-1} \sum_{k=1}^{N} X_k| > \varepsilon) = \infty .$$

Hence Theorems 3 and 4 do not hold for $t = 1$ if $\rho = 0$. It would be of interest to know if they hold in the case $t = 1$ for $\beta(t - s) - \alpha > 0$ or $\rho > 0$, respectively.

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DEPARTMENT OF STATISTICS
UNIVERSITY OF IOWA
IOWA CITY, IOWA 52240