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SOME FAMILIES OF DESIGNS FOR MULTISTAGE EXPERIMENTS: MUTUALLY BALANCED YOUDEN DESIGNS WHEN THE NUMBER OF TREATMENTS IS PRIME POWER OR TWIN PRIMES. I.[1]

BY A. HEDAYAT, E. SEIDEN, AND W. T. FEDERER

Cornell University and Michigan State University

The concepts of balance for ordered and for unordered pairs of treatments are introduced. Methods for constructing multistage experimental designs which are Youden designs at each stage, are given. In the construction of these designs an attempt was made to accommodate as much orthogonality and balance (both in our sense and in the classical sense) as is possible. These constructions are presented in several theorems. In one theorem, we give a uniform method of converting t mutually orthogonal Latin squares of order n into a t-stage balanced $(n-1) \times n$ Youden designs. Since the known methods of constructing orthogonal Latin squares of order $n = 4t + 2$, for $t > 1$, are not uniform, a uniform method for constructing two-stage $(n-1) \times n$ Youden designs for all even n has been developed. In another theorem, a method of constructing $(2\lambda+1)$-stage balanced $(2\lambda+1) \times (4\lambda+3)$ Youden designs, for $4\lambda+3$ a prime power, is presented. A method of constructing $(p^\alpha - 1)$-stage balanced $(v-1)/2 \times v$ Youden designs is given in another theorem for the case when $v = 4\lambda + 3$ and is the product of twin primes, i.e., $v = p^\alpha q^\beta$, $q^\beta = p^\alpha + 2$. Difference sets based on the elements of Galois fields were utilized for these constructions. Other miscellaneous results are given.

1. Introduction and summary. Researchers in industry, biology, education, marketing, psychology, and other fields often conduct two or more experiments on the same set (or on an adjoining set) of experimental units either simultaneously or successively, with different sets of treatments in the different experiments or stages. Such experiments are designated as multistage experimental designs. For the class of multistage experiments considered herein, each single stage contains the three factors, rows, columns, and treatments, that is, the experimental design at each stage involves two-way blocking or control of heterogeneity such as is obtained with the Latin square, the Youden (or Youden square), and other Latin rectangle experimental designs. To satisfy such criteria as minimum variances, equality of variances of differences between treatment effects, efficiency as compared to other experimental designs, associated responses of treatments at different stages, etc., it is desirable to have the treatment effects at any given stage orthogonal, or at least balanced, with respect to all other factors from the preceding stages. A set of t mutually orthogonal Latin squares would achieve this for t stages when it is feasible to use a Latin square design

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at each stage. However, it may be necessary or desirable to use only nk, $k < n$, units in any given stage. For certain values of k a Youden design is possible. (Here it should be noted that the rows of a Youden design form a randomized complete block design and that the columns form a balanced incomplete block design.) The question then arises as to the method of constructing the experimental design in the second and succeeding stages in order to have the treatment effects either orthogonal or balanced with respect to all factors from preceding stages. Some special cases of this type of design and associated statistical analyses have been studied by Clarke [4], [5] and Pearce [10]. These designs can also be considered as multistage three dimensional incomplete block designs. Some constructions and analyses for multidimensional incomplete block designs can be found in papers by Bose and Srivastava [1], Potthoff [11], [12], [13], [14], [15], Causey [2], [3] and Srivastava and Anderson [17].

In this paper the concepts of "balance for ordered and for unordered pairs of treatments" are introduced. Methods for constructing multistage experimental designs which are Youden designs at each stage are given. In the construction of these designs we have tried to accommodate as much orthogonality and balance, both in our sense and the classical sense, as is possible in these multistage experiments. These constructions are given via several theorems of which the following results highlight the contents of the paper. In one theorem we give a uniform method of converting a set of t mutually orthogonal Latin squares of order n into a t-stage balanced (for ordered pairs and also in the classical sense) $(n-1) \times n$ Youden designs. If one wants to apply this theorem he should first construct t mutually orthogonal Latin squares of order n. Unfortunately if $n = 6$ then there are no orthogonal Latin squares of order 6 and, besides, the known methods of construction of orthogonal Latin squares of order $n = 4t + 2$ are not uniform. We have partially overcome these difficulties by giving a uniform method for constructing 2-stage $(n-1) \times n$ Youden designs for all even n. In another theorem we give a method of constructing $(2\lambda+1)$-stage balanced (for unordered pairs and also in the classical sense) $(2\lambda+1) \times (4\lambda+3)$ Youden designs whenever $4\lambda + 3$ is a prime power. A method of construction of $(p^\alpha - 1)$-stage balanced (in the classical sense) $(v-1)/2 \times v$ Youden designs is given in another theorem, whenever $v = 4\lambda + 3 = p^\alpha q^\beta$, $q^\beta = p^\alpha + 2$, p and q primes and α a positive integer. These constructions mainly depend on difference sets based on the elements of Galois fields.

2. Preparatory definitions and result.

DEFINITION 2.1. Let Σ be a collection of v distinct elements (treatments). Then, a balanced incomplete block design with parameters v, b, r, k, and λ on Σ is an arrangement of v distinct elements of Σ in b subsets (blocks) of k elements $(k \leqq v)$ satisfying the condition that any two distinct elements occur in λ blocks. Any symbol occurs in r blocks and

$$vr = bk\,, \qquad \lambda(v-1) = r(k-1)\,, \qquad b \geqq v\,. \tag{1}$$

The customary notation for these designs is BIB(v, b, r, k, λ). (This has also been denoted as a 2-design.) A BIB design is said to be *symmetrical* if $v = b$ and thus $r = k$ and is denoted by BIB(v, k, λ). Some authors call a symmetrical BIB design an SBIB design or a (v, k, λ) configuration.

DEFINITION 2.2. A $k \times v$ Youden design on a set Σ of v distinct treatments is a $k \times v$ matrix D filled out with the elements of Σ with the properties that every row of D is a permutation of the set Σ and that D is a BIB design with respect to the columns.

DEFINITION 2.3. A $k \times s$ Latin rectangle on a set Ω of s distinct elements is a $k \times s$ matrix $[a_{ij}]$, $i = 1, 2, \cdots, k$; $j = 1, 2, \cdots, s$ with the requirement that each row is a permutation of the set Ω and for any j if $i \neq i'$ then $a_{ij} \neq a_{i'j}$.

DEFINITION 2.4. A set of k residues $D = \{d_1, d_2, \cdots, d_k\}$ mod v is called a (v, k, λ)-difference set if for every $a \not\equiv 0 (\text{mod } v)$, there are exactly λ ordered pairs (d_i, d_j), $d_i, d_j \in D$ such that $d_i - d_j \equiv a (\text{mod } v)$.

Now consider a $k \times v$ matrix M. Put $d_i + j (\text{mod } v)$ in the cell (i, j), $i = 1, 2, \cdots, k$; $j = 1, 2, \cdots, v$. Then it is easy to see that the resulting matrix is a $k \times v$ Youden design, and we have:

THEOREM 2.1. *The existence of a (v, k, λ)-difference set implies the existence of a $k \times v$ Youden design.*

Difference sets arise in a natural way in many combinatorial and statistical problems. The literature on difference sets is very extensive. A partial list is given in [6] and [9].

3. Mutually balanced Youden designs for ordered pairs.

DEFINITION 3.1. Let $\Omega = \{D_1, D_2, \cdots, D_t\}$ be a set of t $(v - 1) \times v$ Youden designs on a set Σ consisting of v distinct elements to be designated by $1, 2, \cdots, v$. Then we say Ω is a set of t mutually balanced Youden designs for ordered pairs if upon superposition of D_i on D_j, $i \neq j$, every ordered pair appears once in the resulting array, i.e., every pair (l, k), $l \neq k$, $l, k = 1, 2, \cdots, v$ should appear once and the pair (l, l) should not appear.

LEMMA 3.1. *There can be at most $v - 1$ $(v - 1) \times v$ mutually pairwise balanced Youden designs for ordered pairs.*

The proof is obvious.

As the reader may have noticed there is a direct connection between a set of t mutually orthogonal Latin squares of order v and a set of t $(v - 1) \times v$ mutually balanced Youden designs.

DEFINITION 3.2. A Latin square of order n on a set Σ containing n distinct elements is an $n \times n$ matrix each of whose rows and columns is a permutation of the set Σ. Two Latin squares of order n are said to be orthogonal if, when they are superimposed, each symbol of the first square occurs just once with

each symbol of the second square. A set of t mutually orthogonal Latin squares of order n is a set of t Latin squares of order n any two of which are orthogonal.

THEOREM 3.1. *A set of t mutually orthogonal Latin squares of order v implies a set of t $(v-1) \times v$ mutually balanced Youden designs for ordered pairs.*

Transform these t mutually orthogonal Latin squares of order v into a set in which the first row of each of the members is identical. Then delete the first row of each square. The resulting rectangles produce the desired designs.

Even though there is only one way to complete a $(v-1) \times v$ Youden design into a Latin square of order v, the converse of Theorem 3.1 is trivially false, as may be shown by the following counter example:

$$\begin{array}{cc} 0\ 2\ 3\ 1 & 1\ 0\ 2\ 3 \\ 1\ 0\ 2\ 3 & 2\ 3\ 1\ 0 . \\ 2\ 3\ 1\ 0 & 3\ 1\ 0\ 2 \end{array}$$

Not only are the resulting Latin squares obtained by the completion of these designs not orthogonal, but they are orthogonally mateless [8].

We shall now describe a uniform method of construction for mutually balanced Youden designs of order $(v-1) \times v$ for even v. Our method of construction seems to be valuable because the method of orthogonal Latin squares fails for $v = 6$ and is not simple and uniform for $v = 4t + 2$.

THEOREM 3.2. *Let D_1 and D_2 be two $(v-1) \times v$ matrices, v even. Put in the (i, j) cell of D_1, $i = 0, 1, \cdots, v-2$; $j = 0, 1, 2, \cdots, v-1$,*

$$\begin{aligned} -i/2 + j \pmod v , &\qquad \textit{for } i \textit{ even} \\ (i+1)/2 + j \pmod v , &\qquad \textit{for } i \textit{ odd} , \end{aligned}$$

and in the (i, j) cell of D_2, $i = 0, 1, \cdots, v-2$; $j = 0, 1, 2, \cdots, v-1$

$$\begin{aligned} i/2 + 1 + j \pmod v , &\qquad \textit{for } i \textit{ even} \\ -(i+1)/2 + j \pmod v , &\qquad \textit{for } i \textit{ odd} . \end{aligned}$$

Then $\{D_1, D_2\}$ forms a pair of balanced $(v-1) \times v$ Youden designs for ordered pairs.

PROOF. D_1 and D_2 are clearly Youden designs. Now we show that they are balanced in the sense of Definition 3.1. Consider all the $(v-1)$ cells of D_1 which contain a fixed integer, say α. Then α will be in the cell (i, j) with $j = \alpha + i/2 \pmod v$ if i is even and $j = -(i+1)/2 + \alpha \pmod v$ if i is odd. Then upon superposition of D_1 on D_2 we have the following entries in the corresponding cells of D_2, $(i + 1 + \alpha) \bmod v$ and $(-i - 1 + \alpha) \bmod v$ if i is even and i is odd respectively. Obviously, these entries in D_2 will exhaust $\{0, 1, 2, \cdots, v-1\} - \{\alpha\}$. A similar argument holds if we consider all $v-1$ entries in D_1 corresponding to a fixed element in D_2.

An example will elucidate the method of this theorem. Let $v = 6$. Then

$$D_1 = \begin{matrix} 0\ 1\ 2\ 3\ 4\ 5 \\ 1\ 2\ 3\ 4\ 5\ 0 \\ 5\ 0\ 1\ 2\ 3\ 4 \\ 2\ 3\ 4\ 5\ 0\ 1 \\ 4\ 5\ 0\ 1\ 2\ 3 \end{matrix} \quad \text{and} \quad D_2 = \begin{matrix} 1\ 2\ 3\ 4\ 5\ 0 \\ 5\ 0\ 1\ 2\ 3\ 4 \\ 2\ 3\ 4\ 5\ 0\ 1 \\ 4\ 5\ 0\ 1\ 2\ 3 \\ 3\ 4\ 5\ 0\ 1\ 2 \end{matrix}.$$

Note that if we extend D_1 and D_2 to Latin squares of order n, then the resulting Latin squares will be orthogonally mateless [8].

DEFINITION 3.3. Let F_1 and F_2 be two factors with v_1 and v_2 levels respectively. Now consider a design with n experimental units where each experimental unit received a level of F_1 and a level of F_2. Also, let N be the $v_1 \times v_2$ incidence matrix whose entry in the (i, j) cell, viz., n_{ij} represents the number of experimental units which have received the ith level of F_1 and the jth level of F_2. Then we say F_1 and F_2 are

(a) in a balanced arrangement if for some diagonal matrix T and a scalar c

$$NN' = T + cJ$$

(b) are orthogonal to each other if

$$n_{ij} = (n_{i.})(n_{.j})/n_{..}$$

where J is the $v_1 \times v_1$ matrix of ones, $n_{i.} = \sum_{j=1}^{v_2} n_{ij}$, $n_{.j} = \sum_{i=1}^{v_1} n_{ij}$, and $n_{..} = \sum_{i=1}^{v_1} \sum_{j=1}^{v_2} n_{ij}$.

The family of mutually balanced Youden designs for ordered pairs enjoys the following properties in the sense of Definition 3.3.

1. In each stage

(a) Treatment effects are orthogonal to row effects.
(b) Treatment effects are in balanced arrangement with column effects.
(c) Row effects are orthogonal to column effects.

2. With respect to any two different stages i and j

(a) Treatment effects in the ith stage are orthogonal to row effects in the jth stage.
(b) Treatment effects in the ith stage are in balanced arrangement with column effects in the jth stage.
(c) Row (column) effects in the ith stage are orthogonal to the column (row) effects in the ith stage and to the column (row) effects in the jth stage.
(d) Treatment effects in the ith stage and the jth stage are in a balanced arrangement. Note that in this case the corresponding $NN' = I + (v_1 - 2)J$; I is the identity matrix of order v_1.

The above properties can be easily verified.

4. Mutually balanced Youden designs for unordered pairs.

DEFINITION 4.1. Let $S = \{D_1, D_2, \cdots, D_t\}$ be a set of t $k \times v$ Youden designs on a set Σ containing v distinct elements to be designated by $1, 2, \cdots, v$. Then we say S is a set of t mutually balanced $k \times v$ Youden designs for unordered pairs if upon superposition of D_i on D_j $i \neq j$, every unordered pair appears once, i.e., the pairs (l, l) do not appear and if (l, k) appears then (k, l) should not appear, $l \neq k$, $l, k = 1, 2, \cdots, v$.

LEMMA 4.1. *If S is a set of t mutually balanced $k \times v$ Youden designs for unordered pairs then $k = (v - 1)/2$, where v is of the form $4\lambda + 3$ and $t \leqq v - 1$.*

PROOF. From a set of v distinct elements we can form $v(v - 1)/2$ unordered pairs and therefore $k = (v - 1)/2$. Since each design is a BIB with respect to columns, we have

$$\lambda(v - 1) = k(k - 1) = \left(\frac{v - 1}{2}\right)\left(\frac{v - 3}{2}\right).$$

This implies that $\lambda = (v - 3)/4$ or $v = 4\lambda + 3$. The proof that $t \leqq v - 1$ is obvious. We remark that $v - 1$ is not a good upper bound for t.

Lemma 4.1 indicates that for the purpose of constructing mutually balanced Youden designs for unordered pairs we should limit ourselves to the family of $k \times v$ Youden designs with v of the form $4\lambda + 3$ and $k = (v - 1)/2$.

THEOREM 4.1. *If $v = 4\lambda + 3 = p^\alpha$, p a prime and α a positive integer, then the quadratic residues in* GF(p^α) *form a* $(4\lambda + 3, 2\lambda + 1, \lambda)$ *difference set and hence a* $(2\lambda + 1) \times (4\lambda + 3)$ *Youden design* (see, for example, [7] or [16]).

THEOREM 4.2. *If $v = 4\lambda + 3 = p^\alpha q^\beta$, p and q primes with $q^\beta = p^\alpha + 2$, then there is a* $(4\lambda + 3, 2\lambda + 1, \lambda)$ *difference set and hence a* $(2\lambda + 1) \times (4\lambda + 3)$ *Youden design* [18].

THEOREM 4.3. *If $v = 4\lambda + 3$ is of the form $4x^2 + 27 = p$ prime, then there is a* $(4\lambda + 3, 2\lambda + 1, \lambda)$ *difference set and hence a* $(2\lambda + 1) \times (4\lambda + 3)$ *Youden design* (see, for example, [7]).

Note that the family of designs constructed by Theorem 4.3 will be a subset of designs constructed by Theorem 4.1; however, the method of constructing corresponding difference sets is different. For $v = 4\lambda + 3 \leqq 99$ one can construct all $(2\lambda + 1) \times (4\lambda + 3)$ Youden designs by the method of Theorems 4.1, 4.2 and 4.3, except for $v = 39$, 51, 55, 75, 87 and 95.

THEOREM 4.4. *There exist at least $2\lambda + 1$ mutually balanced $(2\lambda + 1) \times (4\lambda + 3)$ Youden designs for unordered pairs for $4\lambda + 3$ of the form p^α, p a prime and α a positive integer.*

The proof is by construction. Identify the $4\lambda + 3$ treatments with the elements of the GF(p^α) with x as a primitive element. Then, by Theorem 4.1 $H = \{x^0, x^2, x^4, \cdots, x^{4\lambda}\}$ is a $(4\lambda + 3, 2\lambda + 1, \lambda)$ difference set. Note that the

set $x^{2r}H = \{x^{2r}\cdot x^0, x^{2r}\cdot x^2, \cdots\} = H$. Now the $(2\lambda + 1) \times (4\lambda + 3)$ array D_r with entry in the (i, j) position equal to

$$x^{2(r+i)} + (1 - 0^j)x^j, \qquad i = 0, 1, \cdots, 2\lambda; \qquad j = 0, 1, \cdots, 4\lambda + 2$$

forms a $(2\lambda + 1) \times (4\lambda + 3)$ Youden design ($1 - 0^j = 0$ for $j = 0$ and 1 for $j \neq 0$). We prove that D_r and D_s, $r \neq s$, $r, s = 0, 1, \cdots, 2\lambda$, are balanced for unordered pairs. Consider the $2\lambda + 1$ cells of D_r which contain a fixed element of GF(p^α), say $\beta = x^k$. β will occur in row i and column j where j satisfies $(1 - 0^j)x^j = x^k - x^{2(r+i)}$ $i = 0, 1, \cdots, 2\lambda$. The corresponding entries in the cells of D_s will be

$$u_i = x^{2i}(x^{2s} - x^{2r}) + \beta, \qquad i = 0, 1, \cdots, 2\lambda.$$

Conversely entries in $2\lambda + 1$ cells of D_r corresponding to those cells of D_s containing β will be

$$v_i = x^{2i}(x^{2r} - x^{2s}) + \beta, \qquad i = 0, 1, \cdots, 2\lambda.$$

Now clearly $\{u_i, i = 0, 1, \cdots, 2\lambda\} \cup \{v_i, i = 0, 1, \cdots, 2\lambda\} = \text{GF}(p^\alpha) - \{\beta\}$, and this completes the proof.

Let us clarify the method of Theorem 4.4 with an example. Let $4\lambda + 3 = 7$. Then 3 is a primitive element of GF(7). Thus $H = \{1, 2, 4\}$. Hence

$$D_0 = \begin{matrix} 1 & 4 & 3 & 0 & 5 & 6 & 2 \\ 2 & 5 & 4 & 1 & 6 & 0 & 3 \\ 4 & 0 & 6 & 3 & 1 & 2 & 5 \end{matrix}, \quad D_1 = \begin{matrix} 2 & 5 & 4 & 1 & 6 & 0 & 3 \\ 4 & 0 & 6 & 3 & 1 & 2 & 5 \\ 1 & 4 & 3 & 0 & 5 & 6 & 2 \end{matrix}, \quad D_2 = \begin{matrix} 4 & 0 & 6 & 3 & 1 & 2 & 5 \\ 1 & 4 & 3 & 0 & 5 & 6 & 2 \\ 2 & 5 & 4 & 1 & 6 & 0 & 3 \end{matrix}.$$

It is, of course, desirable to have a set of Youden designs which satisfy both balance properties in the sense of Definitions 3.3 and 4.1. The method of construction of Theorem 4.3 guarantees that the constructed $2\lambda + 1$ Youden designs are balanced in both senses. We point out that it is not in general true that if a set of Youden designs are balanced in the sense of Definition 3.3 (Definition 4.1), then they are necessarily balanced in the sense of Definition 4.1 (Definition 3.3). To support this we present the following two counter examples.

The following two designs are balanced in the sense of Definition 3.3 but not 4.1:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{matrix} \qquad \begin{matrix} 1 & 3 & 4 & 5 & 6 & 0 & 2 \\ 4 & 5 & 6 & 0 & 2 & 1 & 3 \\ 2 & 1 & 3 & 4 & 5 & 6 & 0 \end{matrix}$$

The following two designs are balanced in the sense of Definition 4.1 but not 3.3:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 \\ 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix}$$

2	3	4	5	6	7	8	9	10	11	12	13	14	0	1
4	5	6	7	8	9	10	11	12	13	14	0	1	2	3
10	11	12	13	14	0	1	2	3	4	5	6	7	8	9
5	6	7	8	9	10	11	12	13	14	0	1	2	3	4
8	9	10	11	12	13	14	0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	8	9	10	11	12	13	14	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

THEOREM 4.5. *Any pair of Youden designs belonging to the* $2\lambda + 1$ *designs of Theorem* 4.4 *are balanced in the sense of Definition* 3.3, *i.e.*,

$$NN' = (\lambda + 1)I + \lambda J .$$

PROOF. Let D_r and D_s be two Youden designs with the values in cells (i, j) equal to $x^{2(r+i)} + (1 - 0^j)x^j$ and $x^{2(s+1)} + (1 - 0^j)x^j$ respectively, $i = 0, 1, \cdots, 2\lambda$; $j = 0, 1, \cdots, 4\lambda + 2$. Consider two sets of $2\lambda + 1$ entries of D_s corresponding to the entries in D_r which take on exactly two distinct values, say, k and l. Clearly

$$k = x^{2(r+i)} + (1 - 0^j)x^j \qquad \text{for} \quad 2\lambda + 1 \quad \text{pairs} \quad (i, j)$$

and

$$l = x^{2(r+i)} + (1 - 0^j)x^j \qquad \text{for} \quad 2\lambda + 1 \quad \text{pairs} \quad (i, j) .$$

For any j the corresponding entries in D_s will be equal to

$$x^{2(s+i)} + k - x^{2(r+i)}$$

and

$$x^{2(s+i')} + l - x^{2(r+i')}$$

for some i and i'. We want to show that there are λ pairs, (i, i'), for which these expressions coincide, i.e., that the equation $x^{2(s+i)} + k - x^{2(r+i)} = x^{2(s+i')} + l - x^{2(r+i')}$ has λ distinct solutions for variable i, i'. We may rewrite this equation in the form

$$x^{2i} - x^{2i'} = (l - k)/(x^{2s} - x^{2r}) .$$

The right-hand side of this equation is a nonzero element of GF(p^α). The left-hand side is a difference of two elements belonging to the difference set. Hence, the proof. (A result paralleling this theorem has been obtained by Causey [3].)

We note that this family of designs also satisfies all the properties listed under properties 1 and 2 listed at the end of Section 3.

We shall now consider another family of Youden designs for $4\lambda + 3$ of the form $p^\alpha q^\beta$ where p, q are primes and α, β positive integers such that $q^\beta = p^\alpha + 2$. For this family we have the following theorem.

THEOREM 4.6. *There exist at least* $(p^\alpha - 1)$ *mutually balanced* $(v - 1)/2 \times v$ *Youden designs in the sense of Definition* 3.3 *where* $v = 4\lambda + 3 = p^\alpha q^\beta$, $q^\beta = p^\alpha + 2$.

The proof is by construction. Let x be a primitive root of GF(p^α) and y be a

primitive root of GF(q^β). Then, it has been shown by Stanton and Sprott [18] that the following set forms a $(4\lambda + 3, 2\lambda + 1, \lambda)$ difference set:

$$M = \{z^0, z^1, \cdots, z^{d-1}, \hat{0}, w^0, w^1, \cdots, w^{s-2}\}$$

where

$$z^i = (x^i, y^i), \qquad \hat{0} = (0, 0), \qquad w^i = (x^i, 0),$$
$$s = p^\alpha \qquad \text{and} \qquad d = (s^2 - 1)/2 .$$

Addition and multiplication on M are defined by the relations

$$(r_1, t_1) + (r_2, t_2) = (r_1 + r_2, t_1 + t_2),$$
$$(r_1, t_1)(r_2, t_2) = (r_1 r_2, t_1 t_2) .$$

It is clear that $(x^k, y^k)M$, $k = 0, 1, \cdots, s - 2$, is also a $(4\lambda + 3, 2\lambda + 1, \lambda)$ difference set. Indeed, for the given k's $(x^k, y^k)M = M$. Let D_k be the following $(2\lambda + 1) \times (4\lambda + 3)$ array. Name the rows of D_k by $0, 1, \cdots, 2\lambda$ and its columns in any manner by (j_1, j_2) where (j_1, j_2) belongs to M. Put in $(i, (j_1, j_2))$ cells of D_k

$$\begin{aligned} &(x^k, y^k)z^i + (j_1, j_2), && 0 \leqq i \leqq d - 1 \\ &(j_1, j_2), && i = d \\ &(x^k, y^k)w^{i-(d+1)} + (j_1, j_2), && d + 1 \leqq i \leqq 2\lambda . \end{aligned}$$

Now we prove that D_k and D_l are balanced in the sense of Definition 3.3, $k \neq l$, $k, l = 0, 1, \cdots, s - 2$. Note that each row of D_k and D_l exhausts the $4\lambda + 3$ elements of the Galois Domain GD(v), that is, the set of elements (r, t) with r in GF(p^α) and t in GF(q^β). Now consider the $2\lambda + 1$ cells of D_k which contain a fixed element of GD(v), say (r, t). This element occurs in row i and column

$$\begin{aligned} &(j_1, j_2) = (r, t) - (x^k, y^k)z^i , && 0 \leqq i \leqq d - 1 \\ &(j_1, j_2) = (r, t) , && i = d \\ &(j_1, j_2) = (r, t) - (x^k, y^k)w^{i-(d+1)} , && d + 1 \leqq i \leqq 2\lambda \end{aligned}$$

Then the corresponding entries in the cells of D_l will be

$$\begin{aligned} &(x^l, y^l)z^i + (r, t) - (x^k, y^k)z^i , && 0 \leqq i \leqq d - 1 \\ &(r, t) , && i = d \\ &(x^l, y^l)w^{i-(d+1)} + (r, t) - (x^k, y^k)w^{i-(d+1)} , && d + 1 \leqq i \leqq 2\lambda \end{aligned} \tag{2}$$

Now consider all the $2\lambda + 1$ cells of D_k which contain (r', t'). Then the corresponding $2\lambda + 1$ cells in D_l are

$$\begin{aligned} &(x^l, y^l)z^{i'} + (r', t') - (x^k, y^k)z^{i'} , && 0 \leqq i' \leqq d - 1 \\ &(r', t') , && i' = d \\ &(x^l, y^l)w^{i'-(d+1)} + (r', t') - (x^k, y^k)w^{i'-(d+1)} , && d + 1 \leqq i' \leqq 2\lambda . \end{aligned} \tag{3}$$

We shall show analogously to the previous case that the following system of

equations has exactly λ solutions, i.e., there are λ pairs (i, i') such that

$$(x^l, y^l)z^i + (r, t) - (x^k, y^k)z^i$$
$$= (x^l, y^l)z^{i'} + (r', t') - (x^k, y^k)z^{i'}, \qquad 0 \leqq i, \quad i' \leqq d - 1$$
$$(x^l, y^l)w^{i-(d+1)} + (r, t) - (x^k, y^k)w^{i-(d+1)}$$
$$= (x^l, y^l)w^{i'-(d+1)} + (r', t) - (x^k, y^k)w^{i'-(d+1)}, \qquad d + 1 \leqq i, \quad i' \leqq 2\lambda$$

or

$$z^i - z^{i'} = [(r', t') - (r, t)]/[(x^l, y^l) - (x^k, y^k)], \qquad 0 \leqq i, \quad i' \leqq d - 1$$
$$w^{i-(d+1)} - w^{i'-(d+1)}$$
$$= [(r', t') - (r, t)]/[(x^l, y^l) - (x^k, y^k)], \qquad d + 1 \leqq i, \quad i' \leqq 2\lambda.$$

The right-hand side of this system of equations in a nonzero element of the GD(v) and the left-hand side are differences of the elements belonging to the difference set M in GD(v). Hence the proof.

The following example elucidates the method of Theorem 4.6. Let $v = 15 = 3 \times 5$. For this case $x = 2$ and let $y = 2$.

$$M = \{z^0, z^1, z^2, z^3, \hat{0}, w^0, w^1\} = \{(1, 1), (2, 2), (1, 4), (2, 3), (0, 0), (1, 0), (2, 0)\}$$

and

	00	01	02	03	04	10	11	12	13	14	20	21	22	23	24
11	11	12	13	14	10	21	22	23	24	20	01	02	03	04	00
22	22	23	24	20	21	02	03	04	00	01	12	13	14	10	11
14	14	10	11	12	13	24	20	21	22	23	04	00	01	02	03
$D_0 =$ 23	23	24	20	21	22	03	04	00	01	02	13	14	10	11	12
00	00	01	02	03	04	10	11	12	13	14	20	21	22	23	24
10	10	11	12	13	14	20	21	22	23	24	00	01	02	03	04
20	20	21	22	23	24	00	01	02	03	04	10	11	12	13	14

	00	01	02	03	04	10	11	12	13	14	20	21	22	23	24
22	22	23	24	20	21	02	03	04	00	01	12	13	14	10	11
14	14	10	11	12	13	24	20	21	22	23	04	00	01	02	03
23	23	24	20	21	22	03	04	00	01	02	13	14	10	11	12
$D_1 =$ 11	11	12	13	14	10	21	22	23	24	20	01	02	03	04	00
00	00	01	02	03	04	10	11	12	13	14	20	21	22	23	24
20	20	21	22	23	24	00	01	02	03	04	10	11	12	13	14
10	10	11	12	13	14	20	21	22	23	24	00	01	02	03	04

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337 WARREN HALL
CORNELL UNIVERSITY
ITHACA, NEW YORK 14850