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MAXIMUM LIKELIHOOD ESTIMATION OF LIFE-DISTRIBUTIONS FROM RENEWAL TESTING

BY L. H. CROW AND I. N. SHIMI

Florida State University

In 1965 Marshall and Proschan (1965) (see also Grenander (1956)), considered the maximum likelihood estimation for life-distributions with monotone failure rate over the support of the distribution functions. They considered data arising from a testing plan which does not allow censoring, time-truncation or replacements. In the present paper we consider the maximum likelihood estimation of life-distributions with monotone failure rates over the interval $[0, T)$, where T is a fixed positive real number, and no other assumptions about the distribution or its failure rate are given outside that interval. The following renewal type testing plan is used, which allows for time-truncation and replacement. At time zero, the beginning of the testing, n new items from the population to be tested are put on test. When an item fails it is instantaneously replaced with another new item from the same population and at time T all testing is stopped. The maximum likelihood estimates of the distribution function and its failure rate over $[0, T)$ are given and shown to be uniformly strongly consistent as n tends to infinity.

1. Introduction. In reliability studies many different testing plans are used. Most of these testing procedures are designed to allow the experimenter to obtain what he decides is a sufficient amount of information and still limit the number of items tested, the number of items that fail, the total testing time, etc. For example, an experimenter would rarely use a testing plan that did not limit the total testing time when the items being tested can be assumed very reliable, since the testing time would usually be very long. The total testing time must also be limited if project deadlines must be met, or if equipment or personnel used in the testing can only be spared for some specified length of time. This limitation on the testing time need not, however, be detrimental to the goals of the experimenter. For example, if the experimenter can assume that the general form of the life-time distribution belongs to some parametric class defined on the positive real axis, then limiting the testing time to, say T, ($T < \infty$), he can still estimate the unknown parameters of the distribution. This will give him an estimate of the distribution on $[0, \infty)$. Examples of this case are numerous. (See Epstein (1959), Gnedenko *et al.*, ((1969) Chapter 3) and Crow and Shimi (1970), (1971).)

Another situation where limiting the testing time is both necessary and practical is the subject of the present paper. Suppose the experimenter cannot conclude that the life-time distribution has a particular form on $[0, \infty)$ but only on $[0, T)$ ($T < \infty$) and suppose that this interval includes the mission time (the

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period of time during which the items being tested were intended to perform under actual operating conditions) of the items tested. In this case the experimenter must limit his inferences to the interval $[0, T)$ and, for all practical purposes, he need not infer anything about the distribution outside this interval.

The concept of "failure rate" is a very important practical concept in reliability and has motivated several very useful classes of distributions, e.g., Increasing Failure Rate (IFR) class, Decreasing Failure Rate (DFR) class, U-Shaped Failure Rate class. The failure rate $r(\cdot)$ of a distribution function F having derivative f is defined by

$$r(x) = f(x)/[1 - F(x)] \quad \text{for} \quad F(x) < 1$$

and

$$r(x) = \infty \quad \text{for} \quad F(x) = 1 .$$

The estimation problem that we shall be concerned with in this paper can be summarized in the following way. The life-time distribution of the items to be tested is assumed to have an increasing failure rate over the interval $[0, T]$, i.e., IFR on $[0, T]$. No other assumptions about the distribution or its failure rate are given outside that interval. The assumption of increasing failure rate can be changed to decreasing failure rate and the same results will follow with the obvious modifications. Data arise from the following testing plan.

Testing Plan A. At time zero, the beginning of the testing, n new items from the population are put on test. When an item fails it is instantaneously replaced with another new item drawn from the same population and at time T the testing is stopped.

The maximum likelihood estimation for parametric classes from this testing plan has been investigated by Epstein ((1959), page 3.17), Gnedenko *et al.*, ((1969), page 169) and Crow and Shimi (1970, 1971).

The notion of IFR on $[0, T]$ is made more precise by the following definition.

DEFINITION. Let T be a fixed positive real number. A cdf F, $F(0) = 0$, is said to be IFR (*Increasing Failure Rate*) on $[0, T]$ iff it satisfies one of the following conditions: (i) $-\log[1 - F(x)]$ is convex on the intersection of the support of F with $[0, T]$, $[\alpha_F, \beta_F]$, $0 \leqq \alpha_F \leqq \beta_F \leqq T$ and $F(\beta_F) = 1$ if $\beta_F < T$; or (ii) the part of the support of F in $[0, T]$ is empty.

Let $\mathscr{F} = \{F : F \text{ is IFR on } [0, T]\}$.

The class $\mathscr{F}$ includes the usual class of IFR distributions. It is easy to show that there exists no sigma-finite measure relative to which all the distributions in $\mathscr{F}$ are absolutely continuous.

Since we are dealing with a nonparametric family of distributions for which there exists no sigma-finite measure relative to which all the measures induced by $\mathscr{F}$ are absolutely continuous, the usual concept of maximum likelihood estimation cannot be applied. The general definition of MLE due to Kiefer and Wolfowitz (1956) is used in this paper to determine the MLE of the life-time distribution F over $[0, T)$, where $F \in \mathscr{F}$ and data arise from Testing Plan A. It is also shown that this MLE is strongly consistent as n, the number of original items, tends to infinity.

We need to present the following preliminary notation and a couple of obvious theorems.

Plan A can be considered as n independent experiments each beginning at time zero and ending at time T. Each experiment has a random number, K_r, $r = 1, 2, \cdots, n$, of items that are put on test. Let X_{jr} be the time-to-failure of the jth item put on test in the rth experiment. Then K_r is the first integer such that

$$\sum_{j=1}^{K_r} X_{jr} \geqq T .$$

Let

$$Y_{ir} = X_{ir}, \qquad r = 1, \cdots, n,\ i = 1, 2, \cdots, K_r - 1$$

and

$$Y_{K_r r} = T - \sum_{i=1}^{K_r - 1} X_{ir}, \qquad r = 1, 2, \cdots, n .$$

Observe that Y_{jr}'s are "times-on-test" for the items tested. Let $d(n)$ denote the total number of distinct failures in $[0, T)$ in the combined n experiments. Note that

$$0 \leqq d(n) \leqq \sum_{r=1}^{n} (K_r - 1) .$$

Also let $0 = Z_0 < Z_1 < \cdots < Z_{d(n)}$ be the ordered, distinct, failure times X_{jr}, $j = 1, \cdots, K_r - 1$, $r = 1, \cdots, n$. Finally, let $p(n)$ be the number of $Y_{K_r r}$'s, $r = 1, \cdots, n$, strictly greater than $Z_{d(n)}$.

The following theorem is similar to a theorem concerning IFR distribution given by Marshall and Proschan (1965), and we shall omit its proof because of this similarity.

THEOREM. *Suppose* $F \in \mathscr{F}$ *and* $0 < Z < \beta_F$. *Then* F *is absolutely continuous on* $[0, Z]$. *Note that* F *may take a jump at* β_F *if* $\beta_F < T$. *Also, using the definition of failure rate, it follows that*

(i) $F \in \mathscr{F}$ iff $r(\cdot)$ *is non-decreasing on* $[0, \beta_F)$, $0 \leqq \alpha_F \leqq \beta_F \leqq T$, *and* $F(\beta_F) = 1$ *if* $\beta_F \leqq T$.

(ii) *The part of the support of* F *in* $[0, T]$ *is empty* iff $r(x) = 0$ *on* $[0, T]$.

(iii) *If* $F \in \mathscr{F}$, *then for* $x \in [0, \beta_F)$

$$F(x) = 1 - \exp(-R(x)), \qquad \textit{and}$$
$$f(x) = r(x)\exp(-R(x)), \qquad \textit{where}$$
$$R(x) = \int_0^x r(y)\, dy .$$

2. Maximum likelihood estimate. In this section the MLE of that part of a life-time distribution $F \in \mathscr{F}$ over the interval $[0, T)$ will be presented when data arise from Testing Plan A. The following general definition of a maximum likelihood estimate is due to Kiefer and Wolfowitz (1956) and is needed to determine the MLE of $F \in \mathscr{F}$ for the two reasons mentioned earlier.

DEFINITION 2.1. Let Ω be a sample space, $\mathscr{B}$ a σ-field on Ω, $\mathscr{P}$ a family of probability measures on $\mathscr{B}$ and Θ a set indexing the elements of $\mathscr{P}$ by $P(\cdot \mid \theta)$, $\theta \in \Theta$. Let $\mathbf{X}$ be a random vector defined on Ω with distribution function determined by $P(\cdot \mid \theta_0)$, $\theta_0 \in \Theta$. If $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$ denotes a random sample from $P(\cdot \mid \theta_0)$ then the MLE of θ_0 is $\hat{\theta}$ if $\hat{\theta} \in \Theta$ and

$$\sup_{\theta \in \Theta} \frac{\prod_{r=1}^{n} f(\mathbf{X} \mid \theta, \hat{\theta})}{\prod_{r=1}^{n} [1 - f(\mathbf{X} \mid \theta, \hat{\theta})]} = 1$$

where

$$f(\cdot \mid \theta_1, \theta_2)$$

denote the Radon–Nikodym derivative of $P(\cdot \mid \theta_1)$ with respect to $P(\cdot \mid \theta_1) + P(\cdot \mid \theta_2)$.

The Kiefer and Wolfowitz concept of MLE will now be considered within the framework of Testing Plan A and for life-time distributions $F \in \mathscr{F}$. Let $\Omega = \{\phi$, and all finite sequences of nonnegative numbers whose sum is less than $T\}$, where ϕ is the empty set. Also, let $\mathscr{X}_i = \{\mathbf{x} \in \Omega$ which have exactly i elements$\}$, $i = 0, 1, \cdots$. Then $\Omega = \bigcup_{i=0}^{\infty} \mathscr{X}_i$. We define a set $\mathscr{A}$ to be measurable in Ω if and only if $\mathscr{A} = \bigcup_{i=0}^{\infty} \mathscr{A}_i$ and $\mathscr{A}_i$ is Borel measurable in $\mathscr{X}_i$. Let $\mathscr{B}$ be the σ-field of measurable sets in Ω.

For each $F \in \mathscr{F}$ we will define a probability measure $P(\cdot \mid F)$ on $\mathscr{B}$ and will denote the collection of all such measures by $\mathscr{P}$. These probability measures will be defined first on the Borel measurable sets of each $\mathscr{X}_i$. Some preliminary notation is needed. Denote by $\gamma(\cdot \mid i, F)$ the product measure on R^i (Euclidean ith space) induced by F, where $\gamma(\cdot \mid 0, F)$ is defined to be one. Also, let $F(x^-) = \lim_{\varepsilon \to 0} F(x - \varepsilon)$, $\varepsilon > 0$, and define products of the form $\prod_{j=1}^{0}$ and sums of the form $\sum_{j=1}^{0}$ to be 1 and 0, respectively. For each Borel measurable set $\mathscr{A}_i \subset \mathscr{X}_i$ and $F \in \mathscr{F}$ define the measure $P(\cdot \mid F)$ to be

$$P(\mathscr{A}_i \mid F) = \int_{\mathscr{A}_i} \{1 - F([T - \textstyle\sum_{j=1}^{i} x_j]^-)\} \, d\gamma(\mathbf{x} \mid i, F) .$$

For any $\mathscr{A} \in \mathscr{B}$ we define $P(\mathscr{A} \mid F)$ to be

$$P(\mathscr{A} \mid F) = \textstyle\sum_{i=0}^{\infty} P(\mathscr{A}_i \mid F)$$

where $\mathscr{A}_i = \mathscr{A} \cap \mathscr{X}_i$.

This definition is motivated by the following observation. Let H be a distribution function such that $H(0^-) = 0$, $H(0) < 1$. Let $X_1, X_2, \cdots,$ be a sequence of independent random variables with distribution function H and let K be the stopping variable defined to be the first integer such that $\sum_{j=1}^{K} X_j \geqq T$. It follows that, with probability one, $K < \infty$ and hence

$$1 = \sum_{k=1}^{\infty} \int \cdots \int_{\mathscr{C}_k} \prod_{j=1}^{k} dH(x_j) \tag{2.1}$$

where

$$\mathscr{C}_k = \{(x_1, x_2, \cdots, x_k) : \textstyle\sum_{j=1}^{k-1} x_j < T, \sum_{j=1}^{k} x_j \geqq T\} .$$

Since X_k is, in general, unobservable when testing Plan A is used, we integrate out x_k in (2.1) and obtain

$$1 = \sum_{k=1}^{\infty} \int \cdots \int_{\mathscr{X}_{k-1}} \{1 - H([T - \sum_{i=1}^{k-1} x_i]^-)\} \prod_{j=1}^{k-1} dH(x_j) . \tag{2.2}$$

The integrand in the right-hand side of (2.2) yields the likelihood function if H is parametric for Testing Plan A. The integrand motivates the measures in $\mathscr{P}$ for the nonparametric situation.

Note that for each $F \in \mathscr{F}$

$$P(\Omega \mid F) = P(\bigcup_{i=0}^{\infty} \mathscr{X}_i \mid F) = \sum_{i=0}^{\infty} P(\mathscr{X}_i \mid F)$$
$$= \sum_{i=0}^{\infty} \mathrm{Prob}(K = i + 1 \mid F) = 1 .$$

Thus, for each $F \in \mathscr{F}$, $P(\cdot \mid F)$ is a probability measure on $\mathscr{B}$.

The Kiefer and Wolfowitz concept of maximum likelihood estimate together with our definition of the measures $P(\cdot \mid F) \in \mathscr{P}$, $F \in \mathscr{F}$, yields the MLE $\tilde{F}_n$ of F on $[0, T)$ described in the next theorem.

Let $I(\cdot \mid \mathscr{S})$ be the indicator function of $\mathscr{S}$. Also, let $n_r(y)$ denote

$$\sum_{i=1}^{k_r} I(Y_{ir} \mid [y, \infty)) .$$

THEOREM 2.1. *The* MLE $\tilde{F}_n$ *of* F *has failure rate* $\tilde{r}_n$ *where* $\tilde{r}_n$ *is constant over* $[Z_q, Z_{q+1})$, $q = 0, \cdots, d(n)$, *where* $Z_{d(n)+1} = T$, *and*

$$\tilde{r}_n(Z_q) = \min_{d(n)+1 \geqq v \geqq q+1} \max_{0 \leqq u \leqq q} \frac{\sum_{r=1}^{n} \sum_{j=1}^{K_r - 1} I(X_{jr} \mid [Z_u, Z_v))}{\sum_{r=1}^{n} \int_{Z_u}^{Z_v} n_r(y)\, dy} , \tag{2.3}$$

$$\tilde{r}_n(Z_{d(n)}) = \infty \text{ iff } \max\{Y_{K_r r}, r = 1, \cdots, n\} \leqq Z_{d(n)} .$$

PROOF. The proof of this theorem follows in a straightforward manner from the Kiefer–Wolfowitz definition of MLE using the probability measures we introduced above and Brunk's (1958) results.

REMARK 2.1. We will now give a useful method for determining $\tilde{r}_n$. Let T_{qn} be the time on test over $[Z_q, Z_{q+1})$ (i.e., $T_{qn} = \sum_{r=1}^{n} \int_{Z_q}^{Z_{q+1}} n_r(y)\, dy$), $q = 0, \cdots, d(n)$. If $(T_{0n})^{-1} \leqq (T_{1n})^{-1} \leqq \cdots \leqq (T_{d(n)n})^{-1}$ then $\tilde{r}_n(Z_q) = (T_{qn})^{-1}$, $q = 0, \cdots, d(n)$. If for some i, $(T_{in})^{-1} > (T_{(i+1)n})^{-1}$ then replace $(T_{in})^{-1}$ and $(T_{(i+1)n})^{-1}$ by $2(T_{in} + T_{(i+1)n})^{-1}$.

If a reversal still exists, replace by appropriate averages. That is, if $2(T_{in} + T_{(i+1)n})^{-1} > (T_{(i+2)n})^{-1}$, then replace $(T_{in})^{-1}$, $(T_{(i+1)n})^{-1}$ and $(T_{(i+2)n})^{-1}$ by $3(T_{in} + T_{(i+1)n} + T_{(i+2)n})^{-1}$.

Continue averaging whenever there is a reversal. This will yield the monotone increasing sequence $\tilde{r}(Z_0) \leqq \tilde{r}_n(Z_1) \leqq \cdots \leqq \tilde{r}_n(Z_{d(n)})$ given by (2.3).

3. Strong consistency of $\tilde{F}_n$. The main result of this section is that the MLE of F on $[0, T)$ converges uniformly a.s. to F as the number of items put on test at time 0 increases. To accomplish this we will prove a convergence theorem for $\tilde{r}_n(x)$, $x \in [0, T)$, $\tilde{r}_n(x)$ defined in the last section. This result will allow us to easily prove the main result plus several corollaries. Furthermore, since the failure rate of a life distribution is an important practical concept, the convergence theorem for $\tilde{r}_n(x)$ is also a significant practical result.

We will need several theorems before we can prove the convergence theorem for $\tilde{r}_n(x)$. We first rewrite $\tilde{r}_n(x)$, given in the last section, in a form we need to show consistency.

Let $R(u, v)$ denote $\sum_{r=1}^{n} \sum_{j=1}^{K_r-1} I(X_{jr} \mid [u, v))$ and $S(u, v)$ denote $\sum_{r=1}^{n} \int_u^v n_r(y)\, dy$. Then

$$\tilde{r}_n(x) = \inf_{x \leqq v < T} \sup_{u < Z_n(x)} \frac{R(u, v)}{S(u, v)} \tag{3.1}$$

for $x \in [0, T)$ and

$$Z_n(x) = \max_{0 \leqq i \leqq d(n)} \{Z_i \mid Z_i \leqq x\}\,.$$

To show consistency of $\tilde{r}_n$ we need the next two theorems. Let

$$M_n(u, v) = \frac{R(u, v)}{S(u, v)} \qquad 0 < u < v < T\,,$$

and let I_F be the intersection of the support of F with $[0, T]$.

THEOREM 3.1. *Let $0 \leqq u_0 \leqq v_0 \leqq T$ be fixed where, $0 \leqq u_0 < T$ if $I_F = \phi$, $0 \leqq u_0 < v_0 < \beta_F$ if $I_F = [\alpha_F, \beta_F]$. Then, as $n \to \infty$*

(i) *$M_n(u_0, v)$ converges uniformly*, a.s., *in $v_0 \leqq v \leqq T$,*

(ii) *$M_n(u, v_0)$ converges uniformly*, a.s., *in $0 \leqq u \leqq u_0$.*

PROOF. Let $X_1, X_2, \cdots$, be a sequence of independent, identically distributed random variables with cdf F, $F(0) = 0$. Let N_1 be the first integer such that $\sum_{i=1}^{N_1} X_i \geqq T$, N_2 the first integer such that $\sum_{i=N_1+1}^{N_2} X_i \geqq T$, N_3 the first integer such that $\sum_{i=N_1+N_2+1}^{N_3} X_i \geqq T$, and so on. Then $N_1, N_2, \cdots$, is a sequence of i.i.d. random variables. For $N(n) = \sum_{r=1}^{n} N_r$

$$\begin{aligned} R_n(u, v) &= R(u, v)/N(n) \\ &= \frac{\sum_{r=1}^{n} \sum_{i=1}^{N_r} I(X_{ir} \mid [u, v))}{N(n)} - \frac{\sum_{r=1}^{n} I(X_{N_r r} \mid [u, v))}{N(n)} \\ &\to \left(1 - \frac{1}{E(K_1)}\right) [F(v^-) - F(u^-)] \end{aligned} \tag{3.2}$$

uniformly a.s. for $-\infty < u < v < \infty$, from using the Glivenko–Cantelli theorem and the strong law of large numbers.

Similarly one may show that as $n \to \infty$

$$S_n(u, v) = S(u, v)/N(n) \quad \text{converges uniformly a.s.} \tag{3.3}$$

on $0 \leqq u < v \leqq T$.

Observe that for n_0 sufficiently large

$$R_n(u_0, v) \text{ and } (S_n(u_0, v))^{-1} \text{ are uniformly bounded a.s.} \tag{3.4}$$

on $v_0 \leqq v < T$, $n \geqq n_0$.

Also for sufficiently large n_0

$$R_n(u, v_0) \text{ and } (S_n(u, v_0))^{-1} \text{ are uniformly bounded a.s.} \tag{3.5}$$

on $0 \leqq u \leqq u_0$, $n \geqq n_0$.

The proof is completed since (3.2)—(3.4) imply (i) and (3.2), (3.3) and (3.5) imply (ii).

THEOREM 3.2. *Let F be* IFR *on* $[0, T]$ *with failure rate r on* $[0, T)$. *Then, for* $0 \leqq u < v < T$ *fixed, where* $0 \leqq u < \beta$ *if* $I_F = [\alpha, \beta]$.

$$r(u) \leqq \frac{E(\sum_{i=1}^{K-1} I(X_i \mid [u, v)))}{E(\int_u^v n(y)\, dy)} \leqq r(v) \tag{3.6}$$

where $K = K_1$, $n(\cdot) = n_1(\cdot)$ *and* $X_i = X_{i1}$, $i = 1, 2, \cdots$.

PROOF. If $I_F = \phi$ then F has failure rate 0 on $[0, T)$ and (3.6) follows. If $I_F = \{\beta\}$ then $F(\beta) = 1$, and $r(x) = \infty$, $x \geqq \beta$, and $r(x) = 0$, $x < \beta$. Also,

$$E\left(\sum_{i=1}^{K-1} I(X_i \mid [u, v))\right) = 0 \qquad \text{for} \quad u < v \leqq \beta \qquad \text{and}$$
$$E\left(\sum_{i=1}^{K-1} I(X_i \mid [u, v))\right) \geqq 1 \qquad \text{for} \quad u < \beta < v\,.$$

Also,

$$E\left(\int_u^v n(y)\, dy\right) > 0 \qquad \text{for} \quad u < \beta\,.$$

Thus, (3.6) easily follows.

Now, assume $I_F = [\alpha, \beta]$, $0 \leqq \alpha < \beta \leqq T$. Crow and Shimi (1971) show that if H is a cdf with failure rate constant, say, λ, on $[a, b)$, then

$$\frac{E_H(\sum_{i=1}^{K-1} I(X_i \mid [a, b)))}{E_H(\int_a^b n(y)\, dy)} = \lambda\,. \tag{3.7}$$

Case 1.

$$0 < u < v < \beta\,.$$

If F has a non-decreasing step-function failure rate on $[u, v)$ then (3.6) holds by a simple application of (3.7). To prove that (3.6) holds in general for this case, let $r_n(x)$, $n = 1, 2, \cdots$, $x \in [0, \beta)$ be a sequence of real-valued functions such that $r_n(x) = r(x)$, $x \in [0, u)$, $r_n(x)$ is a non-decreasing step-function on $[u, v)$ and $r_n(x) \uparrow r(x)$ on $[u, v)$. Note that $r_n(x) \leqq r(x) \leqq r(v) < \infty$. Thus by the Lebesgue Dominated Convergence theorem, as $n \to \infty$

$$\int_0^y r_n(x)\, dx \to \int_0^y r(x)\, dx\,, \qquad y \in [0, v)\,.$$

Therefore,

$$\begin{aligned} F_n(y) &= 1 - \exp\{-\textstyle\int_0^y r_n(x)\, dx\} \to 1 - \exp\{-\textstyle\int_0^y r(x)\, dx\} \\ &= F(y)\,, \qquad y \in [0, v)\,, \qquad\qquad \text{as} \quad n \to \infty\,. \end{aligned}$$

Let $F_n(y) = F(y)$, $y \geqq v$. Then F_n, $n = 1, 2, \cdots$, is absolutely continuous on $[0, v)$, continuous from the right on $[v, \infty)$, since F is, $F_n(0) = 0$, $F_n(\infty) = 1$. Thus, F_n is a sequence of distribution functions, $F_n(y) \to F(y)$, $y \in (-\infty, \infty)$, as $n \to \infty$.

By the Helly–Bray Theorem (Loève (1963))

$$\begin{aligned} P[K = k \mid F_n] &= \textstyle\int_{\mathscr{C}_k} \prod_{i=1}^k dF_n(x_i) \to \int_{\mathscr{C}_k} \prod_{i=1}^k dF(x_i) \\ &= P[K = k \mid F]\,, \qquad n \to \infty, k = 1, 2, \cdots\,. \end{aligned} \tag{3.8}$$

Let $p_n(k) = P[K = k \mid F_n]$, $n = 1, 2, \cdots$, $k = 1, 2, \cdots$, and $p(k) = P[K = k \mid F]$, $k = 1, 2, \cdots$. By Rao ((1968), page 106) and (3.8)

$$\sum_{k=0}^{\infty} |p_n(k) - p(k)| \to 0 , \qquad n \to \infty . \tag{3.9}$$

Now, note that

$$\sum_{i=1}^{K-1} I(X_i \mid [u, v)) \leqq [T/u] \quad \text{a.s.} , \qquad u > 0 \tag{3.10}$$

and

$$n(y) \leqq [T/y] \quad \text{a.s.} , \qquad y > 0 \tag{3.11}$$

where $[x]$ denotes the largest integer less than or equal to x.

Thus, since $u > 0$ and (3.9) holds

$$\begin{aligned} |E_{F_n}(\textstyle\sum_{i=1}^{K-1} I(X_i \mid [u, v))) - E_F(\sum_{i=1}^{K-1} I(X_i \mid [u, v)))| \\ &= |\textstyle\sum_{k=1}^{\infty} \int_{\mathscr{X}_k} \sum_{i=1}^{k-1} I(x_i \mid [u, v)) \prod_{j=1}^{k} dF_n(x_i) \\ &\quad - \textstyle\sum_{k=1}^{\infty} \int_{\mathscr{X}_k} \sum_{i=1}^{k-1} I(x_i \mid [u, v)) \prod_{j=1}^{k} dF(x_i)| \\ &\leqq [T/u] \textstyle\sum_{k=1}^{\infty} |p_n(k) - p(k)| \to 0 , \qquad \text{as } n \to \infty . \end{aligned}$$

Also,

$$\int_u^v n(y)\, dy \leqq (v - u)n(u) \leqq (v - u)[T/u]$$

by (3.11). Hence,

$$E_{F_n}(\textstyle\sum_{i=1}^{K-1} I(X_i \mid [u, v)) \to E_F(\sum_{i=1}^{K-1} I(X_i \mid [u, v))) \tag{3.12}$$

and

$$E_{F_n}(\textstyle\int_u^v n(y)\, dy) \to E_F(\int_u^v n(y)\, dy) . \tag{3.13}$$

However, from (3.11) it follows that

$$r_n(y) \leqq \frac{E_{F_n}(\sum_{i=1}^{K-1} I(X_i \mid [u, v)))}{E_{F_n}(\int_u^v n(y)\, dy)} \leqq r_n(v) \quad \text{a.s.}$$

Taking limits, and using (3.12) and (3.13) gives (3.6).

Case 2.

$$0 = u < v < \beta .$$

Result (3.6) follows easily using the results of Case 1.

Also it is straightforward to use the results of Case 1 to prove (3.6) for

Case 3.

$$\beta < T , \qquad \beta \leqq v < T .$$

We now give the convergence theorem for the estimate $\tilde{r}_n(x)$ of $r(x)$, $0 \leqq x < T$.

THEOREM 3.3. *Let F be* IFR *on $[0, T]$ with failure rate r on $[0, T)$. Then,*

$$r(x_0^-) \leqq \liminf \tilde{r}_n(x_0) \leqq \limsup \tilde{r}_n(x_0) \leqq r(x_0^+) \quad \text{a.s.}$$

for each $x_0 \in (0, T)$.

PROOF.

Case 1.
$$I_F = \phi .$$
In this case $\tilde{r}_n(x) = 0$ a.s. for $0 \leqq x < T$. Since $r(x) = 0$, $0 \leqq x > T$, the result follows.

Case 2.
$$I_F = [\alpha, \beta] .$$
Let $Z_n(x_0) = \max_{0 \leqq i \leqq d(n)} \{Z_i \mid Z_i \leqq x_0\}$. We will show the right-hand inequality first.

If $\beta < T$ and $\beta \leqq x_0 < T$, then $r(x_0^+) = \infty$, since $F(\beta) = 1$. Hence, assume $0 < x_0 < \beta \leqq T$. Choose v_0, $x_0 < v_0 < \beta$. Then

$$\begin{aligned} \tilde{r}_n(x_0) &= \inf_{x_0 \leqq v} \sup_{u < Z_n(x_0)} M_n(u, v) \\ &\leqq \sup_{u < Z_n(x_0)} M_n(u, v_0) . \end{aligned} \tag{3.14}$$

Let
$$M(a, b) = \frac{E(\sum_{j=1}^{K_1-1} I(X_{j1} \mid [a, b)))}{E(\int_a^b n_1(y)\, dy)} .$$

Since $0 < x_0 < v_0$, we may apply Theorem 3.1 (ii) and conclude that, as $n \to \infty$, $M_n(u, v_0)$ converges uniformly a.s. for $0 \leqq u \leqq x_0$. Thus for arbitrary $\varepsilon > 0$ and $n \geqq N(\varepsilon)$, say,
$$\tilde{r}_n(x_0) \leqq \sup_{u < Z_n(x_0)} (M(u, v_0) + \varepsilon) .$$
Since $u < \beta$ we may apply Theorem 3.2 and conclude that $\limsup \tilde{r}_n(x_0) \leqq r(v_0) + \varepsilon$. This gives $\limsup \tilde{r}_n(x_0) \leqq r(x_0^+)$ a.s. since $x_0 < v_0$ and the right-hand limits exist.

We will now show the left-hand inequality.

Case 2(a).
$$0 < \alpha \quad \text{and} \quad x_0 \in (0, \alpha] .$$
Since $r(x_0^-) = 0$ the left-hand inequality holds.

Case 2(b).
$$\beta < T \quad \text{and} \quad \beta \leqq x_0 < T .$$
If F takes a jump at β then with probability one $Z_n(x_0) = \beta$ for $n \leqq N$, N sufficiently large. But this implies that $\tilde{r}_n(x) = \tilde{r}_n(\beta) = \infty$ for $n \geqq N$, $\beta \leqq x < T$. Thus, $\liminf \tilde{r}_n(x_0) = \infty$ and, hence, $\liminf \tilde{r}_n(x_0) \geqq r(x_0^-)$.

If F does not take a jump at β then $r(\beta^-) = \infty$ and therefore as $n \to \infty$, $Z_n(x_0) \to \beta$ a.s. Choose u_0, $0 < u_0 < \infty$. Then for N sufficiently large, $u_0 < Z_n(x_0) < \beta$ a.s., for $n \geqq N$, and, thus,
$$\begin{aligned} \tilde{r}_n(x_0) &= \inf_{x_0 \leqq v} \sup_{u < Z_n(x_0)} M_n(u, v) \\ &\geqq \inf_{x_0 \leqq v} M_n(u_0, v) \\ &\geqq \inf_{v_0 \leqq v} M_n(u_0, v) \qquad \text{for} \quad u_0 < v_0 < \beta . \end{aligned}$$

Apply Theorem 3.1 (i) and conclude that for arbitrary $\varepsilon > 0$, $N(\varepsilon)$ sufficiently large,

$$\bar{r}_n(x_0) \geqq \inf_{v_0 \leqq v} (M(u_0, v) - \varepsilon) \quad \text{a.s.} \qquad n \geqq N(\varepsilon) .$$

By Theorem 3.2, $\bar{r}_n(x_0) \geqq r(u_0) - \varepsilon$ a.s., $n \geqq N(\varepsilon)$. This gives $\liminf \bar{r}_n(x_0) \geqq r(u_0)$ a.s. for all $u_0 < \beta$. Letting $u_0 \to \beta^-$ gives $\liminf \bar{r}_n(x_0) = \infty$ a.s. Since $\bar{r}_n(x_0^-) = \infty$ for $x_0 \geqq \beta$, we have the desired result for Case 2(b).

Case 2(c).

$$\alpha < x_0 < \beta \leqq T .$$

Choose u_0, $\alpha < u_0 < x_0$. Then for N large enough so that $u_0 < Z_n(x_0) < x_0$,

$$\begin{aligned} \bar{r}_n(x_0) &= \inf_{x_0 \leqq v} \sup_{u < Z_n(x_0)} M_n(u, v) \\ &\geqq \inf_{x_0 \leqq v} M_n(x_0, v) . \end{aligned}$$

Applying Theorems 3.1 and 3.2 in the usual manner gives $\liminf \bar{r}_n(x_0) \geqq r(u_0)$ a.s. for all $\alpha < u_0 < x_0$. The result follows.

This completes the proof.

The main result of this section is

THEOREM 3.4. *Let F be* IFR *on* $[0, T]$ *with failure rate r on* $[0, T)$. *Then* $\tilde{F}_n(t) \to F(t)$ *uniformly* a.s. *in* $t \in [0, T)$, *where*

$$\tilde{F}_n(t) = 1 - \exp(-\textstyle\int_0^t \bar{r}_n(y)\, dy) .$$

PROOF. Let I_F be the support of F on $[0, T]$. If $I_F = \phi$ the conclusion is clearly true. Note, also, that $\tilde{F}_n(0) = 0$ a.s. Suppose then that $I_F = [\alpha, \beta]$. By Theorem 3.3 $\bar{r}_n(t) \to r(t)$, $t \in [0, \beta)$ except possibly on a set of Lebesgue measure zero. Let $t \in [0, \beta)$ and let $t_0 \in [t, \beta)$ be a continuity point of r. For arbitrary $\varepsilon > 0$ and $N = N(t_0, \varepsilon)$ sufficiently large, $\bar{r}_n(x) \leqq \bar{r}_n(t_0) \leqq r(t_0) + \varepsilon$ for $x \in [0, t]$, $n \geqq N$. Thus, by the Lebesgue Dominated Convergence theorem

$$\int_0^t \bar{r}_n(z)\, dz \to \int_0^t r(z)\, dz \quad \text{a.s.} \tag{3.15}$$

Since

$$F(t) = 1 - \exp(-\textstyle\int_0^t r(z)\, dz) , \qquad t \in [0, \beta) ,$$

(3.15) implies that

$$\tilde{F}_n(t) \to F(t) \quad \text{a.s.} \qquad t \in [0, \beta) . \tag{3.16}$$

Case 1. F is continuous on $[0, T)$.

Since Testing Plan A and all related random variables are unaffected by the behavior of F on $[T, \infty)$, we may assume without any loss of generality that F is continuous on $[0, T]$.

Case 1 a.

$$F(T) = 1 , \qquad \beta = T .$$

Extend $\tilde{F}_n$ to $(-\infty, \infty)$ by defining $\tilde{F}_n(x) = 0$, $x < 0$, $\tilde{F}_n(x) = 1$, $x \geqq T$. Then, by (3.16) as $n \to \infty$

$$\tilde{F}_n(t) \to F(t) \quad \text{a.s.} \qquad t \in (-\infty, \infty) . \tag{3.17}$$

Note that $\tilde{F}_n$ is a distribution function. Since F is continuous on $(-\infty, \infty)$, $\tilde{F}_n(t) \to F(t)$ uniformly a.s. $t \in (-\infty, \infty)$, as $n \to \infty$ by Pólya's theorem (Eisen, (1969)).

Case 1 b.

$$F(T) < 1 , \qquad \beta = T .$$

Since $1 - F(T^-) > 0$, there exists a.s. some $m = 1, 2, \cdots$, such that $K_m = 1$. This implies that either $d(n) = 0$, or $p(n) > 1$ and $d(n) \geqq 1$, for $n \geqq m$. In any case, $\tilde{r}_n(x) < \infty$ a.s. $x \in [0, T)$, for $n \geqq m$. Therefore, $\tilde{F}_n(x) < 1$, $x \in [0, T)$, and hence, $\tilde{F}_n$ may be extended to $[0, T]$ in a continuous manner, for $n \geqq m$. Since F is continuous, $\tilde{F}_n(T) \to F(T)$.

Since $\tilde{F}_n(t)$ and $\tilde{F}(t)$ are both non-decreasing for $t \in [0, T)$ and $F(t)$ is continuous and bounded for $t \in [0, T)$, we can apply Pólya's theorem to show that $\tilde{F}_n(t) \to F(t)$ uniformly a.s. for $t \in [0, T)$.

Case 1 c.

$$\beta < T .$$

Let $\beta \leqq x < T$ and $\varepsilon > 0$ be given. By the continuity of F there exists a $0 < z < \beta$ such that $1 - F(z) \leqq \varepsilon$, and by (3.17) there exists a $N = N(z, \varepsilon)$ such that $F(z) - \varepsilon \leqq \tilde{F}_n(z)$, $n \geqq N$. Hence, for $n \geqq N$, $1 - 2\varepsilon \leqq F(z) - \varepsilon \leqq \tilde{F}_n(z) \leqq \tilde{F}_n(x) \leqq 1$. Therefore, $\lim_{n\to\infty} \tilde{F}_n(x) \to F(x) = 1$ a.s. for $x \geqq \beta$. Using (3.16) we have $\tilde{F}_n(x) \to F(x)$ a.s. for $x \in [0, T)$. Using Pólya's theorem again we may conclude that $\tilde{F}_n(x) \to F(x)$ uniformly a.s. for $x \in [0, T)$, as $n \to \infty$.

Case 2. F takes a jump on $[0, T)$.

Since F takes a jump on $[0, T)$ at β, it follows that with probability one $K_m = 1$, for some $m = 1, 2, \cdots$. Thus, $\tilde{r}_n(t) = \infty$, $\beta \leqq t < T$, $n \geqq m$, which implies that $\tilde{F}_n(t) = 1$, $\beta \leqq t < T$, $n \geqq m$. Since $F(t) = 1$, $t \geqq \beta$, we have

$$\tilde{F}_n(t) \to F(t) \text{ uniformly a.s.} \quad \text{for} \quad t \in [\beta, T) , \quad \text{as } n \to \infty . \tag{3.18}$$

We will now show that the convergence is uniform on $[0, T)$. For $n \geqq m$, $\tilde{F}_n$ is a sequence of non-decreasing, bounded, continuous functions. Hence, they may be extended to $[0, \beta]$ in a fashion which will preserve continuity. Similarly, we may extend F to $[0, \beta]$ in a continuous manner.

Applying Pólya's theorem and using (3.16) we can show that $\tilde{F}_n(t) \to F(t)$ uniformly a.s. for $t \in [0, \beta)$, The result follows.

We now give two useful corollaries of Theorems 3.3 and 3.4.

COROLLARY 3.5. *Let $S = [u, v]$ be a closed interval of continuity of r, $0 \leqq u < v < T$. Then, $\tilde{r}_n(x) \to r(x)$ uniformly* a.s. *on S as $n \to \infty$.*

PROOF. By Theorem 3.3

$$\tilde{r}_n(x) \to r(x) \quad \text{a.s.} \quad \text{on} \quad S \qquad \text{as } n \to \infty . \tag{3.19}$$

Since S is closed and bounded, the result follows.

COROLLARY 3.6. *Let $S = [u, v]$ be a closed interval of continuity of r, $0 \leqq u < v < T$. Then,*

$$\tilde{f}_n(x) \to f(x) \quad \textit{uniformly a.s. on } S \qquad \textit{as } n \to \infty$$

where $\tilde{f}_n(x) = \tilde{r}_n(x) \exp(-\int_0^x \tilde{r}_n(y)\, dy)$.

PROOF. The proof follows directly from Theorem 3.4 and Corollary 3.5.

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DEPARTMENT OF STATISTICS
FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA 32306