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ASYMPTOTIC REGIMES FOR THE PARTITION INTO COLONIES OF A BRANCHING PROCESS WITH EMIGRATION

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We consider a spatial branching process with emigration in which children either remain at the same site as their parents or migrate to new locations and then found their own colonies. We are interested in asymptotics of the partition of the total population into colonies for large populations with rare migrations. Under appropriate regimes, we establish weak convergence of the rescaled partition to some random measure that is constructed from the restriction of a Poisson point measure to a certain random region, and whose cumulant solves a simple integral equation.

1. Introduction. Imagine a spatial branching process in which the child of an individual either is a homebody, that is, remains at the same site as its parent, or migrates to a new location which has never been occupied before and then founds its own colony. We assume that the reproduction law is the same for homebodies and migrants and do not depend on the spatial location either, so this is essentially a discrete version of the Virgin Island Model of Hutzenthaler [12] when local competition between individuals is discarded; see also [13] and references therein.

The dynamics of the process are entirely determined by the pair $(\xi^{\mathrm{h}}, \xi^{\mathrm{m}})$ of integer valued random variables giving the number of homebody children and the number of migrant children of a typical individual. The special case where each child chooses to emigrate with a fixed probability $p \in (0, 1)$ and independently of the other children can be interpreted in the framework of the infinite-sites model in population genetics by identifying a spatial location with a locus on a chromosome and p with the rate of neutral mutations. This setting has motivated a number of works in the literature; see in particular [10] and [19]. In a quite different direction, we may also consider for instance the cut-off situation where there is a threshold k such that the first k children of an individual are always homebodies while the next children (if any) are forced to migrate. We may think of many other simple rules as there are no assumptions on the correlation between ξ^{h} and ξ^{m}.

We are interested in statistics of the decomposition of the entire population according to the locations of individuals, which we call the partition into colonies. This partition is naturally endowed with a genealogical tree structure which has

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been described in [4]. Recent work [5] focused on the special case where neutral mutations occur in a Galton–Watson process with a fixed reproduction law which is critical and has a finite variance. In asymptotic regimes where the population is large and the mutation rate small, we established a weak limit theorem for the tree of alleles, that is, in the present setting, the partition into colonies equipped with its genealogical structure. The limit was described in terms of the genealogical tree of a continuous state branching process in discrete time (cf. [14]) with an inverse Gaussian reproduction measure. In a related direction, we also point at recent work by Abraham and Delmas [2] on pruning Lévy continuum random trees.

In the present paper, we shall investigate more generally asymptotics of the partition into colonies (ignoring its genealogical structure) for branching processes with emigration when populations are large and migrations rare. The regimes of interest are related to the well-known limit theorems for rescaled Galton–Watson processes toward continuous state branching processes in continuous time. Our main result (Theorem 2) states that after an appropriate rescaling, the partition into colonies converges weakly to some random point measure. The latter is constructed from the restriction of a Poisson point measure to a certain random region. An important step in our analysis is that, although in general the cumulant of this limiting random measure is not explicitly known, it can be characterized as the unique solution to a rather simple integral equation.

Let us briefly present the plan of this work by explaining our approach. In Section 2, we point at the fact that the cumulant of the partition into colonies solves a certain integral equation. This equation stems from the extended branching property that is fulfilled by the partition, and is given in terms of the distribution of a pair of random variables which arise naturally in this setting. We also recall a useful identity in law which relates the preceding variables to that of passage times in certain random walks.

In Section 3, we consider a sequence of branching processes with emigration and introduce the basic assumptions. These are closely related to the classical limit theorems for rescaled Galton–Watson processes and involve Lévy processes with no negative jumps. Motivated by Section 2, we investigate limits in distribution for passage times of random walks, and point at the role of the Lévy measure of a bivariate subordinator which arises in this setting.

In Section 4, we introduce a family of random point measures which are constructed from Poisson point measures on a product space by restriction to certain random domains. The main feature is that the cumulant of such a random measure can be characterized as the unique solution to another integral equation involving the intensity measure of the underlying Poisson measure.

Our main result for limits in law of partitions into colonies is presented and proved in Section 5. Roughly, we show that the cumulants of the partitions into colonies of a sequence of branching processes with emigration converge after an appropriate rescaling to the unique solution of an equation of the type which appeared in Section 4. More precisely, it corresponds to the case where the intensity of the driving Poisson measure is given by the Lévy measure that has arisen in Section 3.

Finally, Section 6 is devoted to a few (hopefully) interesting examples, partly to demonstrate the variety of possible asymptotic behaviors. Roughly speaking, the common feature in these examples is that the Galton–Watson process for which spatial locations of individuals are ignored has a fixed distribution. We shall consider different natural possibilities for selecting migrants children amongst the progeny of an individual, which will yield different limiting partitions into colonies. In the case corresponding to rare neutral mutations in the infinite alleles branching process, the limiting random partition can be described in terms of certain Poisson–Kingman partitions which have been considered by Pitman [17].

2. Preliminaries on partitions into colonies. In this section, we briefly introduce notation and present some basic properties for Galton–Watson processes with emigration and the induced partitions into colonies. The material is essentially adapted from [4] and [5] to which we refer for details, with the exception of Lemma 1 which is new.

Roughly speaking, we consider a spatial haploid population model with discrete nonoverlapping generations where each individual begets independently of the others, according to a fixed reproduction law which is independent of the location of that individual. We do not specify geometrical details of the space where individuals live as this would be irrelevant for the study; the only implicit assumption is that this space is infinite. A child can either stay at the same site as its parent or migrate to a new site which has never been occupied before and then found its own colony. This child is called a *homebody* in the first case, and a *migrant* in the second. For the sake of simplicity, we shall assume in this work that at the initial time each ancestor lives in a different location, although arbitrary initial conditions could be dealt with more generally. The law of this model is thus entirely determined by the number of ancestors and a pair of integer-valued random variables $(\xi^{\mathrm{h}}, \xi^{\mathrm{m}})$ which should be thought of as the number of homebody children and the number of migrant children of a typical individual. For every $a \in \mathbb{N}$, we use the notation $\mathbb{P}_a$ for the probability measure under which this model starts from a ancestors.

If spatial locations are discarded, then the total number of individuals per generation clearly forms a standard Galton–Watson process with reproduction law given by the distribution of $\xi = \xi^{\mathrm{h}} + \xi^{\mathrm{m}}$. We always assume that this Galton–Watson process is critical or sub-critical, namely, $\mathbb{E}(\xi) \leq 1$, and implicitly exclude the degenerate case where $\xi \equiv 1$, so the population becomes eventually extinct a.s. The main object of interest in this work is the *partition into colonies*, which we represent as a random discrete measure

$$\mathcal{P} = \sum_{j=1}^{\gamma} \delta_{C_j}.$$

Here γ is the total number of colonies (that is occupied sites) and C_j denotes the total number of individuals that lived at the jth colony. Observe that the first moment of $\mathcal{P}$ coincides with the total population of the Galton–Watson process, namely,

$$\zeta := \sum_{j=1}^{\gamma} C_j = a + \sum_{k=1}^{\zeta} \xi_k,$$

where $\xi_k = \xi_k^{\mathrm{h}} + \xi_k^{\mathrm{m}}$ stands for the number of children of the kth individual for some enumeration procedure, and that the mass of $\mathcal{P}$ is just the number of colonies

$$\gamma = a + \sum_{k=1}^{\zeta} \xi_k^{\mathrm{m}}.$$

We denote the cumulant of partition into colonies when there is a single ancestor by

$$K(f) = -\ln \mathbb{E}_1(\exp(-\langle \mathcal{P}, f \rangle)),$$

where $f : \mathbb{N} \to \mathbb{R}_+$ stands for a generic function and

$$\langle \mathcal{P}, f \rangle = \sum_{j=1}^{\gamma} f(C_j).$$

We also point out from the branching property that for an arbitrary number of ancestors $a \in \mathbb{N}$ we have

$$\mathbb{E}_a(\exp(-\langle \mathcal{P}, f \rangle)) = \exp(-aK(f)),$$

hence the cumulant K characterizes the law of $\mathcal{P}$ under $\mathbb{P}_a$ for any $a \geq 1$.

The starting point of our analysis relies on the fact that this cumulant is determined in terms of the distribution of a pair of random variables which appear naturally in the branching process with emigration. Specifically, imagine for a while a variation of the model starting from a single ancestor in which migrants are sterilized (i.e., they have no offspring). We denote by C the total number of individuals that lived at the same site as the ancestor and by M the number of sterilized migrant children. In other words, C is the size of the colony generated by the ancestor and M the number of colonies which have been founded by migrant children of the ancestral colony.

LEMMA 1. *For every function $f:\mathbb{N}\to\mathbb{R}_+$, the cumulant $K(f)$ of $\mathcal{P}$ is the unique solution $\lambda \geq 0$ to the equation*

$$\mathrm{e}^{-\lambda} = \mathbb{E}_1\bigl(\exp\bigl(-f(C) - \lambda M\bigr)\bigr).$$

PROOF. This stems from the branching property which is inherited by the partition into colonies. More precisely, we work under $\mathbb{P}_1$ and decompose the total population into the ancestral colony and families generated by the migrant children of that colony. Because the descent of each migrant child has the same distribution as the initial spatial Galton–Watson process, independently of the other migrant children and of the homebody offspring of the ancestor, this yields

$$\begin{aligned}\exp(-K(f)) &= \mathbb{E}_1(\exp(-\langle\mathcal{P}, f\rangle)) \\ &= \mathbb{E}_1\bigl(\exp(-f(C))\mathbb{E}_M(\exp(-\langle\mathcal{P}, f\rangle))\bigr) \\ &= \mathbb{E}_1\bigl(\exp\bigl(-f(C) - K(f)M\bigr)\bigr).\end{aligned}$$

We refer to Chauvin [6] for a rigorous formulation of the extended branching property of Galton–Watson processes at stopping lines that we have used above, and also to [4] for an alternative argument based on the strong Markov property of random walks.

Uniqueness of the solution follows from the following observation. Suppose first that $f(C) \not\equiv 0$. By Hölder's inequality, the map

$$\lambda \to \lambda + \ln \mathbb{E}_1\bigl(\exp\bigl(-f(C) - \lambda M\bigr)\bigr)$$

is convex and its value at $\lambda = 0$ is negative. Hence it can take the value 0 for a single value of $\lambda > 0$ at most. When $f(C) \equiv 0$, the equation reduces to

$$\mathrm{e}^{-\lambda} = \mathbb{E}_1(\exp(-\lambda M)).$$

Recall that the Galton–Watson process is critical or sub-critical, so $\mathbb{E}_1(M) \leq 1$ according to Corollary 1 of [5]. It is well known that this ensures uniqueness of the solution to the preceding equation. $\square$

Lemma 1 provides an implicit characterization of the law of the partition into colonies through that of the pair of random variables (C, M). In turn, the latter can be conveniently described in terms of a pair of random walks. This has its root in a key observation for Galton–Watson processes that goes back to Harris [11], and will have an important role here for the analysis of asymptotic behaviors. Specifically, consider

$$S^{\mathrm{h}}_k = \xi^{\mathrm{h}}_1 + \cdots + \xi^{\mathrm{h}}_k - k \quad \text{and} \quad S^{\mathrm{m}}_k = \xi^{\mathrm{m}}_1 + \cdots + \xi^{\mathrm{m}}_k, \qquad k \in \mathbb{Z}_+.$$

Next define for every integer $j \geq 0$ the first passage time

$$\tau_j = \inf\{k : S^{\mathrm{h}}_k = -j\}.$$

We lift the following useful identity from Lemma 3 in [5].

LEMMA 2. *The pair* $(\tau_1, S^{\mathrm{m}}_{\tau_1})$ *has the same law as* (C, M).

We refer to Theorem 1(ii) in [4] or to Proposition 1 in [5] for an explicit formula for this distribution which is obtained by a combinatorial argument and extends the well-known result of Dwass [8] for the total population of Galton–Watson processes and passage times of downward skip free random walks. In this direction we also mention that the sequence of the atoms of the partition into colonies has the same distribution under $\mathbb{P}_a$ as

$$(\tau_j - \tau_{j-1} : 1 \leq j \leq \eta_a) \qquad \text{with } \eta_a = \inf\{j : j - S^{\mathrm{m}}_{\tau_j} = a\}. \tag{1}$$

This follows from Section 2 in [4]; see in particular Lemma 4 there. The interested reader may wish to provide an alternative proof of Lemma 1 based on this representation and using the strong Markov property for random walks in place of the extended branching property for Galton–Watson processes.

3. Random walks, Lévy processes and passage times. As our main goal is to investigate limits of partitions into colonies, Lemmas 1 and 2 suggest that we should study asymptotics of first passage times in random walks, which is the purpose of this section. We first introduce the asymptotic regimes that we shall consider later on, and develop some of their consequences for passage times of certain random walks and Lévy processes. Our starting point is a classical result of convergence for rescaled Galton–Watson processes toward continuous state branching processes (in short, CSBP) that we now recall. We refer to the monograph [7] by Duquesne and Le Gall for a complete account, including some terminology which will not be defined here.

We consider for each integer n a sequence $(\xi_{n,k} : k \in \mathbb{N})$ of i.i.d. copies of some $\mathbb{Z}_+$-valued random variable with mean at most 1 which should be thought of as the number of children of a typical individual in the nth population model. The basic assumption is that there exists a sequence $\alpha(n)$ with $\lim_{n\to\infty} \alpha(n)/n = \infty$ and a process $(X_t, t \geq 0)$ such that

$$\frac{1}{n}\big(\xi_{n,1} + \cdots + \xi_{n,[\alpha(n)t]} - [\alpha(n)t]\big) \Longrightarrow X_t \tag{2}$$

for some (and then all) $t > 0$, where the notation $\Longrightarrow$ refers to convergence in distribution as $n \to \infty$. More precisely, (2) then can be reinforced to weak convergence on the space of càdlàg processes endowed with Skorohod's topology; see, for instance, Theorem 16.4 in [15]. Moreover, the limit $X = (X_t : t \geq 0)$ is necessarily a Lévy process which has no negative jumps and does not drift to $+\infty$, that is $\mathbb{E}(X_t) \in [-\infty, 0]$ [this follows from the requirement that $\mathbb{E}(\xi_{n,k}) \leq 1$ for every n].

The law of the Lévy process X is characterized by its Laplace exponent $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is defined by

$$\mathbb{E}(\exp(-qX_t)) = \exp(t\psi(q)), \qquad q \geq 0.$$

We shall further assume that X has infinite variation, or equivalently that

$$\lim_{q\to\infty} q^{-1}\psi(q) = \infty.$$

Next, consider a sequence $(a(n) : n \in \mathbb{N})$ with $a(n)/n \to a$ for some $a > 0$, and denote by $Z^{(n)}$ a Galton–Watson process started from $a(n)$ ancestors and with reproduction law given by the distribution of $\xi_{n,k}$. Then we have

$$\frac{1}{n} Z^{(n)}_{[\alpha(n)t/n]} \Longrightarrow Z_t,$$

where $(Z_t : t \geq 0)$ is a CSBP started from $Z_0 = a$ and with branching mechanism ψ; see, for example, Theorem 2.1.1 in [7].

We now turn our attention to the spatial case where some children of a parent may emigrate. That is we consider an array $((\xi^{\mathrm{h}}_{n,k}, \xi^{\mathrm{m}}_{n,k}) : k, n \in \mathbb{N})$ of random variables with values in $\mathbb{Z}^2_+$, where $\xi^{\mathrm{h}}_{n,k}$ should be thought of as the number of homebody children and $\xi^{\mathrm{m}}_{n,k}$ as the number of migrant children of the kth individual for the nth population model; in particular $\xi_{n,k} = \xi^{\mathrm{h}}_{n,k} + \xi^{\mathrm{m}}_{n,k}$. We assume that for each fixed integer n, the sequence $((\xi^{\mathrm{h}}_{n,k}, \xi^{\mathrm{m}}_{n,k}) : k \in \mathbb{N})$ is i.i.d., and just as in the preceding section, we construct a pair of random walks

$$S^{\mathrm{h}}_{n,k} = \xi^{\mathrm{h}}_{n,1} + \cdots + \xi^{\mathrm{h}}_{n,k} - k \quad \text{and} \quad S^{\mathrm{m}}_{n,k} = \xi^{\mathrm{m}}_{n,1} + \cdots + \xi^{\mathrm{m}}_{n,k}, \qquad k \in \mathbb{Z}_+.$$

Observe that the random walk $S^{\mathrm{m}}_{n,\cdot}$ is nondecreasing while $S^{\mathrm{h}}_{n,\cdot}$ is downward skip free, that is, its increments belong to $\{-1, 0, 1, 2, \ldots\}$. We now reinforce (2) by assuming that the Lévy process X can be decomposed as a sum

$$X_t = X^{\mathrm{h}}_t + X^{\mathrm{m}}_t,$$

where $((X^{\mathrm{h}}_t, X^{\mathrm{m}}_t) : t \geq 0)$ is a bivariate Lévy process, in such a way that for some (and then all) $t > 0$

$$\frac{1}{n}\big(S^{\mathrm{h}}_{n,[\alpha(n)t]}, S^{\mathrm{m}}_{n,[\alpha(n)t]}\big) \Longrightarrow (X^{\mathrm{h}}_t, X^{\mathrm{m}}_t). \tag{3}$$

We point out that again (3) is automatically reinforced to weak convergence in the sense of Skorohod by an appeal to Theorem 16.4 in [15].

We stress that necessarily, the Lévy process X^{h} has no negative jumps, infinite variation, and does not drift to $+\infty$, and that X^{m} must be a subordinator (i.e., an increasing Lévy process); the two may or not be correlated. We denote the bivariate Laplace exponent by Ψ, that is,

$$\mathbb{E}\big(\exp -(qX^{\mathrm{h}}_t + rX^{\mathrm{m}}_t)\big) = \exp(t\Psi(q, r)), \qquad q, r \geq 0.$$

In particular, there is the identity

$$\psi(q) = \Psi(q, q), \qquad q \geq 0;$$

note also that our assumptions force $\Psi(q, q) \geq 0$ whereas $\Psi(0, r) \leq 0$.

Next, we consider the first passage process

$$T_x = \inf\{t \geq 0 : X_t^{\mathrm{h}} < -x\}, \qquad x \geq 0,$$

which is a subordinator whose Laplace exponent is given by the inverse function of $\Psi(\cdot, 0)$; see Theorem VII.1 in [3]. Using $T_\cdot$ as a time-substitution, we also introduce the compound process

$$Y_x = X_{T_x}^{\mathrm{m}}, \qquad x \geq 0.$$

The distribution of the pair (T, Y) can be described as follows.

LEMMA 3. (i) *The process*

$$\big((T_x, Y_x) : x \geq 0\big)$$

is a bivariate subordinator.

(ii) *Its Laplace exponent* $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ *defined by*

$$\mathbb{E}\big(\exp(-qT_x - rY_x)\big) = \exp(-x\Phi(q, r)), \qquad q, r \geq 0,$$

is determined as the unique solution to the equation

$$\Phi(\Psi(q, r), r) = q.$$

(iii) *There exists a unique measure* Λ *on* $\mathbb{R}_+^2 \backslash \{(0, 0)\}$ *with* $\int(1 \wedge (x_1 + x_2))\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2) < \infty$ *such that*

$$\Phi(q, r) = \int (1 - \mathrm{e}^{-qx_1 - rx_2})\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2).$$

In other words, the bivariate subordinator (T_x, Y_x) *has no drift and Lévy measure* Λ.

(iv) *Finally, we also have*

$$\mathbb{E}(Y_1) = \int x_2 \Lambda(\mathrm{d}x_1\,\mathrm{d}x_2) \leq 1.$$

PROOF. The proof is essentially a variation of that of Theorem VII.1 in [3]. The passage times T_x are stopping times in the natural filtration of the bivariate Lévy process $(X^{\mathrm{h}}, X^{\mathrm{m}})$ which are a.s. finite and such that $X_{T_x}^{\mathrm{h}} = -x$ (by the absence of negative jumps for X^{h} and the fact that X^{h} does not drift to $+\infty$). The strong Markov property immediately implies that (T_x, Y_x) is has independent and stationary increments; further this process has clearly càdlàg nondecreasing sample paths in each coordinate. In other words, it is a bivariate subordinator.

The Laplace exponent Φ is then determined by an application of Doob's sampling theorem to the martingale

$$\exp\big(-(qX_t^{\mathrm{h}} + rX_t^{\mathrm{m}}) - t\Psi(q, r)\big), \qquad t \geq 0$$

(recall that $X^{\mathrm{h}}_{T_x} = -x$ a.s.). Observe that our assumptions ensure that for every $r \geq 0$, the function $\Psi(\cdot, r)$ is continuous and convex with $\Psi(0, r) \leq 0$ and $\Psi(\infty, r) = \infty$, so the equation $\Phi(\Psi(q, r), r) = q$ determines Φ on $\mathbb{R}^2_+$.

It remains to check that both subordinators have no drift. We know from Corollary VII.5 in [3] and the fact that X^{h} has unbounded variation that $\lim_{q\to\infty} \Psi(q, 0)/q = \infty$. This implies that $\lim_{q\to\infty} \Phi(q, 0)/q = 0$ and hence $T.$ has no drift. On the other hand, the Lévy–Itô decomposition enables us to express the subordinator X^{m} as the sum of a linear drift and a pure-jump process. The time-substitution by T_x thus yields that Y_x can be expressed as the sum of two pure-jump processes, and hence its drift coefficient must be zero.

The penultimate displayed identity of the statement is just the celebrated Lévy–Khintchine formula. Finally, the assumption that the Lévy process X does not drift to $+\infty$ is equivalent to requiring that its first moment exists and is nonpositive, $\mathbb{E}(X_t) \leq 0$. It follows that $X_t = X^{\mathrm{h}}_t + X^{\mathrm{m}}_t$ is a super-martingale, and since T_x is a stopping time, we deduce from Doob's sampling theorem that for every $x, t \geq 0$,

$$\mathbb{E}(X^{\mathrm{m}}_{t\wedge T_x}) \leq \mathbb{E}(-X^{\mathrm{h}}_{t\wedge T_x}) \leq x,$$

where the second inequality is due to the definition of T_x and the absence of negative jumps for X^{h}. Then it suffices to let $t \to \infty$ to get by monotone convergence that $\mathbb{E}(Y_x) \leq x$, which in turn yields our last claim by an application of the Lévy–Itô decomposition of the subordinator Y and the first-moment formula for Poisson measures. $\square$

Lemmas 1 and 2 suggest that the asymptotic behavior of the distribution of the partition into colonies should be related to that of the first passage times of the downward skip free random walk $S^{\mathrm{h}}_{n,\cdot}$,

$$\tau_{n,j} = \inf\{k : S^{\mathrm{h}}_{n,k} = -j\}, \qquad j \in \mathbb{N}.$$

In this direction, we point at the following limit theorem.

COROLLARY 1. *In the regime* (3), *we have for every bounded continuous function* $g : \mathbb{R}^2_+ \to \mathbb{R}$ *with* $g(x_1, x_2) = O(x_1 + x_2)$ *as* $x_1 + x_2 \to 0$ *that*

$$\lim_{n\to\infty} n\mathbb{E}(g(\alpha(n)^{-1}\tau_{n,1}, n^{-1}S^{\mathrm{m}}_{n,\tau_{n,1}})) = \int_{\mathbb{R}^2_+} g(x_1, x_2)\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2),$$

where the Lévy measure Λ *has been defined in Lemma* 3.

PROOF. It follows from the assumptions (3) and routine arguments [recall that $S^{\mathrm{h}}_{n,\cdot}$ is downwards skip free and that (3) can be reinforced to weak convergence of càdlàg processes] that for an arbitrary $x > 0$

$$(\alpha(n)^{-1}\tau_{n,[nx]}, n^{-1}S^{\mathrm{m}}_{n,\tau_{n,[nx]}}) \Longrightarrow (T_x, Y_x). \tag{4}$$

On the other hand, one readily deduces from the strong Markov property for random walks that for each fixed n,

$$(\tau_{n,k}, S^{\mathrm{m}}_{n,\tau_{n,k}}), \qquad k \geq 0,$$

is a random walk with nondecreasing coordinates. We complete the proof by taking $x = 1$ in (4), and appealing to (i) in Corollary 15.16 in [15] and Lemma 3. □

4. A family of random point measures. In this section, we introduce and develop some properties of a class of random point measures which will arise later on as limits for partitions into colonies. The idea stems from the representation (1) of the sequence of the atoms of the partition into colonies. Indeed, as by the strong Markov property, the increments $\tau_j - \tau_{j-1}$ of the first passage time process in a downward skip free random walk are i.i.d., the combination of (1) and the law of rare events suggest that if a limiting partition exists, then it should be described in terms of a Poisson random measure restricted to a random domain with a boundary given by a first passage time.

Our basic analytic datum is some sigma-finite measure on $\mathbb{R}_+^2$ with no mass at $(0, 0)$ that will be denoted by Λ. Although in subsequent sections Λ will be chosen to be the Lévy measure that arises in Lemma 3, this specification is not required in the present section (of course, the notation introduced here is coherent with that of Section 3). We write $\Lambda^{(1)}$ and $\Lambda^{(2)}$ for the restrictions to $(0, \infty)$ of the two marginals of Λ and assume that $\Lambda^{(1)}$ is sigma-finite and

$$\int_{(0,\infty)} x \Lambda^{(2)}(\mathrm{d}x) \leq 1. \tag{5}$$

We consider a Poisson measure $\mathcal{N}$ on $(0, \infty) \times \mathbb{R}_+^2$ with intensity measure $\mathrm{d}t \otimes \Lambda(\mathrm{d}x_1\, \mathrm{d}x_2)$, and denote by $(t, \Delta_t) = (t, \Delta_t^{(1)}, \Delta_t^{(2)})$ a generic atom of $\mathcal{N}$. Following the classical construction of Lévy and Itô, we introduce the subordinator

$$Y_t = \int_{(0,t]\times\mathbb{R}_+^2} x_2 \mathcal{N}(\mathrm{d}s\, \mathrm{d}x_1\, \mathrm{d}x_2) = \sum_{0<s\leq t} \Delta_s^{(2)}, \qquad t \geq 0.$$

Because Y has no drift and its Lévy measure $\Lambda^{(2)}$ fulfills (5), we have $\mathbb{E}(Y_1) \leq 1$. In particular, the first passage times

$$\sigma_y = \inf\{t \geq 0 : t - Y_t = y\}, \qquad y \geq 0,$$

are finite a.s.

Next, for every $t \geq 0$, we consider the point measure on $(0, \infty)$

$$\mathcal{N}_t^{(1)} = \sum_{0<s\leq t} \delta_{\Delta_s^{(1)}}.$$

Note that $\mathcal{N}_t^{(1)}$ is a Poisson random measure with intensity $t\Lambda^{(1)}$, and from the superposition property of Poisson measure, that the measure-valued process

$(\mathcal{N}_t^{(1)} : t \geq 0)$ has independent and stationary increments. The random point measures we are interested in are defined by time-substitution through the passage times σ_y

$$\mathcal{M}_y = \mathcal{N}^{(1)}_{\sigma_y} = \sum_{0<t\leq\sigma_y} \delta_{\Delta_t^{(1)}}. \tag{6}$$

In words, $\mathcal{M}_y$ is the image of the restriction of $\mathcal{N}$ to the random set $(0, \sigma_y] \times \mathbb{R}_+^2$ by the projection $(s, x_1, x_2) \to x_1$.

In the special case when the intensity measure Λ is carried by the diagonal $\{(x, x) : x > 0\}$, that is, when $\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2) = \delta_{x_1}(\mathrm{d}x_2)\Lambda^{(1)}(\mathrm{d}x_1)$, we have $\Delta_s^{(1)} = \Delta_s^{(2)}$ a.s. and the random point measure $\mathcal{M}_y$ coincides with the empirical measure of the sizes of the jumps performed by the subordinator Y during the time-interval $(0, \sigma_y]$. This case has also a natural interpretation in terms of continuous state branching processes in discrete time (see [14]). More precisely, the well-known correspondence between CSBP in discrete time and subordinators enables us to think of $\mathcal{M}_y$ as the empirical measure of the sizes of siblings in a CSBP in discrete time with reproduction intensity $\Lambda^{(1)}$ and started from an initial population of size y.

We now observe that the property of independence and stationarity of the increments for the process of point measures $(\mathcal{N}_t^{(1)} : t \geq 0)$ is preserved after the time-substitution by σ_y. This claim is essentially a variation of the well-known fact that the first passage process of a real-valued Lévy process with no negative jumps is a subordinator; see, for example, Theorem VII.1 in [3].

LEMMA 4. *The measure-valued process* $(\mathcal{M}_y : y \geq 0)$ *has independent and stationary increments.*

PROOF. Assume for a while that $\int_{(0,\infty)} (1 \wedge x)\Lambda^{(1)}(\mathrm{d}x) < \infty$, which enables us to construct

$$T_t = \langle \mathcal{N}_t^{(1)}, \mathrm{Id} \rangle = \int_{(0,t]\times\mathbb{R}_+^2} x_1 \mathcal{N}(\mathrm{d}s\,\mathrm{d}x_1\,\mathrm{d}x_2) = \sum_{0<s\leq t} \Delta_s^{(1)}, \qquad t \geq 0.$$

Plainly (T, Y) is a pure-jump Lévy process, more precisely it is a bivariate subordinator with no drift. Further, the Poisson measure $\mathcal{N}$ can be recovered from the jump process of (T, Y). The strong Markov property for Lévy processes shows that for every $y > 0$, the shifted process

$$(T', Y')_t = (T, Y)_{\sigma_y+t} - (T, Y)_{\sigma_y}, \qquad t \geq 0,$$

is independent of $((T, Y)_t : t \leq \sigma_y)$ and has the same law as (T, Y). As $\sigma_{y+y'} - \sigma_y$ coincides with the first passage time of the process $t \to t - Y'_t$ at level y', this establishes our claim.

Finally, the assumption that $\int_{(0,\infty)}(1 \wedge x)\Lambda^{(1)}(\mathrm{d}x) < \infty$ can be removed by considering the image of $\mathcal{N}$ by a mapping $(t, x_1, x_2) \to (t, \phi(x_1), x_2)$ for some appropriate bijective map ϕ (recall that the measure $\Lambda^{(1)}$ is sigma-finite). $\square$

We next point at an interesting connection between the distributions of $\mathcal{N}_t^{(1)}$ and $\mathcal{M}_y$ which is an avatar of the classical ballot theorem (compare with Corollary VII.3 in [3]).

PROPOSITION 1. *There is the identity*

$$\mathbb{P}(\mathcal{M}_y \in A, \sigma_y \in \mathrm{d}t)\,\mathrm{d}y = \frac{y}{t}\mathbb{P}\big(\mathcal{N}_t^{(1)} \in A, t - Y_t \in \mathrm{d}y\big)\,\mathrm{d}t, \qquad t > y > 0,$$

where A denotes an arbitrary measurable subset of point measures on $(0, \infty)$.

PROOF. Introduce the random set

$$\mathcal{R} = \Big\{t : t - Y_t = \max_{0\le s\le t}(s - Y_s)\Big\}.$$

The cyclic exchangeability property of the point measure $\mathcal{N}$ enables us to use a variation of the well-known combinatorial argument for the ballot theorem (see [20]) and get

$$t\mathbb{E}\big(g(t - Y_t), \mathcal{N}_t^{(1)} \in A \text{ and } t \in \mathcal{R}\big) = \mathbb{E}\big(g(t - Y_t)(t - Y_t)^+, \mathcal{N}_t^{(1)} \in A\big),$$

where $g : \mathbb{R} \to [0, \infty)$ stands for a generic measurable function. This easily yields the claim. $\square$

REMARK. In the special when the intensity measure Λ is carried by the diagonal, we have $Y_t = \langle \mathcal{N}_t^{(1)}, \mathrm{Id}\rangle$ a.s., and Proposition 1 shows that the distribution of $\mathcal{M}_y$ is essentially a mixture of laws of *Poisson–Kingman partitions* as defined by Pitman [17]. More precisely, suppose for simplicity that for every $t > 0$, the infinitely divisible variable Y_t has an absolutely continuous law with a continuous density, say $\rho_t(\cdot)$. It then follows from Proposition 1 that

$$\mathbb{P}(\mathcal{M}_y \in \cdot) = y\int_{(y,\infty)} \frac{1}{t}\mathbb{P}\big(\mathcal{N}_t^{(1)} \in \cdot \mid \langle \mathcal{N}_t^{(1)}, \mathrm{Id}\rangle = t - y\big)\rho_t(t - y)\,\mathrm{d}t,$$

where $\mathcal{N}_t^{(1)}$ is a Poisson random measure with intensity $t\Lambda^{(1)}$. Up-to a normalization, the conditional Poisson measures $\mathbb{P}(\mathcal{N}_t^{(1)} \in \cdot \mid \langle \mathcal{N}_t^{(1)}, \mathrm{Id}\rangle = a)$ which appear in the integral above belong to the family of Poisson–Kingman partitions studied in depth by Pitman [17].

Next, for every Borel function $f : (0, \infty) \to \mathbb{R}_+$ with compact support, we define the cumulant $\kappa(f) > 0$ by

$$\mathbb{E}(\exp - \langle \mathcal{M}_1, f\rangle) = \exp(-\kappa(f)),$$

with the usual notation

$$\langle \mathcal{M}_1, f\rangle = \sum_{0<t\le\sigma_1} f\bigl(\Delta_t^{(1)}\bigr).$$

Observe from Lemma 4 that for an arbitrary $y > 0$ we have more generally

$$\mathbb{E}(\exp - \langle \mathcal{M}_y, f\rangle) = \exp(-y\kappa(f)). \tag{7}$$

It is well known that the cumulant κ determines the law of $\mathcal{M}_1$, in the sense that any random measure on $(0,\infty)$ having the same cumulant as $\mathcal{M}_1$ is distributed as $\mathcal{M}_1$; see for instance Lemma 12.1 in [15]. We may now state the following basic result which provides the characteristic equation solved by the cumulant:

THEOREM 1. *For every Borel function $f:[0,\infty)\to\mathbb{R}_+$ with compact support in $(0,\infty)$, the equation*

$$\lambda = \int_{\mathbb{R}_+^2} \bigl(1 - \exp\bigl(-f(x_1) - \lambda x_2\bigr)\bigr)\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2)$$

has a unique solution in $[0,\infty)$ which is given by $\lambda = \kappa(f)$.

PROOF. For any random time $R \ge 1$, we see from elementary properties of Poisson random measures that $\mathcal{N}_R^{(1)} = \mathcal{N}_1^{(1)} + \tilde{\mathcal{N}}_{R-1}^{(1)}$ where $(\tilde{\mathcal{N}}_t : t \ge 0)$ is a process of point measures which is independent of the restriction of $\mathcal{N}$ to $[0,1]\times\mathbb{R}_+^2$ and has the same distribution as $(\mathcal{N}_t : t\ge 0)$. We then note that the first passage time σ_1 is bounded from below by 1, and more precisely there is the identity

$$\sigma_1 = 1 + \tilde{\sigma}(Y_1),$$

where

$$\tilde{\sigma}(y) = \inf\{t \ge 0 : t - \tilde{Y}_t = y\} \quad\text{and}\quad \tilde{Y}_t = \int_{[0,t]\times\mathbb{R}_+^2} x_2 \tilde{\mathcal{N}}(\mathrm{d}s\,\mathrm{d}x_1\,\mathrm{d}x_2).$$

Applying the preceding observation, we thus have

$$\mathcal{M}_1 = \mathcal{N}_{\sigma_1}^{(1)} = \mathcal{N}_1^{(1)} + \tilde{\mathcal{N}}_{\tilde{\sigma}(Y_1)}^{(1)}, \tag{8}$$

so we can deduce from (7) that

$$\exp(-\kappa(f)) = \mathbb{E}\bigl(\exp - \bigl(\bigl\langle \mathcal{N}_1^{(1)}, f\bigr\rangle + Y_1\kappa(f)\bigr)\bigr).$$

From the very definitions of $\mathcal{N}^{(1)}$ and Y_1, we can rewrite the preceding identity as

$$\begin{aligned}\exp(-\kappa(f)) &= \mathbb{E}\biggl(\exp\biggl(-\sum_{0<t\le 1}\bigl(f\bigl(\Delta_t^{(1)}\bigr) + \kappa(f)\Delta_t^{(2)}\bigr)\biggr)\biggr)\\ &= \exp\biggl(-\int_{\mathbb{R}_+^2}\bigl(1 - \exp\bigl(-f(x_1) - \kappa(f)x_2\bigr)\bigr)\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2)\biggr),\end{aligned}$$

where the last line is Campbell's identity. Thus $\kappa(f)$ solves (6).

Uniqueness is now easy. Indeed the map

$$F:\lambda \to \lambda - \int_{\mathbb{R}_+^2} \big(1 - \exp\big(-f(x_1) - \lambda x_2\big)\big)\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2)$$

has derivative

$$F'(\lambda) = 1 - \int_{\mathbb{R}_+^2} x_2 \exp\big(-f(x_1) - \lambda x_2\big)\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2)$$

which is positive due to (5). □

We now conclude this section by discussing a simple example. Suppose that Λ has support on the axes, that is,

$$\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2) = \Lambda^{(1)}(\mathrm{d}x_1)\delta_0(\mathrm{d}x_2) + \delta_0(\mathrm{d}x_1)\Lambda^{(2)}(\mathrm{d}x_2).$$

Then the equation in Theorem 1 can be rewritten as

$$\int_{(0,\infty)} \big(1 - \mathrm{e}^{-f(x_1)}\big)\Lambda^{(1)}(\mathrm{d}x_1) = \lambda - \int_{(0,\infty)} \big(1 - \mathrm{e}^{-\lambda x_2}\big)\Lambda^{(2)}(\mathrm{d}x_2).$$

On the other hand, our assumption implies that the subordinator Y and the process of point measures $\mathcal{N}^{(1)}$ are independent, and $\mathcal{M}_1 = \mathcal{N}^{(1)}_{\sigma_1}$ is thus a mixed Poisson measure with intensity $t\Lambda^{(1)}$ and mixing law $\mathbb{P}(\sigma_1 \in \mathrm{d}t)$. In particular we have

$$\mathbb{E}(\exp -\langle \mathcal{M}_1, f\rangle) = \int_{(0,\infty)} \exp\left(-t \int_{(0,\infty)} \big(1 - \mathrm{e}^{-f(x_1)}\big)\Lambda^{(1)}(\mathrm{d}x_1)\right)\mathbb{P}(\sigma_1 \in \mathrm{d}t).$$

Now recall from Theorem VII.1 in [3] that the Laplace transform of the first passage time σ_1 of the Lévy process with no positive jumps $t - Y_t$ is given by

$$\mathbb{E}(\mathrm{e}^{-q\sigma_1}) = \exp(-\varphi(q)), \qquad q \ge 0,$$

where the cumulant φ is the unique solution to

$$q = \varphi(q) - \int_{(0,\infty)} \big(1 - \mathrm{e}^{-\varphi(q)x_2}\big)\Lambda^{(2)}(\mathrm{d}x_2).$$

We conclude that

$$\kappa(f) = \varphi\left(\int_{(0,\infty)} \big(1 - \mathrm{e}^{-f(x_1)}\big)\Lambda^{(1)}(\mathrm{d}x_1)\right),$$

which is thus in agreement with Theorem 1.

5. Limit laws for partitions into colonies. In this section, we state and prove the main limit theorem for distributions of partitions into colonies. We consider for each fixed integer n a Galton–Watson process with emigration started from $a(n)$ ancestors that all occupy different sites, such that the number of homebody children and the number of migrant children $(\xi^{\mathrm{h}}_{n,k}, \xi^{\mathrm{m}}_{n,k})$ of the kth individual is given by an i.i.d. sequence. We write $\mathcal{P}_n$ for the partition into colonies induced by this model.

We also consider a bivariate Lévy process $(X^{\mathrm{h}}, X^{\mathrm{m}})$ such that X^{h} has no negative jumps and infinite variation, X^{m} is a subordinator, and the sum $X = X^{\mathrm{h}} + X^{\mathrm{m}}$ does not drift to $+\infty$. We write Λ for the Lévy measure that arises in Lemma 3, and then $(\mathcal{M}_y : y \geq 0)$ for the process of random point measures which has been studied in Section 4 for this specific choice of Λ.

We are now able to state the main result of this work.

THEOREM 2. *Write $\tilde{\mathcal{P}}_n$ for the image of the partition into colonies $\mathcal{P}_n$ by the rescaling $x \to x/\alpha(n)$, namely,*

$$\langle \tilde{\mathcal{P}}_n, f\rangle = \langle \mathcal{P}_n, \tilde{f}_n\rangle,$$

where $\tilde{f}_n(x) = f(x/\alpha(n))$. Assume that the number of ancestors $a(n)$ fulfills $a(n) \sim an$ for some $a > 0$. Then in the regime (3), $\tilde{\mathcal{P}}_n$ converges weakly on the space of sigma-finite measures on $(0,\infty)$ as $n \to \infty$ toward $\mathcal{M}_a$.

The material developed so far suggests that the proof of Theorem 2 should consist of two steps, namely first a tightness property for the rescaled partitions into colonies, and then uniqueness of the limit of a subsequence that shall be derived by the analysis of cumulants. This is indeed the route that we will follow.

LEMMA 5. *In the regime (3), the sequence of the distributions of the variables $\langle \tilde{\mathcal{P}}_n, \mathrm{Id}\rangle$, for $n \in \mathbb{N}$, is tight on the space of sigma-finite measures on $(0,\infty)$.*

PROOF. Indeed, recall that

$$\langle \tilde{\mathcal{P}}_n, \mathrm{Id}\rangle = \alpha(n)^{-1}\langle \mathcal{P}_n, \mathrm{Id}\rangle = \alpha(n)^{-1}\sum_{j=1}^{\gamma^{(n)}} C_j^{(n)} = \zeta_n/\alpha(n)$$

is simply the size ζ_n of the total population generated by the Galton–Watson process $Z^{(n)}$ renormalized by the factor $1/\alpha(n)$. It is well known that in the regime (3), this quantity converges in distribution as $n \to \infty$ toward the size of the total population of the CSBP Z, that is, equivalently, the first passage time of the Lévy process X at level $-a$. Hence, the sequence in the statement is tight. □

For the second step of the proof of Theorem 2, we write $\tilde{K}_n$ for the cumulant of the rescaled random measure $\tilde{\mathcal{P}}_n$, and fix a Borel function $f : (0,\infty) \to \mathbb{R}_+$

with compact support. For the sake of simplicity, we will suppose in the sequel that $a = 1$, that is, that $a(n) \sim n$, which induces no loss of generality thanks to the branching property and (7).

LEMMA 6. *Let $f : [0, \infty) \to \mathbb{R}_+$ be an arbitrary continuous function with compact support in $(0, \infty)$. Any limit point λ of the sequence $(\tilde{K}_n(f) : n \in \mathbb{N})$ fulfills*

$$\lambda = \int_{\mathbb{R}_+^2} (1 - \exp(-f(x_1) - \lambda x_2)) \Lambda(\mathrm{d}x_1\, \mathrm{d}x_2).$$

PROOF. We work with an increasing subsequence of integers n such that $\tilde{K}_n(f) \to \lambda$. Tracing back the definitions, we get

$$\begin{aligned}\exp(-\tilde{K}_n(f)) &= \exp(-a(n) K_n(\tilde{f}_n)) \\ &= (1 - \mathbb{E}_1^{(n)}(1 - \exp(-\tilde{f}_n(C) - K_n(\tilde{f}_n)M)))^{a(n)},\end{aligned}$$

where we used Lemma 1 for the last equality, and the notation $\mathbb{E}^{(n)}$ refers to the mathematical expectation corresponding to the nth population model. Taking logarithms, we arrive at

$$a(n)\mathbb{E}_1^{(n)}(1 - \exp(-\tilde{f}_n(C) - K_n(\tilde{f}_n)M)) \to \lambda.$$

Then recall from Lemma 2 that

$$\begin{aligned}&\mathbb{E}_1^{(n)}(1 - \exp(-\tilde{f}_n(C) - K_n(\tilde{f}_n)M)) \\ &\quad = \mathbb{E}_1^{(n)}\left(1 - \exp\left(-f(C/\alpha(n)) - \frac{n}{a(n)} \tilde{K}_n(f) \frac{M}{n}\right)\right) \\ &\quad = \mathbb{E}\left(1 - \exp\left(-f(\alpha(n)^{-1}\tau_{n,1}) - \frac{n}{a(n)} \tilde{K}_n(f) n^{-1} S^{\mathrm{m}}_{n,\tau_{n,1}}\right)\right).\end{aligned}$$

Recall also that $n/a(n) \to 1$ and $\tilde{K}_n(f) \to \lambda$. We now see that for every $\varepsilon > 0$

$$\limsup n\mathbb{E}(1 - \exp(-f(\alpha(n)^{-1}\tau_{n,1}) - (\lambda + \varepsilon) n^{-1} S^{\mathrm{m}}_{n,\tau_{n,1}})) \le \lambda,$$

so applying Corollary 1 with $g(x_1, x_2) = 1 - \exp(-f(x_1) - (\lambda + \varepsilon)x_2)$, we get

$$\int_{\mathbb{R}_+^2} (1 - \exp(-f(x_1) - (\lambda + \varepsilon)x_2)) \Lambda(\mathrm{d}x_1\, \mathrm{d}x_2) \le \lambda.$$

By a similar argument, we also obtain

$$\lambda \le \int_{\mathbb{R}_+^2} (1 - \exp(-f(x_1) - (\lambda - \varepsilon)x_2)) \Lambda(\mathrm{d}x_1\, \mathrm{d}x_2).$$

We derive the equation of the statement letting ε tend to 0. $\square$

The proof of Theorem 2 should now be plain. It follows from Lemma 5 that the sequence of the laws of rescaled partitions into colonies $\tilde{\mathcal{P}}_n$ is tight in the space of sigma-finite measures on $(0, \infty)$. We then deduce from Prohorov's lemma (see, e.g., Lemma 16.15 in [15]) that the sequence of the distributions of the random measures $\tilde{\mathcal{P}}_n$ on $(0, \infty)$ is relatively compact. If $\tilde{\mathcal{P}}$ has the law of the limit of some sub-sequence, then we deduce from Lemma 6 that for an arbitrary continuous function $f : [0, \infty) \to \mathbb{R}_+$ with compact support in $(0, \infty)$, the cumulant

$$\tilde{K}(f) = -\ln \mathbb{E}(\exp - \langle \tilde{\mathcal{P}}, f \rangle)$$

solves

$$\tilde{K}(f) = \int_{\mathbb{R}_+^2} \bigl(1 - \exp\bigl(-f(x_1) - \tilde{K}(f)x_2\bigr)\bigr)\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2).$$

We conclude from Theorem 1 that $\tilde{K}(f) = \kappa(f)$, and thus $\tilde{\mathcal{P}}$ has the same distribution as $\mathcal{M}_1$.

REMARK. It may be interesting to point at a different route for establishing Theorem 2, which uses the representation (1) of the partition into colonies. Recall the notation there and the convergence in distribution (4). Invoking Theorem 16.14 in [15], it is easy to check that the latter can be reinforced into weak convergence of càdlàg processes in the sense of Skorohod. One can then deduce from a time-substitution that

$$\bigl(\alpha(n)^{-1}\tau_{n,[nx]}, n^{-1}S^{\mathrm{m}}_{n,\tau_{n,[nx]}}\bigr) \Longrightarrow (T_x, Y_x),$$

where again the convergence holds in the sense of Skorohod. Loosely speaking, this entails the weak convergence of the increments of the random walk $\tau_{n,\cdot}$ rescaled by a factor $1/\alpha(n)$ to the jump-process of the subordinator T. We know from the Lévy–Itô decomposition that the latter can be described as a Poisson random measure whose intensity is expressed in terms of the Lévy measure of T. It remains to recall that in this setting, the number γ_n of colonies fulfills

$$\gamma_n = \min\{k : S^{\mathrm{m}}_{\tau_{n,k}} - k = -a(n)\}$$

and to check that

$$n^{-1}\gamma_n \Longrightarrow \sigma_a = \inf\{t \geq 0 : t - Y_t = a\}.$$

Some technical details needed to justify rigorously this approach may be tedious; they are circumvented here by the appeal to the characterization of the cumulant of the random measure $\mathcal{M}_a$ in Theorem 1 and the simple argument for tightness in Lemma 5.

6. Examples. In this section, we shall illustrate our main results for partitions into colonies by discussing some natural examples. Their common feature is that the distribution of the total number of children (homebodies and migrants) of a typical individual is fixed, that is the Galton–Watson process for which spatial locations of individuals are discarded has a fixed reproduction law. The differences in the models thus only appear through the repartition between homebody and migrant children. One could, of course, deal with much more general examples, however the present ones already exhibit a rich variety of asymptotic behaviors.

Recall (2). Throughout this section, we consider an integer-valued random variable ξ with unit mean, which belongs to the domain of attraction of a (completely asymmetric) stable variable with index $\beta \in (1, 2]$. That is, there is a sequence $(\alpha(n) : n \in \mathbb{N})$ which varies regularly with index β such that

$$n^{-1}\big(\xi_1 + \cdots + \xi_{\alpha(n)} - \alpha(n)\big) \Longrightarrow X_1,$$

where $(\xi_i : i \in \mathbb{N})$ is a sequence of i.i.d. copies of ξ and now $(X_t : t \geq 0)$ a stable(β) Lévy process with no negative jumps. In other words, there is some $b > 0$ such that

$$\mathbb{E}(\exp(-qX_t)) = \exp(tbq^{\beta}), \qquad q \geq 0,$$

i.e. $\psi(q) = bq^{\beta}$. The (continuous version of the) density of the variable X_1 will be denoted by ρ,

$$\mathbb{P}(X_1 \in \mathrm{d}x) = \rho(x)\,\mathrm{d}x,$$

so that, by scaling,

$$\mathbb{P}(X_t \in \mathrm{d}x) = t^{-1/\beta}\rho(t^{-1/\beta}x)\,\mathrm{d}x$$

for every $t > 0$.

6.1. *Allelic partitions for rare neutral mutations*. We first deal with the classical model corresponding to neutral mutations. That is for each fixed integer n, the total number of children of the kth individual is decomposed as $\xi_k = \xi^{\mathrm{h}}_{n,k} + \xi^{\mathrm{m}}_{n,k}$ where conditionally on $\xi_k = \ell$, the variable $\xi^{\mathrm{m}}_{n,k}$ has the binomial distribution with parameter $(\ell, p(n))$ for some $p(n) \in (0, 1)$. In other words, we assume that each child chooses to become a migrant with probability $p(n)$, independently of the other individuals.

If we now suppose that

$$p(n) \sim cn/\alpha(n)$$

for some constant $c > 0$, so that the mutation rate is small when n is large, then (3) clearly holds with $X^{\mathrm{h}}_t = X_t - ct$ and $X^{\mathrm{m}}_t = ct$, and thus

$$\Psi(q, r) = bq^{\beta} + cq - cr, \qquad q, r \geq 0.$$

We now see from Lemma 3(ii) that the bivariate subordinator (T_x, Y_x) has Laplace exponent $\Phi(q,r) = \varphi(q+cr)$ where $\varphi(q) = z$ is given by the nonnegative solution to

$$bz^\beta + cz = q.$$

We stress that $Y_x = cT_x$ a.s., and that the subordinator $(T_x : x \ge 0)$ has Laplace exponent φ.

An easy consequence of a version of the Ballot theorem (more precisely, cf. Corollary VII.3 in [3]) is that the Lévy measure $\Lambda^{(1)}$ of the subordinator of the first passage time of X^{h} can be expressed in terms of the density of X^{h} at 0. More precisely, one gets

$$\begin{aligned}\Lambda^{(1)}(\mathrm{d}t) &= \lim_{x\to 0+} x^{-1}\mathbb{P}(T_x \in \mathrm{d}t)\\ &= \lim_{x\to 0+} \frac{1}{t}\frac{\mathbb{P}(-X_t^{\mathrm{h}} \in \mathrm{d}x)}{\mathrm{d}x}\,\mathrm{d}t\\ &= t^{-1-1/\beta}\rho(ct^{1-1/\beta})\,\mathrm{d}t,\end{aligned}$$

where the last equality follows from the fact that

$$\mathbb{P}(X_t^{\mathrm{h}} \in \mathrm{d}x) = \mathbb{P}(X_t \in ct + \mathrm{d}x) = t^{-1/\beta}\rho\big(t^{-1/\beta}(x+ct)\big)\,\mathrm{d}x.$$

Recall that $Y = cT$; it follows that the Lévy measure of the bivariate subordinator (T, Y) that determines the law of the limiting partition $\mathcal{M}_a$ is then given by

$$\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2) = x_1^{-1-1/\beta}\rho(cx_1^{1-1/\beta})\delta_{cx_1}(\mathrm{d}x_2)\,\mathrm{d}x_1.$$

On the other hand, recall again from Corollary VII.3 in [3] that

$$\frac{\mathbb{P}(T_x \in \mathrm{d}t)}{\mathrm{d}t} = \frac{x}{t}\frac{\mathbb{P}(-X_t^{\mathrm{h}} \in \mathrm{d}x)}{\mathrm{d}x} = xt^{-1-1/\beta}\rho\big(t^{-1/\beta}(ct-x)\big), \qquad t, x > 0.$$

We can combine this identity with the argument in the remark following Proposition 1 to express the distribution of $\mathcal{M}_a$ as a mixture of laws of Poisson measures conditioned on their first moments (i.e., Poisson–Kingman partitions; see [17]). More precisely, we have

$$\mathbb{P}(\mathcal{M}_a \in \cdot) = a\int_{(a,\infty)} \frac{1}{t}\mathbb{P}\big(\mathcal{N}_t^{(1)} \in \cdot \mid t - Y_t = a\big)\frac{\mathbb{P}(t - Y_t \in \mathrm{d}a)}{\mathrm{d}a}\,\mathrm{d}t,$$

where $\mathcal{N}_t^{(1)}$ is a Poisson random measure with intensity $t\Lambda^{(1)}$. We now get, using the identity $Y = cT$, that

$$\begin{aligned}\mathbb{P}(\mathcal{M}_a \in \cdot) &= a\int_{(a,\infty)} \mathbb{P}\big(\mathcal{N}_t^{(1)} \in \cdot \mid T_t = (t-a)/c\big)\\ &\qquad \times \rho\big(t^{-1/\beta}(ct - (t-a)/c)\big)\frac{t-a}{c^2}t^{-2-1/\beta}\,\mathrm{d}t.\end{aligned} \tag{9}$$

In the important case $\beta = 2$, which occurs whenever the reproduction law has finite variance, ρ is simply the Gaussian density and

$$\Lambda^{(1)}(\mathrm{d}t) = \frac{1}{2\sqrt{\pi t^3 b}} \exp\left(-\frac{c^2 t}{4b}\right) \mathrm{d}t.$$

That is $\Lambda^{(1)}$ is the Lévy measure of an inverse Gaussian subordinator, which is merely an exponential transform of the stable(1/2) Lévy measure. Recall that the exponential transform plays no role for the distribution of Poisson measures conditioned on their first moments, which thus reduces the description (9) of the law of $\mathcal{M}_a$ to the more usual stable(1/2) Lévy measure. This situation has been investigated in depth by Pitman who has obtained a number of formulas for distributions related to such Poisson–Kingman partitions; see Section 8 in [17] or Section 4.5 in [18]. In particular Pitman has established sampling formulas in terms of Hermite functions which provide extensions of the celebrated one due to Ewens [9].

6.2. *One-type siblings*. We consider now an example related to the fragmentation process at nodes of the stable tree which has been considered by Miermont [16]; see also [1]. Specifically, we suppose henceforth that $\beta < 2$, and for each fixed value of the parameter $n \in \mathbb{N}$, all the children of an individual are homebodies with a probability that decays exponentially in the size of the number of children, and all children are migrant otherwise. More precisely, conditionally on $\xi_k = \ell$, the event $\xi^{\mathrm{h}}_{n,k} = \ell$ and $\xi^{\mathrm{m}}_{n,k} = 0$ occurs with probability $\mathrm{e}^{-\ell/n}$, while the event $\xi^{\mathrm{h}}_{n,k} = 0$ and $\xi^{\mathrm{m}}_{n,k} = \ell$ occurs with probability $1 - \mathrm{e}^{-\ell/n}$.

In this situation, it is easy to check that (5) holds with

$$X^{\mathrm{m}}_t = \sum_{0<s\le t} \mathbb{1}_{\{\Delta X_s > \epsilon_s\}} \Delta X_s \quad \text{and} \quad X^{\mathrm{h}}_t = X_t - X^{\mathrm{m}}_t,$$

where ΔX_s stands for the size of the jump (if any) of the stable(β) process X at time s and ϵ_s for an independent standard exponential mark which is attached to each jump of X. Well-known properties of Lévy processes imply that the processes X^{h} and X^{m} are independent and that X^{h} is an (Esscher) exponential transform of X. More precisely, the bivariate Laplace exponent of $(X^{\mathrm{h}}, X^{\mathrm{m}})$ is then given by

$$\Psi(q, r) = b\big((q+1)^\beta - 1\big) + b\big(r^\beta + 1 - (r+1)^\beta\big).$$

Since the law of X^{h}_t is simply an exponential transform of that of X_t,

$$\mathbb{P}(X^{\mathrm{h}}_t \in \mathrm{d}x) = t^{-1/\beta} \mathrm{e}^{-tx-bt} \rho(t^{-1/\beta} x)\, \mathrm{d}x,$$

and we deduce from the Ballot theorem that

$$\Lambda^{(1)}(\mathrm{d}t) = \rho(0) t^{-1-1/\beta} \mathrm{e}^{-bt}\, \mathrm{d}t.$$

As the first passage process $T_{\cdot}$ is a subordinator which is independent of X^{m}, and $Y = X^{\mathrm{m}} \circ T$ results from Bochner's subordination, and we get that the Lévy measure Λ of (T, Y) can be expressed in the form

$$\Lambda(\mathrm{d}x_1\,\mathrm{d}x_2) = \rho(0) x_1^{-1-1/\beta} \mathrm{e}^{-bx_1}\,\mathrm{d}x_1\,\mathbb{P}(X^{\mathrm{m}}_{x_1} \in \mathrm{d}x_2).$$

6.3. *Migration forced by cut-off.* In the preceding two examples, the subordinator X^{m} was either deterministic (a pure drift) or independent of the Lévy process X^{h}. Our last example shows that more general situations may arise. Specifically, we consider the parameter n as a threshold and decide that at most n of the children of each individual are homebodies and the rest are migrants. In other words, $\xi^{\mathrm{h}}_{n,k} = \xi_k \wedge n$ and $\xi^{\mathrm{m}}_{n,k} = (\xi_k - n)^+$.

Then (5) is fulfilled with

$$X^{\mathrm{m}}_t = \sum_{0<s\le t} \mathbb{1}_{\{\Delta X_s > 1\}}(\Delta X_s - 1) \quad \text{and} \quad X^{\mathrm{h}}_t = X_t - X^{\mathrm{m}}_t.$$

We stress that the jump times of X^{m} are exactly the times when X^{h} has a jump of size 1; in particular the processes X^{h} and X^{m} are not independent.

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