The Annals of Mathematical Statistics
1972, Vol. 43, No. 5, 1507-1516

BIVARIATE TESTS FOR LOCATION AND THEIR BAHADUR EFFICIENCIES

BY TIMOTHY J. KILLEEN AND THOMAS P. HETTMANSPERGER

The University of Connecticut and The Pennsylvania State University

We consider $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$, random sample from an absolutely continuous bivariate, population with symmetric density $f(x, y)$ and test H_0: $f(x, y)$ symmetric about $(0, 0)$ against H_1: all possible location alternatives.

Hotelling's T^2 statistic is often used for this test. We denote a form of this statistic by $T_n^{(2)}$ and make an exact Bahadur efficiency comparison of $T_n^{(2)}$ with respect to three of its competitors: a new bivariate Wilcoxon signed rank test $T_n^{(1)}$, Hodges' bivariate sign test $T_n^{(3)}$, and Blumen's bivariate sign test $T_n^{(4)}$.

When a bivariate normal alternative with parameter $\Delta = \mu'\Sigma^{-1}\mu$ obtains, it is shown that the exact Bahadur slopes of $T_n^{(1)}$, $T_n^{(2)}$, and $T_n^{(3)}$ are identical to the exact slopes of their univariate analogues with a univariate normal alternative with parameter $\Delta = \mu/\sigma$ obtains. In this case, the exact Bahadur efficiency of $T_n^{(1)}$ is uniformly better than either the exact Bahadur efficiency of $T_n^{(3)}$ or $T_n^{(4)}$ with respect to $T_n^{(2)}$.

1. Introduction. Bahadur (1967) developed the concept of the exact slope of a test statistic T_n. Suppose that $X_1, X_2, \cdots, X_n$ is a random sample from a distribution with distribution function $F_\theta(x)$, $\theta \in \Omega$, and we wish to test the hypothesis H_0: $\theta = \theta_0$ against H_1: $\theta \in \Omega - \{\theta_0\}$. Assume T_n rejects H_0 for $T_n > k_n$. Under appropriate conditions, discussed by Bahadur, it follows that

$$\lim_{n\to\infty} n^{-1} \log P_{\theta_0}(T_n > k_n) = h(k) < 0 \qquad \text{for all} \quad k_n \to k > 0 .$$

Further, if there exists a function $b(\theta_1) > 0$ for $\theta_1 \in \Omega - \{\theta_0\}$ such that

$$\lim_{n\to\infty} P_{\theta_1}(|T_n - b(\theta_1)| > \varepsilon) = 0 ,$$

then the exact slope of T_n evaluated at θ_1 is $C(\theta_1) = -2h(b(\theta_1))$.

We call $h(x)$ the large deviation of T_n and $b(\theta_1)$ the stochastic limit of T_n when θ_1 obtains.

If $T_n^{(1)}$ and $T_n^{(2)}$ are two statistics with exact slopes $C_1(\theta_1)$ and $C_2(\theta_1)$ respectively, then $e_{1,2}(\theta_1) = C_1(\theta_1)/C_2(\theta_1)$ is known as the exact Bahadur efficiency of $T_n^{(1)}$ with respect to $T_n^{(2)}$ evaluated at θ_1.

In this paper we evaluate the Bahadur efficiencies of three bivariate tests with respect to Hotelling's T^2 statistic, Anderson (1958), when the underlying distribution is bivariate normal.

2. Bivariate tests for location. Suppose that $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ is a random sample of size n from a two-dimensional absolutely continuous distribution with density $f(x, y)$. We further suppose that $f(x, y)$ is symmetric about

Received June 28, 1971; revised February 29, 1972.

some point (θ_1, θ_2), that is, $f(\theta_1 + x, \theta_2 + y) = f(\theta_1 - x, \theta_2 - y)$, for all x, y. Here we consider testing H_0: f symmetric about (0, 0) against all possible location alternatives H_1: f symmetric about $(\theta_1, \theta_2) \neq (0, 0)$.

The first test we wish to consider, $T_n^{(1)}$, is the maximum value that the standardized Wilcoxon signed rank test, (Wilcoxon (1945)), attains when calculated using the directed distances of the projections of the sample points on any line passing through the origin. To define the Wilcoxon statistic, let $Z_1, Z_2 \cdots, Z_n$ be random variables, and let R_i be the rank of $|Z_i|$. Further let $a_i = \pm 1$ depending upon whether Z_i is positive or negative. There is an ambiguity in the definition of the a_i if $|Z_i| = |Z_j|$ for some $i \neq j$. However, if the Z_i are independent identically distributed absolutely continuous random variables the probability of these occurrences is 0. Therefore, we will not be concerned with this possibility. Now define $W = (n^{\frac{1}{2}}(n-1))^{-1} \sum_{i=1}^n a_i R_i$ to be the standardized Wilcoxon signed rank statistic calculated on the Z_i.

We extend these concepts to the bivariate situation. Let $Z_i(t)$ be the directed distance to the origin of the projection of the point (X_i, Y_i) on the directed line which is a counterclockwise rotation of t radians of the x axis. Also let $R_i(t)$ be the rank of $|Z_i(t)|$, with $a_i(t) = \pm 1$ depending upon whether $Z_i(t)$ is positive or negative. We let

$$T_n^{(1)} = \max_{0 \leq t \leq 2\pi} (n^{\frac{1}{2}}(n-1))^{-1} \sum_{i=1}^n a_i(t) R_i(t) . \tag{2.1}$$

The asymptotic distribution of $T_n^{(1)}$ is discussed in Killeen (1971). The small sample distribution and hence whether or not the distribution of the statistic is nonparametric is unknown at this time.

If it is plausible to assume that $f(x, y)$ is a bivariate normal density, we would use Hotelling's T^2 statistic to test the hypothesis. The T^2 statistic is of the form $T^2 = n(\bar{X}, \bar{Y})S^{-1}(\bar{X}, \bar{Y})'$, where $\bar{X} = n^{-1} \sum_{i=1}^n X_i$ and $\bar{Y} = n^{-1} \sum_{i=1}^n X_i$, with S the sample convariance matrix. If $f(x, y)$ is the bivariate normal density, then

$$T_n^{(2)} = [2(n-1)]^{-1}(n-2)T^2 \tag{2.2}$$

has an F distribution with 2 and $n - 2$ degrees of freedom. Many competitors of this test have been proposed in the literature. The simplest of these is perhaps the bivariate sign test introduced by Hodges (1955). We will denote this test by $T_n^{(3)}$. The statistic $T_n^{(3)}$ is the maximum number of sample points in the plane lying in any halfplane formed by a line passing through the origin. Joffe and Klotz (1962) give expressions for the exact and asymptotic null distributions of $T_n^{(3)}$. They also calculate the Bahadur approximate efficiency of this test with respect to Hotelling's T^2 when the underlying population is bivariate normal.

Another test we wish to consider is Blumen's (1958) bivariate sign test. To construct this test, let Γ_i be the random angle measured counterclockwise from the nonnegative x axis to the line passing through (X_i, Y_i) and the origin, $0 < \Gamma_i < \pi$. Now let R_i be the rank of Γ_i, and let $a_{R_i} = \pm 1$ according to whether Y_i is positive or negative. Define:

$$T_n^{(4)} = 2[(\textstyle\sum_{j=1}^n a_j \cos(\pi j/n))^2 + (\sum_{j=1}^n a_j \sin(\pi j/n))^2]/n . \tag{2.3}$$

The test rejects H_0 for large values of $T_n^{(4)}$. Blumen shows that, under the null hypothesis, $T_n^{(4)}$ has an asymptotic chi-squared distribution with 2 degrees of freedom and claims that $T_n^{(4)}$ is more efficient that Hodges's $T_n^{(3)}$ test.

We find the exact slopes of the four previously introduced statistics. The large deviation of Hodges' bivariate sign test $T_n^{(3)}$ and our test $T_n^{(1)}$ are obtained by an *ad hoc* method which is particularly suited to this type of test. The large deviation of Hotelling's T^2 is found, with the assumption that the sample is bivariate normal, using the technique of Killeen, Hettmansperger, and Sievers (1972).

The exact slope of Blumen's bivariate sign test $T_n^{(4)}$ seems less accessible. Here we use the techniques of Klotz (1965) and Chernoff and Savage (1958).

3. The exact slopes of $T_n^{(3)}$ and $T_n^{(1)}$. The Hodges bivariate sign test $T_n^{(3)}$ and the new bivariate signed rank test $T_n^{(1)}$ are formulated in a similar fashion. Therefore, we are able to calculate the large deviations of each of these tests using the same technique. We use the following theorem.

THEOREM 3.1. *Suppose that for each* $n = 1, 2, \cdots, X_{1,n}, X_{2,n}, \cdots, X_{m_n,n}$ *are identically distributed random variables. Further, there exists an integer k such that for sufficiently large n,* $1 \leqq m_n \leqq n^k$. *Let* $M_n = \max_{1\leqq i \leqq m_n} X_{i,n}$, *and suppose that*

$$\lim_{n\to\infty} n^{-1} \log P(X_{1,n} \geqq \phi_n) = c \,, \tag{3.1}$$

then

$$\lim_{n\to\infty} n^{-1} \log P(M_n \geqq \phi_n) = c \,. \tag{3.2}$$

PROOF. Since $\{X_{1,n} \geqq \phi_n\} \subset \{M_n \geqq \phi_n\} = \bigcup_{i=1}^{m_n} \{X_{i,n} \geqq \phi_n\}$, it follows that

$$P(X_{1,n} \geqq \phi_n) \leqq P(M_n \geqq \phi_n) \leqq n^k P(X_{1,n} \geqq \phi_n) \,. \tag{3.3}$$

The last inequality results from the fact that the $X_{i,n}$ are identically distributed and $m_n \leqq n^k$ for sufficiently large n. The result now follows from (3.3).

We now state the stochastic limit of $T_n^{(3)}$ as given by Joffe and Klotz (1962). They show that if any particular density $f(x, y)$ obtains, then $T_n^{(3)}/n$ converges almost surely to $\sup_{0\leqq\alpha\leqq 2\pi} P(X_1 \cos(\alpha) + Y_1 \sin(\alpha) > 0) + \frac{1}{2}$. They further show that if $f(x, y)$ is a bivariate normal density with parameter

$$\Delta = (\theta_1, \theta_2)\Sigma^{-1}(\theta_1, \theta_2)' \,,$$

then

$$T_n^{(3)}/n \to \Phi(\Delta) = b_3(\Delta) \,, \quad \text{a.s.} \tag{3.4}$$

where Φ is the standard univariate normal distribution function.

The large deviation of $T_n^{(3)}$ is obtained by means of Theorem 3.1. Let L_t be the directed line passing through the origin such that the angle from the non-negative x axis to the positive half of L_t is $t\pi$, $\frac{1}{2} \leqq t \leqq \frac{5}{2}$; now let the number of positive projections of sample points on L_t be N_t and let the line passing through the origin which is perpendicular to L_t be denoted by L_t'. As t increases, each time that L_t' crosses a sample point, the value of N_t changes by one. Let $X_{1,n} = N_{\frac{1}{2}}$ and let $X_{k,n}$ be the kth value of N_t as t increases from $\frac{1}{2}$. Notice that $T_n^{(2)} = \max_{i=1,\cdots,2n} X_{i,n}$ and we will show that each $X_{k,n}$ is binomially distributed with parameters n and $\frac{1}{2}$ when $(\theta_1, \theta_2) = (0, 0)$ obtains. To accomplish this, we use the following fact:

If X is a random variable with a binomial $(n, \frac{1}{2})$ distribution and Z is a ± 1 random variable such that (i) $P(Z = 1 \mid X = k) = 1 - k/n$, and (ii) $P(Z = -1 \mid X = k) = k/n$, then $X' = X + Z$ is also binomial $(n, \frac{1}{2})$.

If we let $X = X_{1,n}$ and $Z = \pm 1$ depending upon whether the first sample point encountered when the x axis is rotated counterclockwise has a negative or positive y value, then conditions (i) and (ii) are satisfied, and $X_{2,n}$ is binomial $(n, \frac{1}{2})$. We get the same distribution for $X_{k,n}$ inductively, $k \leqq 2n$.

Theorem 3.1 now implies that the large deviation of the sequence $T_n^{(3)}$ is identical to that of a sequence of random variables which are binomial $(n, \frac{1}{2})$. It is well known (Klotz (1965)) that in this case

$$\lim_{n\to\infty} n^{-1} \log P(T_n^{(3)} \geqq na) = -(a)\log(2a) - (1-a)\log(2(1-a)) . \tag{3.5}$$

Combining this with (3.4) we get

$$C_3(\Delta)/2 = a\log(2a) + (1-a)\log(2(1-a)) , \tag{3.6}$$

where $a = \Phi(\Delta)$. We observe that the exact slope of Hodges' bivariate sign test is identical to that of the univariate sign test when the underlying distribution is normal with $\Delta = \mu/\sigma$.

An argument similar to the proof of the Glivenko-Cantelli Theorem yields, when $f(x, y)$ is the bivariate normal density,

$$T_n^{(1)}/n^{\frac{1}{2}} \to \Phi(2^{\frac{1}{2}}\Delta) - \tfrac{1}{2} = b_1(\Delta) , \quad \text{a.s.} \tag{3.7}$$

This result is identical to the stochastic limit of the univariate Wilcoxon statistic as shown by Klotz (1965).

We use Theorem 3.1 to evaluate the large deviation of $T_n^{(1)}$. We will define $m_n \leqq 2n^2$ random variables, $X_{1,n}, X_{2,n}, \cdots, X_{m_n,n}$, which are identically distributed and

$$T_n^{(1)} = \max_{1 \leqq k \leqq m_n} X_{k,n} . \tag{3.8}$$

We also show that

$$X_{k,n} = (n^{\frac{1}{2}}(n-1))^{-1} \textstyle\sum_{i=1}^{n} a_{ki} R_i , \tag{3.9}$$

where, for any fixed set of rankings, the a_{ki} are independent ± 1 random variables, $P(a_{ki} = \pm 1) = \frac{1}{2}$. This implies that the distribution of $X_{i,n}$ is identical to that of W, the standardized Wilcoxon signed rank test, under the hypothesis that the underlying univariate density is symmetric about 0. Theorem 3.1 now implies that the large deviation of $T_n^{(1)}$ is the same as the large deviation of W. To arrive at this result we must define $X_{1,n}, X_{2,n}, \cdots, X_{m_n,n}$, show that each of the $X_{k,n}$ have the desired distribution, and establish (3.8). Assume that $f(x, y)$

is symmetric about (0, 0), and consider the set of points G_n in the plane which consists of the combinations of pairs of sample points, $\pm(X_i, Y_i) \pm (X_j, Y_j)$ for all combinations of $+$'s and $-$'s, $1 \leqq i \leqq j \leqq n$, where addition and subtraction are coordinatewise. There are no more than $2n^2$ such points.

Define $X_{1,n}$ to be W calculated on the projections of the n sample points on the y axis. Now rotate the coordinate axes counterclockwise until the x axis crosses a point in G_n; we then calculate W on the projections of the sample points on the rotated y axis. Call this statistic $X_{2,n}$. Continue rotating the coordinate axis and construct $X_{2,n}, \cdots, X_{m_n,n}$ in the same way, where m_n is the number of points in G_n.

We observe that the $X_{k,n}$ attain all of the values of

$$V_n(t) = (n^{\frac{1}{2}}(n-1))^{-1} \sum_{i=1}^n a_i(t)R_i(t)\,, \qquad 0 \leqq t \leqq 2\pi\,. \tag{3.10}$$

Since $T_n^{(1)}$ is defined by (2.1), (3.8) follows.

Choose k, $0 \leqq k \leqq m_n$, $\beta_1, \beta_2, \cdots, \beta_n$ a fixed sequence of ± 1's and $N_1, N_2, \cdots, N_n$ some fixed rearrangement of the integers from 1 to n. Now consider the set A which is the collection of possible samples of size n which yield $a_i = \beta_i$ and $R_i = N_i$ in the calculation of $X_{k,n}$. For the a_i to be independent, $P(a_i = \pm 1) = \frac{1}{2}$, it is equivalent that any sequence of ± 1's is equally likely. Therefore, consider any other sequence $\beta_1', \beta_2', \cdots, \beta_n'$. We let $\beta_1' = \alpha_i\beta_i$, where $\alpha_i = \pm 1$, for $i = 1, 2, \cdots, n$. Notice that if $((x_1, y_1), (x_2, y_2)\cdots(x_n, y_n)) \in A$, then $((\alpha_1 x_1, \alpha_1 y_1), (\alpha_2 x_2, a_2 y_2), \cdots, (\alpha_n x_n, \alpha_n y_n)) \in A'$ where A' is the collection of possible samples of size n which yield $a_i = \beta_i'$ and $R_i = N_i$ in the calculation of $X_{k,n}$. The probability that a given sample of size n is a member of A is the same as the probability that the sample is a member of the set A'. This follows since $f(x, y)$ is symmetric about the origin. The original sample points are independent, and neither the ranks of the projections of the sample points on lines through the origin nor the points of G_n are altered when we reflect sample points through the origin. Therefore, all arrangements of ± 1's are equally likely and the $X_{k,n}$ have the desired distribution. Theorem 3.1 now implies that the large deviations of T_n and W are identical. Recall that $b_1(\Delta)$ is given by (3.7) when a normal alternative with parameter Δ obtains, which is identical to the univariate stochastic limit of $W/n^{\frac{1}{2}}$ when a normal alternative obtains with $\Delta = \mu/\sigma$. Therefore, the exact slope of $T_n^{(1)}$ when Δ obtains is identical to the exact slope of the Wilcoxon signed rank test when the sample is taken from a normal population with $\Delta = \mu/\sigma$.

4. Hotelling's T^2. Since $T_n^{(2)}$, defined by (2.2), is a monotonic function of T^2, it has the same exact slope. Therefore, we proceed to find the exact slope of $T_n^{(2)}$.

We observe that

$$T_n^{(2)}/n \to_p \Delta^2/2 = b_2(\Delta)\,. \tag{4.1}$$

Since $(\bar{X}, \bar{Y})$ and S converge in probability to (θ_1, θ_2) and Σ respectively.

To calculate the large deviation of $T_n^{(2)}$, we use the main result of Killeen, Hettmensperger and Sievers (1972). The statistic $T_n^{(2)}$ satisfies the necessary conditions and it follows that when $f(x, y)$ is a bivariate normal density with $(\theta_1, \theta_2) = (0, 0)$, then $n^{-1} \log f_n(nx) - n^{-1} \log P(T_n^{(2)} \geqq nx) = o(1)$, as $n \to \infty$, where $f_n(\cdot)$ is the null density of $T_n^{(2)}$. Straightforward manipulations now yield $\lim_{n\to\infty} n^{-1} \log P(T_n^{(2)} \geqq nx) = -(\frac{1}{2}) \log(1 + 2x) = h_2(x)$.

Now using these results we finally have

$$C_2(\Delta) = \log(1 + \Delta^2) \tag{4.2}$$

as the exact slope of $T_n^{(2)}$ evaluated at the alternative Δ. Observe that this is identical to the univariate results for the t test (Klotz (1965)) with $\Delta = \mu/\sigma$.

5. Blumen's bivariate sign test. Using (2.3) we see that $T_n^{(4)}/n$ is twice the square of the distance of the centroid of the set of points $\{(a_j \cos(\pi j/n), a_j \sin(\pi j/n)$: $j = 1, 2, \cdots, n\}$ from the origin. Let C_n be the centroid of these points.

LEMMA 5.1. *Suppose that B_n and D_n be any two sets in the plane such that there exists an integer l, $0 \leqq l \leqq 2n$, with $D_n = \tau(B_n)$, where τ is a counterclockwise rotation about the origin of $l\pi/n$ radians. If $(\theta_1, \theta_2) = (0, 0)$ obtains then $P(C_n \in B_n) = P(C_n \in D_n)$.*

PROOF. The statistic has discrete distribution in the plane. Suppose that $c = (x, y)$ is a point of positive probability. This implies that $P(C_n = c) = k/2^n$, where k is the number of distinct arrangements of $a_j = \pm 1$, $j = 1, 2, \cdots, n$, such that $C_n = c$, since all possible arrangements are equally likely. Notice that there are exactly k arrangements of $a_j = \pm 1$ such that $C_n = \tau(c)$. This may be seen if we let $k_j = (j + l - 1) \bmod (n) + 1$, then the sequence $a_1, a_2, \cdots, a_n$ yields $C_n = c$ if and only if the sequence $a_{k_1}, a_{k_2}, \cdots, a_{k_n}$ yields $C_n = \tau(c)$. We have constructed a 1 to 1 correspondence between the sequences which give $C_n = c$ and the sequences which give $C_n = \tau(c)$ and the proof of Lemma 5.1 is complete.

We use Lemma 5.1 to derive the large deviation of $T_n^{(4)}$. Choose a sequence $\rho_n \to \rho > 0$ and define sets A_{nj}, for $n = 1, 2, \cdots, j = 0, 1, \cdots, 2n - 1$ such that $A_{nj} = \{(x, y) \mid r \geqq \rho_n, \pi j/n \leqq \alpha \leqq \pi(j + 1)/n\}$, where $x = r\cos(\alpha)$, $y = r\sin(\alpha)$. Observe that

$$P(C_n \in A_{nj}) = P(C_n \in A_{nk}), \quad \text{for} \quad 0 \leqq k \leqq 2n - 1, \tag{5.1}$$

since A_{nj} and A_{nk} satisfy Lemma 5.1. Now choose ε, $0 < \varepsilon < \rho$, then for large n,

$$P(C_n \in A_{n0}) \leqq P(\textstyle\sum_{j=1}^n a_j \cos(\pi j/n) \geqq (\rho_n - \varepsilon)n). \tag{5.2}$$

We use (5.1) and (5.2) to simplify the large deviation of $T_n^{(4)}$, and get

$$\begin{aligned}
&\lim_{n\to\infty} n^{-1} \log P(T_n^{(4)}/2 \geqq \rho_n^2 n) \\
&\quad = \lim_{n\to\infty} n^{-1} \log(2nP(C_n \in A_{n0})) \\
&\quad = \lim_{n\to\infty} n^{-1} \log P(C_n \in A_{n0}) \\
&\quad \leqq \lim_{n\to\infty} n^{-1} \log P(\textstyle\sum_{j=1}^n a_j \cos(\pi j/n) \geqq (\rho_n - \varepsilon)n).
\end{aligned}$$

This implies that if $\lim_{n\to\infty} n^{-1} \log P(\sum_{j=1}^{n} a_j \cos(\pi j/n) \geqq \rho_n n) = h(\rho)$, for all $\rho_n \to \rho$ and for all $\rho > 0$, then

$$\lim_{n\to\infty} n^{-1} \log P(T_n^{(4)}/2 \geqq \rho_n^2 n) \leqq h(\rho - \varepsilon)\,, \tag{5.3}$$

for $\varepsilon > 0$. We also observe that $P(T_n^{(4)}/2 \geqq \rho_n^2 n) \geqq P(\sum_{j=1}^{n} a_j \cos(\pi j/n) \geqq \rho_n n)$ and $\lim n^{-1} \log P(T_n^{(4)}/2 \geqq \rho_n^2 n) \geqq h(\rho)$. If $h(\rho)$ is continuous, then combining (5.3) and (5.4) yields $\lim n^{-1} \log P(T_n^{(4)}/2 \geqq \rho_n^2 n) = h(\rho)$. It remains to evaluate $h(\rho)$, and to show that it is continuous.

The random variables $Z_{nj} = a_j \cos(\pi j/n)$ are independent but not identically distributed. Therefore, the quantity $h(\rho)$ may be evaluated by an application of Theorem 1 of Feller (1943). The argument is similar to that of Klotz (1965) or Stone (1967). Let $E_{nj} = \cos(\pi j/n)$ and introduce the random variables $Z_{nj}(v)$ defined by

$$\begin{aligned} Z_{nj}(v) &= E_{nj} \qquad \text{with probability} \quad \exp(vE_{nj})[2\cosh(vE_{nj})]^{-1} \\ &= -E_{nj} \qquad \text{with probability} \quad \exp(-vE_{nj})[2\cosh(vE_{nj})]^{-1}\,. \end{aligned}$$

Let $S_n(v) = \sum_{j=1}^{n} Z_{nj}(v)$. Then

$$\begin{aligned} \rho_n &= E(n^{-1}S_n(v)) = n^{-1} \textstyle\sum_{j=1}^{n} \cos(\pi j/n) \tanh(v \cos(\pi j/n)) \\ &\to \frac{1}{\pi} \int_0^\pi \cos(y) \tanh(v\cos(y))\, dy \qquad \text{as} \quad n \to \infty\,. \end{aligned} \tag{5.4}$$

Now let $\operatorname{Var}(S_n(v)) = \sigma^2$ and $Y = (S_n(v) - ES_n(v))/\sigma$. An inductive argument on n gives us

$$P(\textstyle\sum_{i=1}^{n} Z_{nj} > \rho_n n) = \prod_{i=1}^{n} \cosh(vE_{ni}) \int_0^\infty \exp(-v(n\rho_n + y\sigma))\, dF_Y(y)\,.$$

Using an argument identical to Klotz (1965) it may now be shown that $h(\rho) = -\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} (\log\cosh(vE_{ni}) + v\rho_n)$, which yields

$$h(\rho) = v\rho - \frac{1}{\pi} \int_0^\pi \log\cosh(v\cos(y))\, dy\,, \tag{5.5}$$

and from (5.4) we see that

$$\rho = \frac{1}{\pi} \int_0^\pi \cos(y) \tanh(v\cos(y))\, dy\,. \tag{5.6}$$

Therefore the large deviation of $T_n^{(4)}/2$ is given by (5.5) subject to condition (5.6). The resulting integrals are nonelementary and were evaluated by numerical methods on the I.B.M. 360 model 67.

We now obtain the stochastic limit of $T_n^{(4)}/2$. Chernoff and Savage (1958) show the asymptotic normality of a certain class of test statistics, and we employ their method to evaluate this stochastic limit. Recall that $T_n^{(4)}$ is defined by (2.3), so that if the bivariate normal alternative with parameter Δ obtains and

$$\textstyle\sum_{j=1}^{n} a_j \cos(\pi j/n)/n \to_P b_c(\Delta) \tag{5.7}$$

and

(5.8) $$\sum_{j=1}^{n} a_j \sin(\pi j/n)/n \to_P b_s(\Delta)\,,$$

then

(5.9) $$T_n^{(4)}/(2n) \to_P b_c^2(\Delta) + b_s^2(\Delta)\,.$$

We evaluate (5.7); the argument for (5.8) is very similar. Since $n^{-1}\sum_{j=1}^{n}\cos(\pi j/n) \to 0$ as $n \to \infty$, (5.8) holds if and only if $Z_n/n \to_P b_c(\Delta)$ where $Z_n = \sum_{j=1}^{n}(a_j + 1)\cos(\pi j/n)$. Recall that Γ_i is the angle that the line joining (X_i, Y_i) to the origin makes with the nonnegative x axis, $0 < \Gamma_i < \pi$. Let $\delta_i = \Gamma_i$ if $Y_i \geqq 0$ and $\Gamma_i - \pi$ if $Y_i < 0$. Now let $F^*(\delta)$ be the marginal distribution function of δ_i with support on $(-\pi, \pi)$. Let λ_n be the number of $\delta_i > 0$, $i = 1, 2, \cdots, n$, divided by n. The strong law of large numbers insures that $\lambda_n \to P(\delta_1 > 0) = 1 - F^*(0) = p$, a.s. The statistic Z_n is a one-sample Chernoff-Savage statistic. A discussion of the one-sample Chernoff-Savage statistic appears in Puri and Sen (1971). In order to apply these results we define the following quantities: $F(\delta) = (F^*(\delta) - F^*(0))/p$, for $0 < \delta < \pi$, $G(\delta) = F^*(\delta - \pi)/F^*(0)$, for $0 < \delta < \pi$, $J(\bar{H}) = 2\cos(\bar{H})$, and

(5.10) $$\begin{aligned}\bar{H}(\delta) &= \lambda_n F(\delta) + (1 - \lambda_n)G(\delta)\\ &= \lambda_n(F^*(0))/p + (1 - \lambda_n)F^*(\delta - \pi)/F^*(0)\,.\end{aligned}$$

We may replace λ_n by p. Then (5.10) becomes $\bar{H}(\delta) = F^*(\delta) - F^*(0) + F^*(\delta - \pi)$. Further, if

(5.11) $$\begin{aligned}\mu(\Delta) &= p\int_{-\infty}^{\infty} J(\bar{H}(\delta))\,dF(\delta)\\ &= \int_0^{\pi} 2\cos(\pi(F^*(\delta) - F^*(0) + F^*(\delta - \pi)))\,dF^*(\delta)\,,\end{aligned}$$

then $(Z_n - n\mu(\Delta))/n^{\frac{1}{2}}$ is asymptotically normal with 0 mean and finite variance. Therefore, from (5.11) it follows that $Z_n/n \to_P \mu(\Delta) = b_c(\Delta)$. Similarly, we see that

(5.12) $$b_s(\Delta) = \int_0^{\pi} 2\sin(\pi(F^*(\delta) - F^*(0) + F^*(\delta - \pi)))\,dF^*(\delta)\,.$$

Now we are able to calculate the large deviation of $T_n^{(4)}$ by combining (5.9) and (5.11)—(5.12). The integrals involved in this calculation are nonelementary and were evaluated by numerical methods on the I.B.M. 360 model 67.

Letting $\rho = b_c^2(\Delta) + b_s^2(\Delta)$, (5.5) gives the expression for the exact slope of $T_n^{(4)}$ when the alternative Δ obtains.

6. Conclusions. We have considered four bivariate tests of location: a new bivariate signed rank test $T_n^{(1)}$, Hotelling's statistics $T_n^{(2)}$, Hodges' bivariate sign test $T_n^{(3)}$, and Blumen's bivariate sign test $T_n^{(4)}$.

It is interesting to note that when the underlying distribution is bivariate normal if the test statistic $T_n^{(i)}$ has a univariate analogue, its exact slope is identical to the exact slope of its univariate analogue when the univariate alternative is normal with parameters $\Delta = \mu/\sigma$. One-half times the exact slope

TABLE I
One-half times exact slopes for normal alternatives

Δ	$\frac{1}{2}C_4(\Delta)$	Δ	$\frac{1}{2}C_4(\Delta)$
.125	.00611	1.750	.5462
.250	.02412	1.875	.5744
.375	.05300	2.000	.5961
.500	.09116	2.125	.6116
.625	.1365	2.250	.6299
.750	.1870	2.375	.6431
.875	.2401	2.500	.6528
1.000	.2937	2.625	.6622
1.125	.3457	2.750	.6688
1.250	.3944	2.875	.6810
1.375	.4398	3.000	.6889
1.500	.4706	∞	.6931
1.625	.5166		

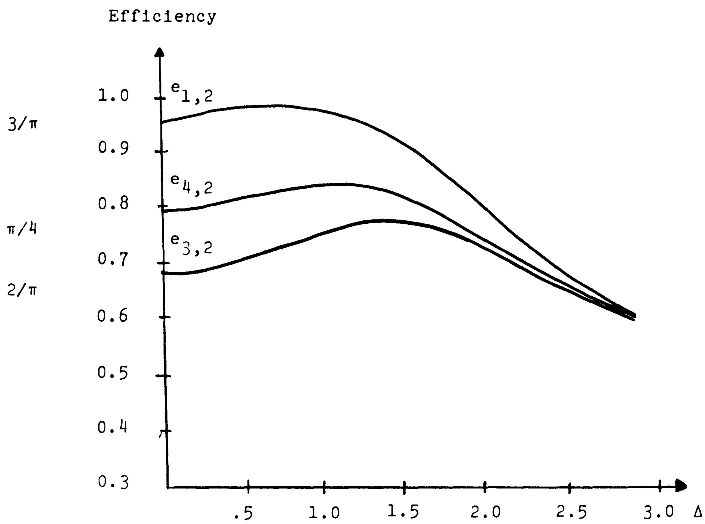

FIG. 1. Efficiencies for normal alternatives.

for $T_n^{(1)}$, $T_n^{(2)}$, and $T_n^{(3)}$ are found in Table I in Klotz (1965) for normal alternatives. Table I gives one-half times the exact slope for $T_n^{(4)}$.

The exact Bahadur efficiencies of $T_n^{(1)}$, $T_n^{(3)}$, and $T_n^{(4)}$ with respect to $T_n^{(2)}$ for bivariate normal alternatives are compared in Figure 1. The bivariate signed rank test $T_n^{(1)}$ is superior to $T_n^{(3)}$ and $T_n^{(4)}$ when we use exact Bahadur efficiency with these alternatives. The limit of the exact Bahadur efficiencies as Δ tends to 0, of $T_n^{(1)}$, $T_n^{(3)}$, and $T_n^{(4)}$ with respect to $T_n^{(2)}$ are $3/\pi$, $2/\pi$, and $\pi/4$. Notice that the bivariate signed rank test easily beats the other two.

7. Acknowledgment. The authors would like to express their appreciation for the many valuable suggestions of the referee.

REFERENCES

[1] ANDERSON, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley New York.

[2] BAHADUR, R. R. (1967). Rates of convergence of estimates and statistics. *Ann. Math. Statist.* **38** 303–325.

[3] BLUMEN, I. (1958). A new bivariate sign test. *J. Amer. Statist. Assoc.* **53** 448–456.

[4] CHERNOFF, H. and SAVAGE, I. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *Ann. Math. Statist.* **29** 972–994.

[5] FELLER, W. (1943). Generalization of a probability limit theorem of Cramér. *Trans. Amer. Math. Soc.* **54** 361–372.

[6] HODGES, J. L. JR. (1955). A bivariate sign test. *Ann. Math. Statist.* **26** 523–527.

[7] JOFFE, A. and KLOTZ, J. (1962). Null distribution and Bahadur efficiency of the Hodges bivariate sign test. *Ann. Math. Statist.* **33** 803–807.

[8] KILLEEN, T., HETTMANSPERGER, T., and SIEVERS, G. (1972). An elementary theorem on the probability of large deviations. *Ann. Math. Statist.* To appear.

[9] KILLEEN, T. (1971). Bivariate tests for location and their Bahadur efficiencies. The Pennsylvania State University. Ph. D. dissertation.

[10] KLOTZ, J. H. (1964). Small sample power of the bivariate sign tests of Blumen and Hodges. *Ann. Math. Statist.* **35** 1576–1582.

[11] KLOTZ, J. H. (1965). Alternative efficiencies for signed rank tests. *Ann. Math. Statist.* **36** 1759–1766.

[12] PURI, M. L. and SEN, P. K. (1971). *Nonparametric Methods in Multivariate Analysis.* Wiley, New York.

[13] STONE, M. (1967). Extreme tail probabilities of the two-sample Wilcoxon statistic. *Biometrika* **54** 629–641.

[14] WILCOXON, F. (1945). Individual comparisons by ranking methods. *Biometrics Bull.* **1** 80–83.

DEPARTMENT OF STATISTICS
UNIVERSITY OF CONNECTICUT
STORRS, CONNECTICUT 06268