

RANK SPECTRAL PROCESSES AND TESTS FOR SERIAL DEPENDENCE¹

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Rank analogues of the integrated periodogram spectral process are introduced and used to generate distribution-free tests for independence of a set of random variables. Under simple autoregressive alternatives, the rank spectral process with normal scores yields a test of Kolmogorov-Smirnov type whose local asymptotic efficiency relative to the analogous test based on the integrated periodogram is at least one. Moreover, the same rank test has good local asymptotic efficiency relative to tests based on optimally lagged rank serial correlation coefficients.

1. Introduction. The basic statistical problem studied in this paper is that of testing a set of observations for randomness against alternatives under which the observations are serially dependent. Simple tests that have been proposed for the problem include the classical serial correlation test and its rank analogues (Wald and Wolfowitz [15], Aiyar [1]). While such tests can be effective against alternatives of practical importance, their successful use requires knowledge of the proper lag to be introduced into the serial correlation coefficient. In the absence of such precise information, it is prudent to use tests which have reasonable power over a larger set of alternatives. The development of spectral methods in time series analysis has provided a means of constructing tests that satisfy this requirement.

Let X_1, X_2, \dots, X_N be the observed random variables and let H_N be the hypothesis of randomness, under which the $\{X_i\}$ are independent identically distributed with common continuous distribution function F . Suppose in addition that $E_{H_N}(X_i) = \mu$, $\text{Var}_{H_N}(X_i) = \sigma^2$, and μ, σ^2 are known. Then, under H_N , the random variables $\{Z_i = \sigma^{-1}(X_i - \mu), 1 \leq i \leq N\}$ form a sample from a stationary random sequence with mean 0 and spectral distribution function $G(\lambda) = \lambda/\pi$ on $[0, \pi]$. The integrated periodogram estimate of $G(\lambda)$ is (see Grenander and Rosenblatt [7])

$$(1.1) \quad \begin{aligned} G_N(\lambda) &= (\pi N)^{-1} \int_0^\lambda \left| \sum_{j=1}^N Z_j e^{-i\mu j} \right|^2 d\mu \\ &= N^{-1} \sum_{j=1}^N \sum_{k=1}^N a_{|j-k|}(\lambda) Z_j Z_k, \end{aligned}$$

where

$$(1.2) \quad \begin{aligned} a_m(\lambda) &= \frac{\sin(\lambda m)}{\pi m} & m \geq 1 \\ &= \frac{\lambda}{\pi} & m = 0. \end{aligned}$$

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Tests for H_N versus serial dependence can therefore be based upon appropriate functionals of the spectral process $N^{1/2}[G_N(\lambda) - \lambda/\pi]$. Under mild conditions, such tests are consistent against stationary alternatives with spectral distribution function differing from λ/π .

The weak convergence in $C[0, \pi]$ of $N^{1/2}[G_N(\lambda) - G(\lambda)]$ to a Gaussian process has been studied under assumptions on the $\{Z_i\}$ more general than independence. Grenander and Rosenblatt [7] treated the case of a linear process for which the eighth moment exists. Ibragimov [10] considered stationary Gaussian processes with square integrable spectral densities. Brillinger [3] proved weak convergence in a topology finer than the uniform topology, assuming stationarity, a mixing condition, and existence of all moments. These results are applicable to the testing problem described above with μ, σ^2 known.

In practice, however, the moments μ, σ^2 are not known. One way of overcoming this difficulty in testing for H_N is to substitute the estimates $\bar{X} = N^{-1} \sum_{i=1}^N X_i, s^2 = N^{-1} \sum_{i=1}^N (X_i - \bar{X})^2$ for μ, σ^2 in $N^{1/2}[G_N(\lambda) - \lambda/\pi]$. Together with a rescaling, this step yields the spectral process

$$(1.3) \quad V_N(\lambda) = (N^{1/2}s^2)^{-1} \sum_{j \neq k}^N c_{|j-k|}(\lambda)(X_j - \bar{X})(X_k - \bar{X}),$$

where

$$(1.4) \quad c_m(\lambda) = \frac{\sin(m\lambda)}{2^{1/2}\pi m}, \quad m \geq 1.$$

Another approach, the one studied in this paper, is to replace the observations by their ranks $(R_{N1}, R_{N2}, \dots, R_{NN})$ and form a rank analogue of $V_N(\lambda)$:

$$(1.5) \quad S_N(\lambda) = (N^{1/2}||\varphi||^2)^{-1} \sum_{j \neq k}^N c_{|j-k|}(\lambda)a_N(R_{Nj}, R_{Nk}).$$

The scores $\{a_N(\cdot, \cdot)\}$ satisfy regularity conditions and $||\varphi||^2$ is a normalizing constant (see Section 2).

Tests for H_N against serial dependence can be based upon functionals of the spectral processes $S_N(\lambda)$ or $V_N(\lambda)$. The tests based upon $S_N(\lambda)$ have the advantage of being distribution-free, making it practical to tabulate their exact critical values for small sample sizes. In addition, our results in this paper suggest that for testing H_N versus autoregressive alternatives, the spectral process $S_N(\lambda)$ with normal scores is at least as efficient asymptotically as is $V_N(\lambda)$. This dominance is established formally for tests based on $\max_{\lambda} S_N(\lambda)$ over their counterparts based on $\max_{\lambda} V_N(\lambda)$. Furthermore, it is shown that these particular rank tests fare reasonably well in comparison with asymptotically most powerful rank tests, at least for the low-order autoregressive alternatives of greatest practical interest.

The remainder of this paper is organized as follows. Section 2 describes the principal formal results: the weak convergence, asymptotic power, and asymptotic efficiency theorems which justify the remarks above. Proofs of these results are given in Sections 3 and 4. Section 3 deals with proofs under H_N while

Section 4 treats proofs under simple autoregressive alternatives. Similar methods can be used to study signed-rank spectral processes and rank cross-spectral processes.

2. Principal results. Before stating the main results of the paper, we introduce some notation and assumptions. Let (\cdot, \cdot) and $\|\cdot\|$ denote, respectively, the inner product and norm in $L_2[0, 1]$. Let X_1, X_2, \dots be a sequence of real-valued random variables defined on the space and σ -algebra (Ω, \mathscr{A}) and let R_{N_i} denote the rank of X_i among (X_1, X_2, \dots, X_N) . Under H_N , the hypothesis of randomness for N observations, (X_1, X_2, \dots, X_N) are assumed to be independent identically distributed with common continuous distribution function F .

If $U_i = F(X_i)$ for $1 \leq i \leq N$, the random variables (U_1, U_2, \dots, U_N) are independent and uniformly distributed on $(0, 1)$ under H_N . Corresponding to each $\varphi \in L_2[0, 1]$, a sequence of rank scores $\{a_N^\varphi(\cdot, \cdot)\}$ may be defined by

$$(2.1) \quad a_N^\varphi(j, k) = E_{H_N}[\varphi(U_1)\varphi(U_2) | R_{N1} = j, R_{N2} = k], \quad 1 \leq j \neq k \leq N.$$

The rank spectral process $S_N^\varphi(\lambda)$ associated with these scores is

$$(2.2) \quad S_N^\varphi(\lambda) = (N^2\|\varphi\|^2)^{-1} \sum_{j \neq k}^N c_{|j-k|}(\lambda) a_N^\varphi(R_{Nj}, R_{Nk}),$$

where the $\{c_m(\lambda)\}$ are defined as in (1.4). The sample paths of $S_N^\varphi(\lambda)$ belong to $C[0, \pi]$, the space of all continuous functions whose domain is the interval $[0, \pi]$. The Gaussian process $Z(\lambda)$ defined on $[0, \pi]$, satisfying

$$(2.3) \quad \begin{aligned} E[Z(\lambda)] &= 0 \\ E[Z(\lambda)Z(\mu)] &= \min\left(\frac{\lambda}{\pi}, \frac{\mu}{\pi}\right) - \frac{\lambda\mu}{\pi^2}, \end{aligned}$$

and having all its sample paths in $C[0, \pi]$ is called the Brownian bridge on $[0, \pi]$.

THEOREM 2.1. *Suppose that $\varphi \in L_{2+\delta}[0, 1]$ for some $\delta > 0$ and that $(\varphi, 1) = 0$. Then, under H_N as $N \rightarrow \infty$, $S_N^\varphi(\lambda)$ converges in distribution in $C[0, \pi]$ to the Brownian bridge.*

Thus, if h is a continuous functional defined on $C[0, \pi]$ and if the conditions of Theorem 2.1 are satisfied, then the asymptotic distribution of $h(S_N^\varphi(\lambda))$ coincides with the distribution of $h(Z(\lambda))$. In particular, the statistics $\max_\lambda S_N^\varphi(\lambda)$, $\max_\lambda |S_N^\varphi(\lambda)|$ and $\pi^{-1} \int_0^\pi [S_N^\varphi(\lambda)]^2 d\lambda$ have the same limiting distributions under H_N as, respectively, the one-sided and two-sided Kolmogorov-Smirnov statistics and the Cramér-von Mises statistic. Each of these three functionals of $S_N^\varphi(\lambda)$ will provide a test for H_N versus serial dependence. Another expression for the statistic of Cramér-von Mises type is

$$(2.4) \quad \pi^{-1} \int_0^\pi [S_N^\varphi(\lambda)]^2 d\lambda = (4\pi^2)^{-1} \sum_{t=1}^{N-1} \gamma_N^2(t)/t^2,$$

where

$$(2.5) \quad \gamma_N(t) = (N^2\|\varphi\|^2)^{-1} \sum_{|j-k|=t}^N a_N^\varphi(j, k).$$

In the class of score functions $\{a_N^\varphi(\cdot, \cdot) : \varphi \in L_2[0, 1]\}$, two choices stand out. Simplicity suggests taking $\varphi(u) = u - \frac{1}{2}$, which yields

$$(2.6) \quad a_N^\varphi(j, k) = \frac{jk + \min(j, k)}{(N+1)(N+2)} - \frac{j+k}{2(N+1)} + \frac{1}{4}.$$

Another choice, $\varphi(u) = \Phi^{-1}(u)$, where Φ is the standardized normal distribution function, is supported by the power considerations to which we turn now.

Let K_N denote the following alternative to the hypothesis of randomness: under K_N , the joint density of (X_1, X_2, \dots, X_N) is $\prod_{i=1}^N f(x_i - \theta_N x_{i-r})$, where $r \geq 1$, $\theta_N = N^{-1/2}\theta$, $\theta \neq 0$, $x_0 = x_{-1} = \dots = x_{1-r} = 0$, and f is the density of F , which is now assumed absolutely continuous. In other words, the observations are generated by a simple r th order autoregressive scheme. If f is absolutely continuous, we may define on $(0, 1)$ the function

$$(2.7) \quad \varphi_F(u) = -\frac{f' \cdot F^{-1}(u)}{f \cdot F^{-1}(u)}.$$

THEOREM 2.2. *Suppose that f is absolutely continuous, that for some $\delta > 0$, $\varphi, \varphi_F, F^{-1} \in L_{2+\delta}[0, 1]$, and that $(\varphi, 1) = 0$. Let*

$$(2.8) \quad \mu_\theta(\lambda) = \left[\frac{2^{1/2}\theta \sin(r\lambda)}{\pi r \|\varphi\|^2} \right] (\varphi, F^{-1})(\phi, \phi_F).$$

Then under K_N as $N \rightarrow \infty$, $S_N^\varphi(\lambda) - \mu_\theta(\lambda)$ converges in distribution in $C[0, \pi]$ to the Brownian bridge.

Let h be a continuous functional on $C[0, \pi]$ and let $D(\alpha)$ be a Borel set for which $P[h(Z(\lambda)) \in D(\alpha)] = \alpha$. The test which rejects H_N if $h(S_N^\varphi(\lambda)) \in D(\alpha)$ has asymptotic size α when Theorem 2.1 applies. The asymptotic power of this test under K_N is defined to be

$$(2.9) \quad \beta(\alpha, \theta, f) = \lim_{N \rightarrow \infty} P_{K_N}[h(S_N^\varphi(\lambda)) \in D(\alpha)].$$

If the assumptions of Theorem 2.2 are satisfied,

$$(2.10) \quad \beta(\alpha, \theta, f) = P[h(Z(\lambda) + \mu_\theta(\lambda)) \in D(\alpha)].$$

The evaluation of (2.10) is very complex in practice. A method described by Hájek and Šidák [8] yields $\beta_s'(\alpha, 0, f)$, the derivative at $\theta = 0$ of the asymptotic power of the test which rejects H_N whenever $\max_\lambda S_N^\varphi(\lambda) \geq (-2^{-1} \log \alpha)^{1/2}$. Since $P[\max_\lambda Z(\lambda) \geq x] = \exp(-2x^2)$ for all $x \geq 0$, this test has asymptotic size α . Higher derivatives and a Taylor expansion for the asymptotic power can be obtained using results by Anděl [2].

THEOREM 2.3. *Suppose that f is absolutely continuous, that for some $\delta > 0$, $\varphi, \varphi_F, F^{-1} \in L_{2+\delta}[0, 1]$, and that $(\varphi, 1) = 0$. Let*

$$(2.11) \quad \tau = (\varphi, F^{-1})(\varphi, \varphi_F) \|\varphi\|^{-2}$$

and

$$(2.12) \quad \phi(\alpha, t) = 2\Phi[(-2^{-1} \log \alpha)^{1/2}(1-2t)[t(1-t)]^{-1/2}] - 1.$$

Then

$$(2.13) \quad \beta_s'(\alpha, 0, f) = 2\tau\alpha(-\log \alpha)^{\frac{1}{2}} \int_0^1 \cos(\pi rt)\phi(\alpha, t) dt \quad \text{if } r \text{ is odd}$$

$$= 0 \quad \text{if } r \text{ is even.}$$

The vanishing of $\beta_s'(\alpha, 0, f)$ when r is even reflects the fact that for such r , the test with critical region $\{\max_{\lambda} S_N^{\varphi}(\lambda) \geq (-2^{-1} \log \alpha)^{\frac{1}{2}}\}$ has reasonable power against alternatives $\theta < 0$ as well as $\theta > 0$. (This may be checked heuristically by considering the spectral distribution function of a stationary r th order autoregressive process.) Thus, to characterize the local behavior of $\beta_s(\alpha, \theta, f)$ for even r , it would be necessary to evaluate the second derivative at $\theta = 0$.

Suppose $T_N^{(1)}, T_N^{(2)}$ are two test statistics which generate one-sided tests for H_N versus K_N with asymptotic powers $\beta_1(\alpha, \theta, f), \beta_2(\alpha, \theta, f)$ respectively. If both $\beta_1'(\alpha, 0, f)$ and $\beta_2'(\alpha, 0, f)$ are not zero, the local asymptotic efficiency of $T_N^{(1)}$ relative to $T_N^{(2)}$ at size α is defined to be

$$(2.14) \quad e(\alpha; T_N^{(1)}, T_N^{(2)}) = \left[\frac{\beta_1'(\alpha, 0, f)}{\beta_2'(\alpha, 0, f)} \right]^2.$$

This concept, introduced in [8], is compatible with the notion of Pitman efficiency for one-sided tests in the following sense: if the limiting distributions of $T_N^{(1)}, T_N^{(2)}$ are normal under H_N and under K_N , then $e(\alpha; T_N^{(1)}, T_N^{(2)})$ equals the Pitman efficiency for all $\alpha \in (0, 1)$.

Theorem 2.3 gives a way of calculating the local asymptotic efficiency of the test with critical region $\{\max_{\lambda} S_N^{\varphi}(\lambda) \geq (-2^{-1} \log \alpha)^{\frac{1}{2}}\}$ relative to other tests for H_N versus K_N , provided r is odd. Two competitors of interest are the test with critical region $\{\max_{\lambda} V_N(\lambda) \geq (-2^{-1} \log \alpha)^{\frac{1}{2}}\}$ and the test with critical region $\{W_N \geq \Phi^{-1}(1 - \alpha)\}$, where

$$(2.15) \quad W_N = (N^{\frac{1}{2}} \|\varphi_F\| \|F^{-1}\|)^{-1} \sum_{k=j}^N \sum_{j=r}^N b_N^F(R_{Nj}, R_{Nk})$$

and

$$(2.16) \quad b_N^F(j, k) = E_{H_N}[F^{-1}(U_1)\varphi_F(U_2) | R_{N1} = j, R_{N2} = k], \quad 1 \leq j \neq k \leq N.$$

The latter rank test is asymptotically most powerful at level α for H_N versus K_N when $\theta > 0$.

Analogues for $V_N(\lambda)$ of Theorems 2.1 through 2.3 may be obtained by replacing φ with F^{-1} in all assumptions and conclusions except $(\varphi, 1) = 0$, and by noting that $(\varphi_F, F^{-1}) = 1$. Thus, the asymptotic power $\beta_V(\alpha, 0, f)$ of the test with critical region $\{\max_{\lambda} V_N(\lambda) \geq (-2^{-1} \log \alpha)^{\frac{1}{2}}\}$ satisfies

$$(2.17) \quad \beta_V'(\alpha, 0, f) = 2\alpha(-\log \alpha)^{\frac{1}{2}} \int_0^1 \cos(\pi rt)\phi(\alpha, t) dt \quad \text{if } r \text{ is odd}$$

$$= 0 \quad \text{if } r \text{ is even.}$$

On the other hand, the asymptotic power of the test with critical region $\{W_N \geq \Phi^{-1}(1 - \alpha)\}$ is

$$(2.18) \quad \beta_W(\alpha, \theta, f) = 1 - \Phi[\Phi^{-1}(1 - \alpha) - \theta \|\varphi_F\| \|F^{-1}\|].$$

THEOREM 2.4. *Suppose that f is absolutely continuous, that for some $\delta > 0$, $\varphi, \varphi_F, F^{-1} \in L_{2+\delta}[0, 1]$, that $(\varphi, 1) = 0$, and that r is odd. Then*

$$(2.19) \quad e(\alpha; \max_{\lambda} S_N^\varphi(\lambda), \max_{\lambda} V_N(\lambda)) = [(\varphi, \varphi_F)(\varphi, F^{-1})|\varphi|^{-2}]^2$$

and

$$(2.20) \quad \min_F e(\alpha; \max_{\lambda} S_N^\varphi(\lambda), \max_{\lambda} V_N(\lambda)) = \frac{9\pi^4}{1024} \quad \text{if } \varphi(u) = u - \frac{1}{2}$$

$$= 1 \quad \text{if } \varphi(u) = \Phi^{-1}(u).$$

The second value in (2.20) is attained only if F is normal. Moreover

$$(2.21) \quad e(\alpha; \max_{\lambda} S_N^\varphi(\lambda), W_N) = 8\pi\rho_1^2\rho_2^2\alpha^2(-\log \alpha) \exp \{[\Phi^{-1}(1-\alpha)]^2\} \int_1^0 \cos(\pi rt)\phi(\alpha, t) dt,$$

where

$$\rho_1 = (\varphi, F^{-1})|\varphi|^{-1}\|F^{-1}\|^{-1}, \quad \rho_2 = (\varphi, \varphi_F)|\varphi|^{-1}\|\varphi_F\|^{-1}.$$

Also

$$(2.22) \quad \lim_{\alpha \rightarrow 0} e(\alpha; \max_{\lambda} S_N^\varphi(\lambda), W_N) = (8/\pi^2 r^2)\rho_1^2\rho_2^2.$$

Let

$$J_N = (N^{\frac{1}{2}}|\varphi|^2)^{-1} \sum_{k-j=r}^N \sum a_N^\varphi(R_{Nj}, R_{Nk}),$$

with $a_N^\varphi(\cdot, \cdot)$ defined by (2.1). The asymptotic efficiency of the test with critical region $\{J_N \geq \Phi^{-1}(1-\alpha)\}$ relative to the analogous test based on W_N is $\rho_1^2\rho_2^2$. Thus, the factor $\rho_1^2\rho_2^2$ in (2.22) represents the efficiency loss due to choice of score function $a_N^\varphi(\cdot, \cdot)$. On the other hand, the factor $8/\pi^2 r^2$ in (2.22) is attributable to using a test based on $\max_{\lambda} S_N^\varphi(\lambda)$ in place of one based on J_N . In effect, this factor is the cost of not knowing the parameter r .

It appears likely that a rate of efficiency loss proportional to r^{-2} will occur also for tests based on other functionals of $S_N^\varphi(\lambda)$ or $V_N(\lambda)$. (Equation (2.4) is particularly suggestive in this respect.) The effect of this loss in practice may not be severe, since serial dependence in empirical time series often can be fitted with an autoregressive model that has some nonvanishing coefficients for low order lags.

It is interesting to contrast asymptotic power under K_N with that achieved under contiguous regression alternatives. Let K'_N be an alternative under which X_1, X_2, \dots, X_N are independent random variables with joint density $\prod_{i=1}^N f(x_i - c_{Ni})$, where $(c_{N1}, c_{N2}, \dots, c_{NN})$ is a vector of constants. Let $\bar{c}_N = N^{-1} \sum_{i=1}^N c_{Ni}$. If $\lim_{N \rightarrow \infty} \max_{1 \leq i \leq N} (c_{Ni} - \bar{c}_N)^2 = 0$, $\lim_{N \rightarrow \infty} \sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2 = b^2 < \infty$, f is absolutely continuous, $\varphi, \varphi_F \in L_{2+\delta}[0, 1]$ for some $\delta > 0$, and $(\varphi, 1) = 0$, then under K'_N as $N \rightarrow \infty$, $S_N^\varphi(\lambda)$ converges in distribution to the Brownian bridge. Consequently, any test for H_N based on $S_N^\varphi(\lambda)$ has asymptotic power against K'_N that equals the asymptotic size. If φ is replaced by F^{-1} in the assumptions above,

excluding $(\varphi, 1) = 0$, the same conclusion extends to $V_N(\lambda)$. This phenomenon is just the reverse of the insensitivity that linear rank tests for H_N versus K_N' display toward the contiguous autoregressive alternatives K_N when $(F^{-1}, 1) = 0$ (see Aiyar [1]).

THEOREM 2.5. *Suppose that $\varphi \in L_2[0, 1]$, $(\varphi, 1) = 0$, $a_N^\varphi(\cdot, \cdot)$ is defined as in (2.1), and $a_N(\cdot, \cdot)$ is another score function which satisfies*

$$(2.23) \quad \lim_{N \rightarrow \infty} N^{-\frac{3}{2}} \log(N) \sum_{j \neq k}^N \sum a_N(j, k) = 0$$

$$(2.24) \quad \lim_{N \rightarrow \infty} N^{-2} (\log(N))^2 \sum_{j \neq k}^N \sum [a_N(j, k) - \bar{a}_N - a_N^\varphi(j, k)]^2 = 0,$$

where

$$(2.25) \quad \bar{a}_N = [N(N - 1)]^{-1} \sum_{j \neq k}^N \sum a_N(j, k).$$

Then Theorems 2.1 through 2.4 apply also to the spectral process

$$(2.26) \quad S_N(\lambda) = (N^{\frac{1}{2}} \|\varphi\|^2)^{-1} \sum_{j \neq k}^N \sum c_{|j-k|}(\lambda) a_N(R_{Nj}, R_{Nk}).$$

It may be noted that both (2.23) and (2.24) are implied by

$$(2.27) \quad \lim N^{-1} (\log(N))^2 \sum_{j \neq k}^N \sum [a_N(j, k) - a_N^\varphi(j, k)]^2 = 0.$$

This theorem can sometimes be used to replace the scores $a_N^\varphi(\cdot, \cdot)$ by more convenient ones. For example, if $\varphi(u) = u - \frac{1}{2}$, the scores

$$(2.28) \quad a_N(j, k) = \left(\frac{j}{N+1} - \frac{1}{2} \right) \left(\frac{k}{N+1} - \frac{1}{2} \right)$$

can be used in place of the corresponding $\{a_N^\varphi(\cdot, \cdot)\}$, given in (2.6).

3. Asymptotics under H_N . This section contains a number of ancillary results as well as a proof of Theorem 2.1.

LEMMA 3.1. *The following inequalities hold for all $\lambda, \mu \in [0, \pi]$.*

$$(3.1) \quad \sum_{t=1}^{\infty} \frac{\sin(\lambda t) \sin(\mu t)}{t^2} = \frac{1}{2} [\pi \min(\lambda, \mu) - \lambda \mu],$$

$$(3.2) \quad N^{-1} \sum_{j \neq k}^N \sum \frac{\sin^2[\lambda(k-j)]}{(k-j)^2} \leq \frac{\pi^2}{4},$$

$$(3.3) \quad \left| \sum_{k \neq j}^N \frac{\sin[\lambda(k-j)]}{(k-j)} \right| \leq 2(\pi + 1), \quad 1 \leq j \leq N.$$

PROOF. Equation (3.1) is well known (Bromwich [5]) and implies (3.2). Inequality (3.3) is a consequence of the bound

$$(3.4) \quad \left| \sum_{t=1}^n \frac{\sin(\lambda t)}{t} \right| \leq \pi + 1,$$

which holds for all integer $n \geq 1$, all $\lambda \in [0, \pi]$. This bound can be established by using the Fejér formula for Cesàro sums.

The next lemma is a variant of Lemma A1 in Jogdeo [11].

LEMMA 3.2. (Jogdeo) *Let the $\{b_N(\cdot, \cdot)\}$ be any scores and let the $\{d_{jk}\}$ be constants satisfying*

$$(3.5) \quad \sum_{j \neq k}^N \sum d_{jk}^2 < C_1$$

$$(3.6) \quad \sum_{j=1}^N (\sum_{k \neq j}^N d_{jk})^2 < C_2, \quad \sum_{k=1}^N (\sum_{j \neq k}^N d_{jk})^2 < C_3,$$

where C_1, C_2, C_3 are constants not depending on N . Let

$$\bar{d} = [N(N - 1)]^{-1} \sum_{j \neq k}^N d_{jk}.$$

Then there exists a constant C_4 , depending only on C_1, C_2, C_3 , such that

$$(3.7) \quad E_{H_N} \left[\sum_{j=k}^N \sum (d_{jk} - \bar{d}) b_N(R_{Nj}, R_{Nk}) \right]^2 \leq C_4 E_{H_N} [b_N^2(R_{N1}, R_{N2})].$$

PROOF. Because of (3.5), $|\bar{d}| \leq C_1^{1/2}$. It is easily verified that (3.5) and (3.6) imply similar bounds in terms of the constants $\{d_{jk} - \bar{d}\}$, whereupon the result follows from the first part of the proof of Lemma A1 in [11].

LEMMA 3.3. *Suppose $\varphi \in L_2[0, 1]$, $(\varphi, 1) = 0$, and $a_N^\varphi(\cdot, \cdot)$ is defined as in (2.1). Then*

$$(3.8) \quad E_{H_N} [a_N^\varphi(R_{N1}, R_{N2})] = 0,$$

$$(3.9) \quad \lim_{N \rightarrow \infty} E_{H_N} [a_N^\varphi(R_{N1}, R_{N2}) - \varphi(U_1)\varphi(U_2)]^2 = 0.$$

PROOF. Let $\mathcal{F}_N \subset \mathcal{A}$ be the σ -algebra generated by $(R_{N1}, R_{N2}, \dots, R_{NN})$ and let $\mathcal{F}_\infty \subset \mathcal{A}$ be the smallest σ -algebra containing $\bigcup_{N=1}^\infty \mathcal{F}_N$. Evidently

$$(3.10) \quad a_N^\varphi(R_{N1}, R_{N2}) = E_{H_N} [\varphi(U_1)\varphi(U_2) | \mathcal{F}_N],$$

so that (3.8) is immediate.

By V, T18 of Meyer [12], $\{a_N^\varphi(R_{N1}, R_{N2}), \mathcal{F}_N; N \geq 1\}$ is a uniformly integrable martingale. Moreover there exists an essentially unique \mathcal{F}_∞ measurable random variable Y such that $a_N^\varphi(R_{N1}, R_{N2}) = E_{H_N} [Y | \mathcal{F}_N]$ and $\lim_{N \rightarrow \infty} a_N^\varphi(R_{N1}, R_{N2}) = Y$ a.s. Since U_1, U_2 may be represented as a.s. limits of \mathcal{F}_∞ measurable random variables (see [8] page 157), it follows that

$$(3.11) \quad \lim_{N \rightarrow \infty} a_N^\varphi(R_{N1}, R_{N2}) = \varphi(U_1)\varphi(U_2) \quad \text{a.s.}$$

By Jensen's inequality

$$(3.12) \quad [a_N^\varphi(R_{N1}, R_{N2})]^2 \leq E_{H_N} [\varphi^2(U_1)\varphi^2(U_2) | \mathcal{F}_N] \quad \text{a.s.},$$

which implies uniform integrability of the random variables $\{[a_N^\varphi(R_{N1}, R_{N2})]^2\}$. Hence (3.11) implies (3.9).

The next lemma is due to Whittle [16].

LEMMA 3.4. (Whittle) Let Z_1, Z_2, \dots, Z_N be independent identically distributed random variables, each assuming the values ± 1 with respective probabilities $\frac{1}{2}$. Then for any set of constants $\{b_j\}$ and any $s \geq 2$,

$$(3.13) \quad E|\sum_{j=1}^N b_j Z_j|^s \leq C(s)[\sum_{j=1}^N b_j^2]^{s/2},$$

where

$$(3.14) \quad C(s) = 2^{s/2} \pi^{-\frac{1}{2}} \Gamma\left(\frac{s+1}{2}\right).$$

A useful bound on the moments of a quadratic form in independent identically distributed random variables that has no diagonal terms is given by the following lemma. It can be proved by simple modification of Whittle's [16] argument for his Theorem 2.

LEMMA 3.5. Let Z_1, Z_2, \dots, Z_N be independent random variables, let the $\{d_{jk}\}$ be constants, and let

$$A = \sum_{j \neq k}^N \sum_{j \neq k} d_{jk} Z_j Z_k.$$

Suppose that $s \geq 2$ and $\mu_j(s) = E|Z_j|^s < \infty$ for $1 \leq j \leq N$. Then

$$(3.15) \quad E|A - E(A)|^s \leq 2^{3s} C^2(s) \left[\sum_{j \neq k}^N \sum_{j \neq k} d_{jk}^2 \mu_j^{2/s}(s) \mu_k^{2/s}(s) \right]^{s/2}.$$

We establish next a central limit theorem for quadratic forms of this type. A related theorem for quadratic forms with nonvanishing diagonal terms was proved by Whittle [17], using a different method. The absence of diagonal terms reduces the moment conditions needed in the theorem.

THEOREM 3.1. Let Z_1, Z_2, \dots be a sequence of independent identically distributed random variables defined on (Ω, \mathcal{A}) . Suppose that $E(Z_j) = 0, E(Z_j^2) = 1$, and that for some $\delta \in (0, 2], E|Z_j|^{2+\delta} < \infty$. Let the $\{d_m\}$ be constants satisfying $\sum_{m=1}^\infty d_m^2 < \infty$. Then the distribution of

$$(3.16) \quad A_N = \sum_{j \neq k}^N \sum_{j \neq k} d_{|j-k|} Z_j Z_k$$

is asymptotically normal $(0, N\gamma^2)$ with

$$(3.17) \quad \gamma^2 = 4 \sum_{m=1}^\infty d_m^2.$$

PROOF. Let $\mathcal{G}_i \subset \mathcal{A}$ be the σ -algebra generated by (A_1, A_2, \dots, A_i) . It may be verified, as in Varberg [14], that $\{A_i, \mathcal{G}_i; i \geq 1\}$ is a martingale. Let

$$(3.18) \quad \begin{aligned} Y_1 &= A_1 = 0, \\ Y_i &= A_i - A_{i-1} = 2Z_i \sum_{j < i} d_{|i-j|} Z_j, \end{aligned} \quad i \geq 2,$$

and let $\gamma_N^2 = \sum_{i=1}^N E(Y_i^2)$. To establish the asymptotic normality of the martingale $\{A_i, \mathcal{G}_i\}$, it is sufficient to show (see Heyde and Brown [9]) that for some $\delta \in (0, 2]$,

$$(3.19) \quad \lim_{N \rightarrow \infty} \gamma_N^{-2-\delta} \sum_{i=1}^N E|Y_i|^{2+\delta} = 0$$

and

$$(3.20) \quad \lim_{N \rightarrow \infty} E|\gamma_N^{-2} \sum_{i=1}^N Y_i^2 - 1|^{1+\delta/2} = 0.$$

For notational convenience, set $s = 1 + \delta/2$ in the remainder of the proof. Since

$$(3.21) \quad \gamma_N^2 = 4N \sum_{m=1}^{N-1} d_m^2(1 - m/N),$$

it follows from (3.17) that

$$(3.22) \quad \lim_{N \rightarrow \infty} N\gamma_N^2/\gamma_N^2 = 1.$$

Applying Lemma 3.5, we see that for each $i \geq 1$,

$$(3.23) \quad E|Y_i|^{2s} \leq 2^{8s}C^2(2s)\mu^2(2s)[\sum_{m=1}^{\infty} d_m^2]^s =: K_1.$$

Consequently,

$$(3.24) \quad \gamma_N^{-2s} \sum_{i=1}^N E|Y_i|^{2s} \leq K_1 N\gamma_N^{-2s},$$

which yields (3.19) because of (3.22).

To prove (3.20), let (V_1, V_2, \dots, V_N) be an independent copy of (Y_1, Y_2, \dots, Y_N) . By a well-known symmetrization inequality (see Rosén [13]),

$$(3.25) \quad \begin{aligned} E|\sum_{i=1}^N Y_i^2 - E(\sum_{i=1}^N Y_i^2)|^s &\leq E|\sum_{i=1}^N (Y_i^2 - V_i^2)|^s \\ &= E|\sum_{i=1}^N (Y_i - V_i)(Y_i + V_i)|^s. \end{aligned}$$

Let $\mathcal{B}_N \subset \mathcal{A}$ denote the σ -algebra generated by $(|Y_1 - V_1|, \dots, |Y_N - V_N|, Y_1 + V_1, \dots, Y_N + V_N)$. Since the conditional distribution of $(Y_1 - V_1, \dots, Y_N - V_N)$ given $(Y_1 + V_1, \dots, Y_N + V_N)$ is the same as that of $(V_1 - Y_1, \dots, V_N - Y_N)$ given $(Y_1 + V_1, \dots, Y_N + V_N)$, we may use Lemma 3.4 to justify the second inequality below

$$(3.26) \quad \begin{aligned} E[|\sum_{i=1}^N (Y_i - V_i)(Y_i + V_i)|^s | \mathcal{B}_N] \\ \leq \{E[|\sum_{i=1}^N (Y_i - V_i)(Y_i + V_i)|^{2s} | \mathcal{B}_N]\}^{1/2} \\ \leq C^{1/2}(2s)[\sum_{i=1}^N |Y_i - V_i|^2 |Y_i + V_i|^2]^{s/2}. \end{aligned}$$

From (3.26) and the c_r inequality, there follows

$$(3.27) \quad \begin{aligned} E|\sum_{i=1}^N (Y_i - V_i)(Y_i + V_i)|^s &\leq C^{1/2}(2s)E[\sum_{i=1}^N |Y_i - V_i|^2 |Y_i + V_i|^2]^{s/2} \\ &\leq C^{1/2}(2s) \sum_{i=1}^N E|Y_i^2 - V_i^2|^s. \end{aligned}$$

On the other hand, from (3.23)

$$(3.28) \quad E|Y_i^2 - V_i^2|^s \leq 2^s K_1 = K_2.$$

Combining (3.25), (3.27) and (3.28) shows that

$$(3.29) \quad E|\sum_{i=1}^N Y_i^2 - E(\sum_{i=1}^N Y_i^2)|^s \leq NK_3.$$

From this and (3.22) follows (3.20) and hence the asymptotic normality. This completes the proof.

PROOF OF THEOREM 2.1. We prove first the weak convergence of all finite-dimensional distributions, then tightness. An arbitrary linear combination $\sum_{i=1}^P \alpha_i S_N^\varphi(\lambda_i)$ may also be written as

$$(3.30) \quad \sum_{i=1}^P \alpha_i S_N^\varphi(\lambda_i) = N^{-\frac{1}{2}} \sum_{j \neq k}^N d_{|j-k|} a_N^\varphi(R_{Nj}, R_{Nk}),$$

with

$$(3.31) \quad d_m = \sum_{i=1}^P \frac{\alpha_i \sin(m\lambda_i)}{2^{\frac{1}{2}} \pi m \|\varphi\|^2}.$$

The Cauchy-Schwarz inequality and (3.2) yield

$$(3.32) \quad N^{-1} \sum_{j \neq k}^N \sum_{j \neq k}^N d_{|j-k|}^2 \leq K_4,$$

while another simple argument using (3.3) gives

$$(3.33) \quad N^{-1} \sum_{j=1}^N (\sum_{k \neq j}^N d_{|j-k|})^2 = N^{-1} \sum_{k=1}^N (\sum_{j \neq k}^N d_{|j-k|})^2 \leq K_5.$$

Let $T_N(\lambda)$ be defined as

$$(3.34) \quad T_N(\lambda) = (N^{\frac{1}{2}} \|\varphi\|^2)^{-1} \sum_{j \neq k}^N c_{|j-k|}(\lambda) \varphi(U_j) \varphi(U_k),$$

so that

$$(3.35) \quad \sum_{i=1}^P \alpha_i T_N(\lambda_i) = N^{-\frac{1}{2}} \sum_{j \neq k}^N \sum_{j \neq k}^N d_{|j-k|} \varphi(U_j) \varphi(U_k).$$

Let T_{N1} be defined as

$$(3.36) \quad T_{N1} = N^{-\frac{1}{2}} \sum_{j \neq k}^N (d_{|j-k|} - \bar{d}) \varphi(U_j) \varphi(U_k),$$

where

$$(3.37) \quad \bar{d} = [N(N-1)]^{-1} \sum_{j \neq k}^N \sum_{j \neq k}^N d_{|j-k|}.$$

If $\mathbf{U}^{(\cdot)}$ denotes the order statistics of (U_1, U_2, \dots, U_N) , then (3.32), (3.33), Lemma 3.2, and (3.8) imply

$$(3.38) \quad \begin{aligned} & E_{HN}[(\sum_{i=1}^N \alpha_i S_N^\varphi(\lambda_i) - T_{N1})^2 | \mathbf{U}^{(\cdot)} = \mathbf{u}^{(\cdot)}] \\ &= E_{HN} \left[N^{-\frac{1}{2}} \sum_{j \neq k}^N (d_{jk} - \bar{d}) [a_N^\varphi(R_{Nj}, R_{Nk}) - \varphi(u^{(RNj)}) \varphi(u^{(RNk)})] \right]^2 \\ &\leq K_6 E_{HN} [(a_N^\varphi(R_{N1}, R_{N2}) - \varphi(U_1) \varphi(U_2))^2 | \mathbf{U}^{(\cdot)} = \mathbf{u}^{(\cdot)}]. \end{aligned}$$

Hence (3.9) and (3.38) yield

$$(3.39) \quad \lim_{N \rightarrow \infty} E_{HN} [\sum_{i=1}^P \alpha_i S_N^\varphi(\lambda_i) - T_{N1}]^2 = 0.$$

On the other hand, it follows from (3.3) that $|\bar{d}| \leq K_7(N-1)^{-1}$. Therefore

$$(3.40) \quad \begin{aligned} E_{HN} [T_{N1} - \sum_{i=1}^P \alpha_i T_N(\lambda_i)]^2 &= (N-1) \|\varphi\|^4 \bar{d}^2 \\ &\leq K_8(N-1)^{-1}. \end{aligned}$$

From (3.39) and (3.40) we conclude that

$$(3.41) \quad \lim_{N \rightarrow \infty} E_{H_N} [\sum_{i=1}^P \alpha_i S_N^\varphi(\lambda_i) - \sum_{i=1}^P \alpha_i T_N(\lambda_i)]^2 = 0.$$

Now (3.1) implies that

$$(3.42) \quad \sum_{m=1}^\infty d_m^2 = (4\|\varphi\|^4)^{-1} \sum_{i=1}^P \sum_{j=1}^P \alpha_i \alpha_j [\min(\lambda_i/\pi, \lambda_j/\pi) - \lambda_i \lambda_j / \pi^2].$$

Therefore, by Theorem 3.1, $\sum_{i=1}^P \alpha_i T_N(\lambda_i)$ is asymptotically normal $(0, 4\|\varphi\|^4 \sum_{m=1}^\infty d_m^2)$. From (3.42) and (3.41), we find that the asymptotic distribution of $\sum_{i=1}^N \alpha_i S_N^\varphi(\lambda_i)$ coincides with the distribution of $\sum_{i=1}^N \alpha_i Z(\lambda_i)$.

To prove tightness of the distributions of the $\{S_N^\varphi(\lambda)\}$, let $\mathcal{F}_N \subset \mathcal{A}$ denote the σ -algebra generated by $(R_{N1}, R_{N2}, \dots, R_{NN})$. Since $S_N^\varphi(\lambda) = E_{H_N}[T_N | \mathcal{F}_N]$ by (2.1), Jensen's inequality on conditional expectations implies that for all $\lambda, \mu \in [0, \pi]$

$$(3.43) \quad E_{H_N} |S_N^\varphi(\lambda) - S_N^\varphi(\mu)|^{2s} \leq E_{H_N} |T_N(\lambda) - T_N(\mu)|^{2s},$$

where $s = 1 + \delta/2$. By Lemma 3.5 and (3.1),

$$(3.44) \quad \begin{aligned} E_{H_N} |T_N(\lambda) - T_N(\mu)|^{2s} &\leq K_9 N^{-s} \left[\sum_{j \neq k}^N \sum (c_{|j-k|}(\lambda) - c_{|j-k|}(\mu))^2 \right]^s \\ &\leq 2^s K_9 \left[\sum_{m=1}^\infty (c_m(\lambda) - c_m(\mu))^2 \right]^s \\ &\leq K_{10} |\lambda - \mu|^s. \end{aligned}$$

Tightness follows from (3.43) and (3.44). This completes the proof (see Billingsley [3]).

REMARK 3.1. Under H_N and the assumption that for some $\delta > 0$, $F^{-1} \in L_{2+\delta}[0, 1]$, the spectral process $V_N(\lambda)$ defined in (1.3) also converges in $C[0, \pi]$ to the Brownian bridge. To verify this, assume also without loss in generality, that $E(X_i) = 0, \text{Var}(X_i) = 1$. Let

$$(3.45) \quad V_{N1}(\lambda) = (N^{\frac{1}{2}} S_N^2)^{-1} \sum_{j \neq k}^N c_{|j-k|}(\lambda) X_j X_k$$

where

$$(3.46) \quad S_N^2 = N^{-1} \sum_{i=1}^N (X_i - \bar{X})^2.$$

Since $S_N^2 \rightarrow 1$ a.s. under H_N , an argument like that for Theorem 2.1 establishes the weak convergence of $V_{N1}(\lambda)$ to the Brownian bridge.

On the other hand, if

$$(3.47) \quad \begin{aligned} \gamma_{N1}(t) &= N^{-\frac{1}{2}} \sum_{|j-k|=t}^N X_j X_k \\ \gamma_{N2}(t) &= N^{-\frac{1}{2}} \sum_{|j-k|=t}^N (X_j - \bar{X})(X_k - \bar{X}), \end{aligned}$$

then

$$(3.48) \quad \begin{aligned} E_{H_N} [\sup_\lambda |S_N^2 (V_{N1}(\lambda) - V_N(\lambda))|] \\ \leq (2^{\frac{1}{2}} \pi)^{-1} \sum_{t=1}^{N-1} t^{-1} E_{H_N} |\gamma_{N1}(t) - \gamma_{N2}(t)|. \end{aligned}$$

Since

$$(3.49) \quad \sum_{t=1}^{N-1} t^{-1} \leq 1 + \log(N - 2)$$

and, by direct calculation,

$$(3.50) \quad E_{H_N} |\gamma_{N1}(t) - \gamma_{N2}(t)| \leq 6N^{-1/2},$$

the bound in (3.48) tends to zero as $N \rightarrow \infty$. Therefore $V_N(\lambda)$ and $V_{N1}(\lambda)$ have the same limiting distribution in $C[0, \pi]$.

4. Asymptotics under K_N . Under K_N , the joint density of (X_1, X_2, \dots, X_N) is assumed to be $\prod_{i=1}^N f(x_i - \theta_N x_{i-r})$, where $r \geq 1$, $\theta_N = N^{-1/2}\theta$, $\theta \neq 0$, $x_0 = x_{-1} = \dots = x_{1-r} = 0$, and f is the density of F . To prove Theorem 2.2 by contiguity methods, it is necessary to study the asymptotic behavior under H_N of the log-likelihood ratio

$$(4.1) \quad L_N = \sum_{i=1}^N \log [f(X_i - \theta_N X_{i-r})/f(X_i)].$$

This in turn requires study of the statistics

$$(4.2) \quad B_N = 2 \sum_{i=1}^N \{ [f(X_i - \theta_N X_{i-r})/f(X_i)]^{1/2} - 1 \},$$

and

$$(4.3) \quad C_N = - \sum_{i=1}^N \theta_N X_{i-r} f'(X_i)/f(X_i).$$

Since L_N, B_N, C_N are each sums of r -dependent random variables, it is advantageous to introduce also

$$(4.4) \quad L_{Nj} = \sum_{i=j(\text{mod } r+1)} \log [f(X_i - \theta_N X_{i-r})/f(X_i)], \quad 1 \leq j \leq r + 1$$

and analogously defined sums B_{Nj}, C_{Nj} . The summands in L_{Nj} are mutually independent under H_N and

$$(4.5) \quad L_N = \sum_{j=1}^{r+1} L_{Nj}.$$

Similar expressions hold for B_N, C_N in terms of the $\{B_{Nj}\}$ and $\{C_{Nj}\}$ respectively.

Let φ_F be defined as in (2.7). The next three lemmas obtain for the alternatives K_N the type of result already known for translation alternatives (see Hájek and Šidák [8]). Aiyar [1] treated the case $r = 1$ under stronger assumptions.

LEMMA 4.1. *Suppose that f is absolutely continuous and $\varphi_F, F^{-1} \in L_2[0, 1]$. Then for $1 \leq j \leq r + 1$,*

$$(4.6) \quad \lim_{N \rightarrow \infty} E_{H_N} [B_{Nj} - C_{Nj} + (4(r + 1))^{-1} \theta^2 \|\varphi_F\|^2 \|F^{-1}\|^2] = 0.$$

PROOF. Let $w(x) = [f(x)]^{1/2}$. Under the assumptions of the lemma, w is absolutely continuous,

$$(4.7) \quad \begin{aligned} w'(x) &= [4f(x)]^{-1/2} f'(x), & \int_{-\infty}^{\infty} [w'(x)]^2 dx &= 4^{-1} \|\varphi_F\|^2, \\ \int_{-\infty}^{\infty} \left[\frac{w(x-h) - w(x)}{h} \right]^2 dx &\leq 4^{-1} \|\varphi_F\|^2 < \infty \end{aligned}$$

for all $h \neq 0$, and

$$(4.8) \quad \lim_{h \rightarrow 0} \int_{-\infty}^{\infty} \left[\frac{w(x-h) - w(x)}{h} - w'(x) \right]^2 dx = 0.$$

Because the summands are independent,

$$(4.9) \quad \begin{aligned} & \text{Var}_{H_N} (B_{Nj} - C_{Nj}) \\ & \leq 4 \sum_{i=j(\bmod r+1)} E_{H_N} \left[\frac{w(X_i - \theta_N X_{i-r})}{w(X_i)} - 1 + 2^{-1} \theta_N X_{i-r} \frac{f'(X_i)}{f(X_i)} \right]^2 \\ & = [4 \sum_{i=j(\bmod r+1)} \theta_N^2] \int_{-\infty}^{\infty} y^2 f(y) \\ & \quad \times \int_{-\infty}^{\infty} \left[\frac{w(x - \theta_N y) - w(x)}{\theta_N y} - w'(x) \right]^2 dx dy. \end{aligned}$$

By integrating separately over $\{|y| \leq c\}$ and $\{|y| > c\}$ and making use of (4.7), (4.8) and the assumptions, we arrive at

$$(4.10) \quad \lim_{N \rightarrow \infty} \text{Var}_{H_N} (B_{Nj} - C_{Nj}) = 0.$$

On the other hand, $E(C_{Nj}) = 0$ because $(\varphi_F, 1) = 0$, and

$$(4.11) \quad \begin{aligned} E_{H_N}(B_{Nj}) &= 2 \sum_{i=j(\bmod r+1)} E_{H_N} \left[\frac{w(X_i - \theta_N X_{i-r})}{w(X_i)} - 1 \right] \\ &= [- \sum_{i=j(\bmod r+1)} \theta_N^2] \int_{-\infty}^{\infty} y^2 f(y) \\ & \quad \times \int_{-\infty}^{\infty} \left[\frac{w(x - \theta_N y) - w(x)}{\theta_N y} \right]^2 dx dy. \end{aligned}$$

From this a simple argument shows that

$$(4.12) \quad \lim_{N \rightarrow \infty} E_{H_N}(B_{Nj}) = -(4(r+1))^{-1} \theta^2 \|F^{-1}\|^2 \|\varphi_F\|^2.$$

The lemma is a consequence of (4.10) and (4.12).

LEMMA 4.2. *Suppose that f is absolutely continuous and $\varphi_F, F^{-1} \in L_2[0, 1]$. Then for any $\varepsilon > 0$,*

$$(4.13) \quad \lim_{N \rightarrow \infty} \max_{1 \leq i \leq N} P_{H_N} \left[\left| \frac{f(X_i - \theta_N X_{i-r})}{f(X_i)} - 1 \right| > \varepsilon \right] = 0.$$

PROOF. Since f is absolutely continuous,

$$(4.14) \quad \begin{aligned} E_{H_N} \left| \frac{f(X_i - \theta_N X_{i-r})}{f(X_i)} - 1 \right| \\ \leq \int_{-\infty}^{\infty} f(y) \text{sgn}(\theta_N y) \int_{-\infty}^{\infty} \int_0^{\theta_N y} |f'(x-t)| dt dx dy. \end{aligned}$$

Applying first Fubini's theorem to the inner pair of integrals and then the Cauchy-Schwarz inequality yields

$$(4.15) \quad E_{H_N} \left| \frac{f(X_i - \theta_N X_{i-r})}{f(X_i)} - 1 \right| \leq N^{-1} |\theta| \|\varphi_F\| \|F^{-1}\|.$$

The lemma follows.

LEMMA 4.3. *Suppose that f is absolutely continuous and $\varphi_F, F^{-1} \in L_2[0, 1]$. Then for any $\varepsilon > 0$,*

$$(4.16) \quad \lim_{N \rightarrow \infty} P_{H_N} [|L_N - C_N + 2^{-1}\theta^2|\varphi_F|^2|F^{-1}|^2 > \varepsilon] = 0 .$$

Moreover L_N is asymptotically normal $(-2^{-1}\theta^2|\varphi_F|^2|F^{-1}|^2, \theta^2|\varphi_F|^2|F^{-1}|^2)$ under H_N and the alternatives K_N are contiguous to H_N .

PROOF. Let $\beta^2 = \theta^2|\varphi_F|^2|F^{-1}|^2$. Since C_{Nj} is a sum of independent identically distributed random variables and $E_{H_N}(C_{Nj}) = 0, \text{Var}_{H_N}(C_{Nj}) = (r + 1)^{-1}\beta^2$, the distribution of C_{Nj} is asymptotically normal $(0, (r + 1)^{-1}\beta^2)$. Then, because of Lemma 4.1, B_{Nj} is asymptotically normal $(-(4(r + 1))^{-1}\beta^2, (r + 1)^{-1}\beta^2)$. From this fact, Lemma 4.2, and a slight extension of LeCam's second lemma as given in [8], we conclude that for any $\varepsilon > 0$,

$$(4.17) \quad \lim_{N \rightarrow \infty} P_{H_N} [|L_{Nj} - B_{Nj} + (4(r + 1))^{-1}\beta^2| > \varepsilon] = 0, \quad 1 \leq j \leq r + 1 .$$

The assertions of the lemma follow from (4.17), Lemma 4.1, LeCam's first lemma, and a central limit theorem for r -dependent random variables (see [6]).

PROOF OF THEOREM 2.2. We prove first the weak convergence of all finite-dimensional distributions, then tightness. Let D_N be an arbitrary linear combination

$$(4.18) \quad D_N = \sum_{i=1}^p \alpha_i S_N^\varphi(\lambda_i) + \alpha_{p+1} L_N .$$

By (3.41) and (4.16), $D_N + 2^{-1}\alpha_{p+1}\theta^2|\varphi_F|^2|F^{-1}|^2$ has the same asymptotic distribution under H_N as

$$(4.19) \quad E_N = \sum_{i=1}^p \alpha_i T_N(\lambda_i) + \alpha_{p+1} C_N \\ = N^{-\frac{1}{2}} \left[\sum_{j \neq k} \sum d_{|j-k|} \varphi(U_j)\varphi(U_k) + \alpha_{p+1} \theta \sum_{j=1}^N F^{-1}(U_{j-r})\varphi_F(U_j) \right]$$

where d_m is defined as in (3.31) and $F^{-1}(U_0) = F^{-1}(U_{-1}) = \dots = F^{-1}(U_{1-r}) = 0$. If

$$(4.20) \quad c^2 = \sum_{i=1}^p \sum_{j=1}^p \alpha_i \alpha_j [\min(\lambda_i/\pi, \lambda_j/\pi) - \lambda_i \lambda_j / \pi^2] \\ + \alpha_{p+1}^2 \theta^2 |F^{-1}|^2 |\varphi_F|^2 + 4\alpha_{p+1} \theta d_r(\varphi, \varphi_F)(\varphi, F^{-1}) ,$$

an argument analogous to that used in proving Theorem 3.1 shows that E_N is asymptotically normal $(0, c^2)$ under H_N and consequently, D_N is asymptotically normal $(-2^{-1}\alpha_{p+1}\theta^2|\varphi_F|^2|F^{-1}|^2, c^2)$ under H_N . In particular, therefore, the joint asymptotic distribution of $(\sum_{i=1}^p \alpha_i S_N^\varphi(\lambda_i), L_N)$ under H_N is bivariate normal (μ, Σ) with

$$(4.21) \quad \mu = (0, -2^{-1}\theta^2|\varphi_F|^2|F^{-1}|^2)' \\ \Sigma = (\sigma_{ij})$$

where

$$(4.22) \quad \sigma_{11} = \sum_{i=1}^p \sum_{j=1}^p \alpha_i \alpha_j [\min(\lambda_i/\pi, \lambda_j/\pi) - \lambda_i \lambda_j / \pi^2] \\ \sigma_{22} = \theta^2 |\varphi_F|^2 |F^{-1}|^2 \\ \sigma_{12} = 2\theta d_r(\varphi, \varphi_F)(\varphi, F^{-1}) = \sum_{i=1}^p \alpha_i \mu_\theta(\lambda_i) .$$

It follows from LeCam's third lemma (see [8]) that the asymptotic distribution of $\sum_{i=1}^p \alpha_i [S_N^\varepsilon(\lambda_i) - \mu_\theta(\lambda_i)]$ under K_N coincides with the distribution of $\sum_{i=1}^p \alpha_i Z(\lambda_i)$. Thus, finite-dimensional distributions of $S_N^\varepsilon(\lambda)$ converge as asserted under K_N .

To establish tightness of the distributions under K_N of the $\{S_N^\varepsilon(\lambda)\}$, it is enough to show (see [3]) that for every $\varepsilon > 0$, $\eta > 0$, there exist $\delta \in (0, \pi)$ and integer N_0 such that

$$(4.23) \quad P_{K_N}[\sup_{|\lambda - \mu| < \delta} |S_N(\lambda) - S_N(\mu)| > \varepsilon] \leq \eta \quad \text{for } N \geq N_0.$$

The analogous statement under H_N was implied by (3.43) and (3.44). Since K_N is contiguous to H_N according to Lemma 4.3, (4.23) follows. This completes the proof of Theorem 2.2.

REMARK 4.1. An analogue of Theorem 2.2 for the process $V_N(\lambda)$ defined in (1.3) is obtained by replacing φ with F^{-1} in both assumptions and conclusions. The result may be justified as follows. Suppose without loss of generality that $E_{H_N}(X_i) = 0$ and $\text{Var}_{H_N}(X_i) = 1$. Then $s_N^2 = N^{-1} \sum_{i=1}^N (X_i - \bar{X})^2$ converges in probability to 1 under K_N as well as under H_N because of contiguity. Let $V_{N1}(\lambda)$ be the process defined in (3.45) and let

$$(4.24) \quad \nu_\theta(\lambda) = \frac{2^{1/2} \theta \sin(r\lambda)}{\pi r}.$$

The proof of Theorem 2.2, the observation above concerning s_N^2 , and the fact $(\varphi_F, F^{-1}) = 1$ imply that under K_N , $V_{N1}(\lambda) - \nu_\theta(\lambda)$ converges in distribution in $C[0, \pi]$ to the Brownian bridge.

Moreover, for any $\varepsilon > 0$,

$$(4.25) \quad \lim_{N \rightarrow \infty} P_{K_N}[\sup_\lambda |V_N(\lambda) - V_{N1}(\lambda)| > \varepsilon] = 0,$$

because the corresponding result under H_N was established in Remark 3.1 and K_N is contiguous to H_N by Lemma 4.3. Hence $V_N(\lambda) - \nu_\theta(\lambda)$ and $V_{N1}(\lambda) - \nu_\theta(\lambda)$ have the same asymptotic distribution on $C[0, \pi]$ under K_N .

PROOF OF THEOREM 2.3. This follows from Theorems 2.1, 3.1, and 4.1 of Anděl [2], applied to the result in Theorem 2.2 of this paper.

REMARK 4.2. Consider the statistic W_N defined in (2.15). Under H_N and the assumption $\varphi \in L_2[0, 1]$, it follows from Lemma 3.2, Lemma 3.3, and a central limit theorem for r -dependent random variables (Diananda [6]) that the asymptotic distribution of W_N is $N(0, 1)$. Under K_N and the additional assumptions that f is absolutely continuous, $\varphi_F, F^{-1} \in L_2[0, 1]$, the asymptotic distribution of W_N is normal $(\theta \|\varphi_F\| \|F^{-1}\|, 1)$. This is proved by contiguity arguments akin to those used for Theorem 2.2. Consequently the asymptotic power of the test which rejects H_N whenever $W_N \geq \Phi^{-1}(1 - \alpha)$ is

$$(4.26) \quad \beta_w(\alpha, \theta, f) = 1 - \Phi[\Phi^{-1}(1 - \alpha) - \theta \|\varphi_F\| \|F^{-1}\|],$$

as asserted in (2.18).

On the other hand, Lemma 4.3 and LeCam's third lemma imply that (4.26) is also the asymptotic power of the size α likelihood ratio test, provided $\theta > 0$. Hence, W_N generates an asymptotically most powerful test at level α for H_N versus K_N when $\theta > 0$.

PROOF OF THEOREM 2.4. The validity of (2.17) is established by the argument of Theorem 2.3 applied to the results concerning $V_N(\lambda)$ noted in Remark 4.1. Then the efficiency (2.19) is immediate from (2.13) and (2.17). The minimization in (2.20) has been carried out in a different context by Aiyar [1]. For details of a limit computation similar to (2.22), see Hájek and Šidák [8] page 272.

REMARK 4.3. Consider now the statistic

$$(4.27) \quad J_N = (N^{\frac{1}{2}} \|\varphi\|^2)^{-1} \sum_{k-j=r}^N a_{N^\varphi}(R_{Nj}, R_{Nk}).$$

An argument like that sketched in Remark 4.2 for W_N shows that the asymptotic power of the test which rejects H_N whenever $J_N \geq \Phi^{-1}(1 - \alpha)$ is

$$(4.28) \quad \beta_J(\alpha, \theta, f) = 1 - \Phi[\Phi^{-1}(1 - \alpha) - (\varphi, \varphi_F)(\varphi, F^{-1})\|\varphi\|^{-2}].$$

Comparison of (4.26) and (4.28) gives

$$(4.29) \quad e(\alpha; J_N, W_N) = \rho_1^2 \rho_2^2,$$

as stated in the remarks following Theorem 2.4.

PROOF OF THEOREM 2.5. It is sufficient to show that

$$(4.30) \quad \lim_{N \rightarrow \infty} E_{H_N}[\sup_\lambda |S_{N^\varphi}(\lambda) - S_N(\lambda)|] = 0,$$

because contiguity of K_N to H_N then implies that for any $\varepsilon > 0$,

$$(4.31) \quad \lim_{N \rightarrow \infty} P_{K_N}[\sup_\lambda |S_{N^\varphi}(\lambda) - S_N(\lambda)| > \varepsilon] = 0.$$

To prove (4.30), let

$$(4.32) \quad \begin{aligned} \gamma_{N3}(t) &= N^{-\frac{1}{2}} \sum_{k-j=t}^N a_{N^\varphi}(R_{Nj}, R_{Nk}) \\ \gamma_{N4}(t) &= N^{-\frac{1}{2}} \sum_{k-j=t}^N a_N(R_{Nj}, R_{Nk}). \end{aligned}$$

Since

$$(4.33) \quad S_N(\lambda) = (2^{\frac{1}{2}} \pi \|\varphi\|^2)^{-1} \sum_{|t|=1}^{N-1} t^{-1} \sin(\lambda t) \gamma_{N4}(t)$$

and a similar expression is possible for $S_{N^\varphi}(\lambda)$, we find

$$(4.34) \quad \begin{aligned} E_{H_N}[\sup_\lambda |S_{N^\varphi}(\lambda) - S_N(\lambda)|] \\ \leq (2^{\frac{1}{2}} \pi \|\varphi\|^2)^{-1} \sum_{|t|=1}^{N-1} t^{-1} E_{H_N} |\gamma_{N3}(t) - \gamma_{N4}(t)|. \end{aligned}$$

Let

$$(4.35) \quad \tilde{\gamma}_N(t) = E_{H_N}[\gamma_{N4}(t)].$$

Applying Lemma 3.2 with

$$(4.36) \quad \begin{aligned} d_{jk} &= N^{-\frac{1}{2}} && \text{if } k - j = t \\ &= 0 && \text{otherwise} \end{aligned}$$

leads to the bound

$$(4.37) \quad \begin{aligned} E_{H_N}[\tilde{\gamma}_{N3}(t) - \gamma_{N4}(t) + \tilde{\gamma}_N(t)]^2 \\ \leq K_{13}[N(N-1)]^{-1} \sum_{j \neq k}^N [a_N^e(j, k) - a_N(j, k) + \bar{a}_N]^2. \end{aligned}$$

By direct calculation,

$$(4.38) \quad |\tilde{\gamma}_N(t)| \leq N^{-\frac{3}{2}} \left| \sum_{j \neq k}^N a_N(j, k) \right|.$$

The limit in (4.30) follows from (4.34), (4.37), (4.38), (3.49), the assumptions of the theorem.

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