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STOCHASTIC VORTEX METHOD FOR FORCED THREE-DIMENSIONAL NAVIER–STOKES EQUATIONS AND PATHWISE CONVERGENCE RATE[1]

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We develop a McKean–Vlasov interpretation of Navier–Stokes equations with external force field in the whole space, by associating with local mild L^p-solutions of the 3d-vortex equation a generalized nonlinear diffusion with random space–time birth that probabilistically describes creation of rotation in the fluid due to nonconservativeness of the force. We establish a local well-posedness result for this process and a stochastic representation formula for the vorticity in terms of a vector-weighted version of its law after its birth instant. Then we introduce a stochastic system of 3d vortices with mollified interaction and random space–time births, and prove the propagation of chaos property, with the nonlinear process as limit, at an explicit pathwise convergence rate. Convergence rates for stochastic approximation schemes of the velocity and the vorticity fields are also obtained. We thus extend and refine previous results on the probabilistic interpretation and stochastic approximation methods for the nonforced equation, generalizing also a recently introduced random space–time-birth particle method for the 2d-Navier–Stokes equation with force.

1. Introduction. The Navier–Stokes equation for a homogeneous and incompressible fluid in the whole plane or space, subject to an external force field $\mathbf{F}$, is given by

$$\begin{aligned}&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla\mathbf{p} + \mathbf{F};\\ &\operatorname{div}\mathbf{u}(t,x) = 0; \qquad \mathbf{u}(t,x)\to 0 \qquad \text{as } |x|\to\infty.\end{aligned}\tag{1}$$

Here, $\mathbf{u}$ denotes the velocity field, $\mathbf{p}$ is the (unknown) pressure function and $\nu > 0$ is the (constant) viscosity coefficient. When $\mathbf{F} = 0$ or, more generally, when $\mathbf{F} = \nabla\Psi$ is a conservative field, a probabilistic interpretation of (1) in space dimension two was first developed in 1982 by Marchioro and Pulvirenti [19]. Their approach was based on the vortex equation satisfied by the (scalar) field $\operatorname{curl}\mathbf{u}$, which in 2d and for the case of a conservative external field, was interpreted as a nonlinear Fokker–Planck (or McKean–Vlasov) equation with signed initial condition.

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This was associated with a nonlinear diffusion process in the sense of McKean, involving singular interactions through the kernel of Biot–Savart. (For a general background on the McKean–Vlasov model, we refer the reader to Sznitman [26] and Méléard [20].) This approach led them to the definition of a stochastic system of particle or vortices with "mollified" mean field interaction, for which the time-marginal empirical measures converge to a solution of the vortex equation associated with (1). The convergence on the path space of that particles system (or, equivalently, the propagation of chaos property) was proved later by Méléard in [21]. Those works provided a rigorous mathematical meaning of Chorin's vortex algorithm, heuristically proposed in [3] as a probabilistic method to simulate the solution of the 2d-Navier–Stokes equation (see also [4]).

In dimension 3, the vorticity field $\mathbf{w} = \operatorname{curl} \mathbf{u}$ is a solution of the vectorial nonlinear equation

$$\begin{aligned} &\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{w} = (\mathbf{w} \cdot \nabla)\mathbf{u} + \nu \Delta \mathbf{w} + \mathbf{g}, \\ &\operatorname{div} w_0 = 0, \end{aligned} \tag{2}$$

where $\mathbf{g} = \operatorname{curl} \mathbf{F}$ and where the relation

$$\mathbf{u}(t,x) = \mathbf{K}(\mathbf{w})(t,x) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \wedge \mathbf{w}(t,y)\,dy \tag{3}$$

holds, thanks to the incompressibility condition $\operatorname{div} \mathbf{u} = 0$ and the Biot and Savart law. Here, $\wedge$ stands for the vectorial product in $\mathbb{R}^3$, $K(x)\wedge := -\frac{1}{4\pi}\frac{x}{|x|^3}\wedge$ is the three-dimensional Biot–Savart kernel and $\mathbf{K}$ is the Biot–Savart operator in 3d. (We refer to Bertozzi and Majda [18] for this and for background on vorticity.)

In absence of external forces, the problem of proving the approximation of solutions of the 3d-Navier–Stokes equations by a stochastic system of mean field interacting particles was first addressed by Esposito and Pulvirenti [7]. In that work, an approximation result of local solutions by a stochastic system of three-dimensional vortices with cutoff and mollified interactions was obtained for each time instant, for initial vorticities that belonged to L^1 together with their Fourier transform. The convergence held for mollifying parameters that depended on the realizations of the empirical measures of the paths of the driving Brownian motions.

Recently, we considered in [9] the mild version of the 3d-vortex equation with $\mathbf{g} = 0$ in the L^p spaces for $p > \frac{3}{2}$. We proved local (in time) well posedness and regularity results for that equation, and, under an additional L^1 assumption on w_0, we showed the equivalence between such solutions and a generalized nonlinear McKean–Vlasov process with values in $\mathbb{R}^3 \times \mathbb{R}^{3\otimes 3}$ and singular drift term at $t = 0$. We then introduced a system of stochastic 3d vortices with cutoff and mollified interaction, and proved the pathwise propagation of chaos property with as limit the nonlinear process, deducing moreover stochastic particle approximation results for the velocity and vorticity fields. (We refer to [10] for a rectification of the discussion in [9] about the work [7].) During the preparation of this work, we have also become aware of the more recent work of Philipowski [22], who obtained (also in the case $\mathbf{g} = 0$) a convergence rate for a mean field particle approximation of the vorticity field, for a simpler variation of the system introduced in [9]. (The pathwise propagation of chaos property was not addressed.)

In presence of an external force field, the additional additive term $\mathbf{g} = \operatorname{curl} \mathbf{F}$ in the (2d or 3d) vortex equation is physically interpreted as creation of rotation in the fluid. In order to describe this phenomenon probabilistically, a nonlinear McKean–Vlasov diffusion process with random space–time birth was recently associated with the 2d-vortex equation in Fontbona and Méléard [11]. More precisely, the law $P_0(dt, dx)$ of the instant and position of birth was suitable, defined in terms of the initial vorticity and of the external field $\operatorname{curl} \mathbf{F}$, and it was shown that a scalar-weighted version of the time marginal law of this process after its birth time was equal to the solution to the 2d-vortex equation (with L^1 data) in a given interval. The propagation of chaos property was established for an approximating system of interacting vortices, which were given birth independently at random positions and times following the law P_0, and a pathwise convergence rate was obtained under slight additional integrability assumptions on the data.

The first purpose of the present paper is to extend the results of [9] and [11] to the 3d-Navier–Stokes equation with nonconservative external force field. More precisely, fix $T > 0$ and assume that $w_0 : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{g} : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$ are divergence-free L^1-fields. Denote by I_3 the identity matrix in $\mathbb{R}^3$ and let (B_t) be a standard 3d-Brownian motion. Our main goal will be to study the well posedness on $[0, T]$ of the following nonlinear process, with singular interaction kernel and values in $\mathbb{R}^3 \times \mathbb{R}^{3\otimes 3}$:

$$\begin{aligned} X_t &= X_0 + \sqrt{2\nu} \int_0^t \mathbf{1}_{\{s \geq \tau\}} \, dB_s + \int_0^t \mathbf{K}(\tilde{\rho})(s, X_s) \mathbf{1}_{\{s \geq \tau\}} \, ds, \\ \Phi_t &= I_3 + \int_0^t \nabla \mathbf{K}(\tilde{\rho})(s, X_s) \Phi_s \mathbf{1}_{\{s \geq \tau\}} \, ds, \end{aligned} \tag{4}$$

where: (τ, X_0) is a random variable in $[0, T] \times \mathbb{R}^3$ (independent of B) with law

$$P_0(dt, dx) \propto \delta_0(dt) |w_0(x)| \, dx + |\mathbf{g}(t, x)| \, dx \, dt,$$

$\tilde{\rho} = \tilde{\rho}(t, x)$ is defined for each t from the law of (τ, X, Φ) as

$$\int_{\mathbb{R}^3} \mathbf{f}(y) \tilde{\rho}(t, y) \, dy := E\big(\mathbf{f}(X_t) \Phi_t h(\tau, X_0) \mathbf{1}_{\{t \geq \tau\}}\big) \qquad \text{for } \mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3, \tag{5}$$

and h in (5) is the density with respect to P_0 of the vectorial measure $\delta_0(dt) \times w_0(x) \, dx + \mathbf{g}(t, x) \, dx \, dt$. [We observe that it is (4) *together* with relation (5) that specify a "nonlinear process" in McKean's sense.]

As we shall see, there will exist a correspondence between mild $L^p(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$-solutions $\mathbf{w}$ of (2) for $p > \frac{3}{2}$, and suitable solutions of the nonlinear stochastic differential equation (4) and (5), through the relation $\mathbf{w} = \tilde{\rho}$. Thus, (5) provides a representation formula for solutions $\mathbf{w}$ of (2) which extends the one obtained in [9] when $\mathbf{g} \equiv 0$ (or $\tau \equiv 0$). In the present case, this representation can be intuitively understood as follows. A point vortex is given birth at random instant and position (τ, X_0), rotating in direction $h(\tau, X_0) \in \mathbb{R}^3$. It then evolves under the effect of diffusion and of the velocity field $\mathbf{K}(\mathbf{w})$ in (4), while its rotation direction and magnitude are changed under the action of the matrix process Φ_t which accounts for the vortex stretching proper to dimension 3. Averaging the rotation vectors on the position of infinitely "already born vortices" yields a macroscopic vorticity field $\mathbf{w}(t) = \tilde{\rho}(t)$, weakly defined by (5). The velocity field instantaneously experienced by each individual vortex is finally recovered from $\mathbf{w}$ as a mean field effect through the interaction kernel of Biot–Savart.

We will adapt the ideas and analytic techniques in [9] to first establish local well-posedness and regularity results for the mild formulation of the vortex equation. Based on this, we shall then prove local [i.e., for small enough $T > 0$ or data $(w_0, \mathbf{g})$] pathwise well posedness for the nonlinear stochastic differential equation (4) and (5), which will have singular drift terms at $t = 0$.

We shall then introduce a stochastic system of n particles in $\mathbb{R}^3 \times \mathbb{R}^{3\otimes 3}$ (or 3d-vortices) with cutoff and mollified interaction kernels, and with random space–time births. The second goal of this paper will be to prove the strong pathwise convergence of each of these particles as n goes to ∞, towards the nonlinear process, at an explicit rate. To that end, we will improve the techniques used in [9] to study the nonlinear process, which relied on tightness estimates for approximating processes and martingale problem characterization. More precisely, by a fine use of regularity properties of the equation, and inspired by ideas introduced in [11], we will show that the approximating "mollified processes" converge pathwise at the same rate at which mollified versions of the vortex equation converge to the original one. We will be able to exhibit that rate for a large class of mollified kernels, thanks to classic regularization techniques in Raviart [23] (which are also similar to those used in [22]). These results will imply the propagation of chaos in a strong norm and, classically, an explicit rate in some pathwise Wasserstein distance $\mathcal{W}$. From this we will also deduce convergence rates for approximation schemes of the vorticity and velocity fields. Unfortunately, the mollifying parameter will be required to go very slowly to 0 as n goes to ∞, which will yield a very slow (but not necessarily optimal) rate for the particles convergence.

Finally, we point out that our regularity results on the mild equation in L^p will ensure that the stochastic flow

$$\xi_{s,t}(x) = x + \sqrt{2\nu}(B_t - B_s) + \int_s^t \mathbf{u}(r, \xi_{s,r}(x))\, dr \tag{6}$$

is of class $C^1(\mathbb{R}^3)$, and so one can write

$$(X_t, \Phi_t)\mathbf{1}_{\{t \geq \tau\}} = (\xi_{\tau,t}(X_0), \nabla_x \xi_{\tau,t}(X_0))\mathbf{1}_{\{t \geq \tau\}}. \tag{7}$$

Equation (5) can thus be seen as a stochastic analog for the 3d-Navier–Stokes equation of the “Lagrangian representation” of the vorticity of the 3d-Euler equation $\nu = 0$ (see, e.g., [5], Chapter 1), an analogy established in [7, 9] when $\mathbf{g} \equiv 0$. Lagrangian representations of the 3d-Navier–Stokes equations as stochastic analogues to representations formulae for the Euler equation have been studied by several authors, some of which have led to (local) well-posedness results for the equation. See, for example, Esposito et al. [6] and, for more recent developments, Busnello et al. [2] and Iyer [14]. The latter works follow approaches that are in some sense “dual” to ours, establishing representations of strong solutions of the vortex or Navier–Stokes equations in terms of expectations of the initial data, after being transported and modified by the stochastic flow. A related stochastic approach is adopted in Gomes [13] to establish a variational formulation of the Navier–Stokes equation, analogous to Arnold’s variational characterization of the Euler equation. A seemingly very different further probabilistic point of view, providing global well posedness for small initial data, was introduced by Le Jan and Sznitman in [16], who associated with the Fourier transform of the velocity field a multitype branching process or stochastic cascade. See, for example, Bhattacharya et al. [1] for more recent developments in that direction.

The remainder of this work is organized as follows. In Section 2 we first present a weak formulation of (4) and (5) in terms of a nonlinear martingale problem, and discuss its connection with (2). In Section 3, we shall obtain local well-posednes and regularity results for the mild version of the vortex equation in L^p, for $p \in (\frac{3}{2}, 3)$. In Section 4 we state some results about a nonlinear Fokker–Planck equation with external field associated with the process with random space–time birth X in (4). We use this and the previous results to show strong local-in-time well posedness for the nonlinear stochastic differential equation (4) and (5). We, moreover, obtain the pathwise convergence result and estimates for approximating mollified versions of that problem. In Section 5, we introduce the system of 3d-stochastic vortices with random space–time birth, and deduce the propagation of chaos property and its rate. We also prove approximation results for the velocity and the vorticity of the forced 3d-Navier–Stokes equation with their corresponding convergence rates. In Section 6 we shall discuss how these rates of convergence are slightly improved when Sobolev regularity of the initial condition and external field is assumed.

Let us establish some notation:

- By $\mathcal{M}eas^T$ we denote the space of measurable real-valued functions on $[0, T] \times \mathbb{R}^3$.
- $C^{1,2}$ is the set of real-valued functions on $[0, T] \times \mathbb{R}^3$ with continuous derivatives up to the first order in $t \in [0, T]$ and up to the second order in $x \in \mathbb{R}^3$. $C_b^{1,2}$ is the subspace of bounded functions in $C^{1,2}$ with bounded derivatives.
- $\mathcal{D}$ is the space of compactly supported functions on $\mathbb{R}^3$ with infinitely many derivatives.

- For all $1 \le p \le \infty$ we denote by L^p the space $L^p(\mathbb{R}^3)$ of real-valued functions on $\mathbb{R}^3$. By $\|\cdot\|_p$ we denote the corresponding norm, and p^* stands for the Hölder conjugate of p. We write $W^{1,p} = W^{1,p}(\mathbb{R}^3)$ for the Sobolev space of functions in L^p with partial derivatives of first order in L^p.
- If E is a space of real-valued functions (defined on $\mathbb{R}^3$ or on $[0,T] \times \mathbb{R}^3$), then the notation $(E)^3$ is used for the space of $\mathbb{R}^3$-valued functions with scalar components in E. If E has a norm, the norm in $(E)^3$ is denoted in the same way.
- For notational simplicity, if $\mathbf{f}, \mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^3$ are vector fields and $Z : \mathbb{R}^3 \to \mathbb{R}^{3\otimes 3}$ is a matrix function, we will write $\mathbf{f}\mathbf{g} := \sum_i^3 \mathbf{f}_i \mathbf{g}_i$ and $\mathbf{f}Z$ for the row-vector $(\mathbf{f}^t Z)_i := \sum_{j=1}^3 \mathbf{f}_j Z_{j,i}$. By $\nabla \mathbf{f}$ we denote the gradient of $\mathbf{f}$, that is, the matrix $(\nabla \mathbf{f})_{i,j} := \frac{\partial \mathbf{f}_i}{\partial x_j}$. We will simply write $(\nabla \mathbf{f})\mathbf{g}$ for the column-vector $(\sum_j \frac{\partial \mathbf{f}_i}{\partial x_j} \mathbf{g}_j)_i$ [instead of the usual "$(\mathbf{g} \cdot \nabla)\mathbf{f}$"].
- C and $C(T)$ are finite positive constants that may change from line to line.

2. The weak 3d-vortex equation and a probabilistic interpretation of the external field. Let us recall a that vector field $w : \mathbb{R}^3 \to \mathbb{R}^3$ with components in $\mathcal{D}'$, and such that $\int_{\mathbb{R}^3} \nabla f(x) w(x)\, dx = 0$ for all $f \in \mathcal{D}$, is said to have *null divergence in the distribution sense*. We write it $\operatorname{div} w = 0$.

If the following two conditions hold, we shall say that $w_0 : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{g} : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfy the hypothesis:

(H_p):

- there exists $p \in [1, \infty[$ such that $w_0 \in (L^p(\mathbb{R}^3))^3$ and $\mathbf{g}(t,\cdot) \in (L^p(\mathbb{R}^3))^3$ for all $t \in [0,T]$, and $\sup_{t\in[0,T]} \|\mathbf{g}(t,\cdot)\|_p < \infty$;
- $\operatorname{div} w_0 = 0$ and $\operatorname{div} \mathbf{g}(t,\cdot) = 0$ for all $t \in [0,T]$.

A necessary assumption for our probabilistic approach will be that (H_p) holds with $p = 1$. We then denote

$$\|\mathbf{g}\|_{1,T} := \int_0^T \int_{\mathbb{R}^3} |\mathbf{g}(s,x)|\, dx\, ds.$$

In that functional setting, the following notion of solution to (2) will appear to be natural:

DEFINITION 2.1. Let w_0 and $\mathbf{g}$ satisfy (H_1). A function $\mathbf{w} \in L^\infty([0,T], (L^1(\mathbb{R}^3))^3)$ is a weak solution on $[0,T]$ of the vortex equation with initial condition w_0 and external field $\mathbf{g}$ (or "weak solution") if:

(i) For $i, j, k = 1, 2, 3$,

$$\begin{aligned} &\int_{[0,T]\times\mathbb{R}^3} |\mathbf{w}_i(t,x)||\mathbf{K}(\mathbf{w})_j(t,x)|\, dx\, dt < \infty, \\ &\int_{[0,T]\times\mathbb{R}^3} |\mathbf{w}_i(t,x)| \left| \frac{\partial \mathbf{K}(\mathbf{w})_j}{\partial x_k}(t,x) \right| dx\, dt < \infty. \end{aligned} \tag{8}$$

(ii) For any $\mathbf{f} \in (C_b^{1,2})^3$,

$$
\begin{aligned}
\int_{\mathbb{R}^3} \mathbf{f}(t,y)\mathbf{w}(t,y)\,dy \\
&= \int_{\mathbb{R}^3} \mathbf{f}(0,y)w_0(y)\,dy + \int_0^t \int_{\mathbb{R}^3} \mathbf{f}(s,y)\mathbf{g}(s,y)\,dy\,ds \\
&\quad + \int_0^t \int_{\mathbb{R}^3} \biggl[\frac{\partial \mathbf{f}}{\partial s}(s,y) + \nu\Delta\mathbf{f}(s,y) \\
&\qquad + \nabla\mathbf{f}(s,y)\mathbf{K}(\mathbf{w})(s,y) + \mathbf{f}(s,y)\nabla\mathbf{K}(\mathbf{w})(s,y)\biggr]\mathbf{w}(s,y)\,dy\,ds.
\end{aligned} \tag{9}
$$

REMARK 2.2. We observe that for any function $\mathbf{v}:\mathbb{R}^3 \to \mathbb{R}^3$ in L^1, the functions $\mathbf{K}(\mathbf{v})$ and $\nabla\mathbf{K}(\mathbf{v})$ are defined a.e. on $\mathbb{R}^3$. Indeed, the first one can be bounded by a (scalar) Riesz potential operator (see Stein [24]), and thus belongs to a suitable weak Lebesgue space. The second one is defined through a singular integral operator acting on $\mathbf{v}$ (see, e.g., [18] for this fact), and this implies (see also [24]) that it is an almost everywhere defined function of some other weak Lebesgue space.

We next introduce the central probabilistic objects we shall be dealing with, which extend the ideas introduced in two dimensions in [11].

DEFINITION 2.3. We write $\mathcal{C}_T := [0,T] \times C([0,T], \mathbb{R}^3 \times \mathbb{R}^{3\otimes 3})$. The canonical process in $\mathcal{C}_T$ will be denoted by (τ, X, Φ), and the space of probability measures on $\mathcal{C}_T$ is written $\mathcal{P}(\mathcal{C}_T)$.

For an element $P \in \mathcal{P}(\mathcal{C}_T)$, we write $P^\circ = \mathrm{law}(X)$ for the second marginal and $P' = \mathrm{law}(\Phi)$ for the third marginal.

We shall also denote

$$
\bar{w}_0(x) = \frac{|w_0(x)|}{\|w_0\|_1 + \|\mathbf{g}\|_{1,T}} \quad \text{and} \quad
\bar{\mathbf{g}}(t,x) = \frac{|\mathbf{g}(t,x)|}{\|w_0\|_1 + \|\mathbf{g}\|_{1,T}}. \tag{10}
$$

We then define a probability measure $P_0(dt,dx)$ on $[0,T] \times \mathbb{R}^3$ by

$$
P_0(dt,dx) = \delta_0(dt)\bar{w}_0(x)\,dx + \bar{\mathbf{g}}(t,x)\,dx\,dt, \tag{11}
$$

together with the vectorial weight function

$$
\begin{aligned}
h(t,x) &= \mathbf{1}_{\{t=0\}} \frac{w_0(x)}{|w_0(x)|}(\|w_0\|_1 + \|\mathbf{g}\|_{1,T}) \\
&\quad + \frac{\mathbf{g}(t,x)}{|\mathbf{g}(t,x)|}(\|w_0\|_1 + \|\mathbf{g}\|_{1,T})\mathbf{1}_{\{t>0\}},
\end{aligned} \tag{12}
$$

where $\mathbf{1}$ denotes the indicator function and the convention "$\frac{0}{0}=0$" is made. We notice that $|h(t,x)| = \|w_0\|_1 + \|g\|_{1,T}$ or 0. Moreover, we have

REMARK 2.4. For measurable bounded functions $\mathbf{f}: [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$, we have

$$\int_{[0,T]\times\mathbb{R}^3} \mathbf{f}(s,x)h(s,x)P_0(ds,dx) = \int_{\mathbb{R}^3} \mathbf{f}(0,x)w_0(x)\,dx + \int_{[0,T]\times\mathbb{R}^3} \mathbf{f}(s,x)\mathbf{g}(s,x)\,dx\,ds.$$

Consider now $Q \in \mathcal{P}(\mathcal{C}_T)$ such that for all $\in [0,T]$, $\mathbb{E}^Q(|\Phi_t|) < \infty$. Then, we can associate with Q a family of $\mathbb{R}^3$-valued vector measures $(\tilde{Q}_t)_{t\in[0,T]}$ on $\mathbb{R}^3$, defined for all bounded measurable function $\mathbf{f}: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\tilde{Q}_t(\mathbf{f}) = \mathbb{E}^Q\big(\mathbf{f}(X_t)\Phi_t h(\tau, X_0)\mathbf{1}_{\{\tau\le t\}}\big). \tag{13}$$

Moreover, $\tilde{Q}_t$ is absolutely continuous with respect to Q_t°, with

$$\frac{d\tilde{Q}_t}{dQ_t^\circ}(x) = E^Q\big(\Phi_t h(\tau, X_0)\mathbf{1}_{\{\tau\le t\}} | X_t = x\big), \tag{14}$$

and its total mass is bounded by $(\|w_0\|_1 + \|\mathbf{g}\|_{1,T})\mathbb{E}^Q(|\Phi_t|)$.

DEFINITION 2.5. We denote by $\mathcal{P}_b(\mathcal{C}_T)$ the subset of probability measures $Q \in \mathcal{P}(\mathcal{C}_T)$ under which the process Φ belongs to $L^\infty([0,T]\times\Omega, dt\otimes Q)$.

Then, we consider the following nonlinear martingale problem:

(MP): to find $P \in \mathcal{P}_b(\mathcal{C}_T)$ such that:

- $X_t = X_0$ in $[0,\tau]$, P-almost surely.
- The law of (τ, X_0) under P is P_0 given by (11), and $\tilde{P}_t$ constructed according to (13) has a bi-measurable density family $(t,x) \mapsto \tilde{\rho}(t,x)$.
- $f(t,X_t) - f(0,X_0) - \int_0^t \frac{\partial f}{\partial s}(s,X_s) + [\nu\triangle f(s,X_s) + \mathbf{K}(\tilde{\rho})(s,X_s)\nabla f(s, X_s)]\mathbf{1}_{s\ge\tau}\,ds$, $0\le t\le T$, is a continuous P-martingale for all $f \in \mathcal{C}_b^{1,2}$ w.r.t. the filtration $\mathcal{F}_t = \sigma(\tau, (X_s,\Phi_s), s\le t)$.
- $\Phi_t = I_3 + \int_0^t \nabla\mathbf{K}(\tilde{\rho})(s,X_s)\Phi_s\mathbf{1}_{s\ge\tau}\,ds$, for all $0\le t\le T$, P-almost surely.

The following statement partially explains the relation between (MP) and (2), and will be useful later on:

LEMMA 2.6. *Assume that the problem (MP) has a solution $P \in \mathcal{P}_b(\mathcal{C}_T)$ satisfying*

$$E\left(\int_0^T |\mathbf{K}(\tilde{\rho})(t,X_t)|\,dt\right) < \infty \tag{15}$$

and

$$E\left(\int_0^T |\nabla \mathbf{K}(\tilde{\rho})(t, X_t)|\, dt\right) < \infty. \tag{16}$$

Then, $\tilde{\rho}$ is a weak solution of the vortex equation with external force field (9).

PROOF. The assumptions on P imply that point (i) in Definition 2.1 is satisfied and, moreover, that $\int_0^t \mathbf{K}(\tilde{\rho})(s, X_s)\, ds$ and $\int_0^t \nabla \mathbf{K}(\tilde{\rho})(s, X_s)\, ds$ are both processes with integrable variation (and thus absolutely continuous on $[0, T]$). Since under P the process Φ_t is almost surely bounded in $[0, T]$, it follows that it has finite variation too.

On the other hand, the martingale associated with $f \in \mathcal{C}_b^{1,2}$ in (MP) equals

$$\begin{aligned} &f(t, X_t) - f(\tau \wedge t, X_0) \\ &\quad - \int_0^t \left[\frac{\partial f}{\partial s}(s, X_s) + \nu \triangle f(s, X_s) + \mathbf{K}(\tilde{\rho})(s, X_s) \nabla f(s, X_s)\right] \mathbf{1}_{s \geq \tau}\, ds \end{aligned}$$

thanks to the first condition of (MP).

Therefore, by Itô's product rule, we see that for each $\mathbf{f} \in (C_b^{1,2})^3$

$$\begin{aligned} &\mathbf{f}(t, X_t)\Phi_t - \mathbf{f}(\tau \wedge, X_0) \\ &\quad - \int_0^t \left[\frac{\partial \mathbf{f}}{\partial s}(s, X_s) + \nu \triangle \mathbf{f}(s, X_s) + \nabla \mathbf{f}(s, X_s) \mathbf{K}(\tilde{\rho})(s, X_s) \right. \\ &\qquad\qquad \left. + \mathbf{f}(s, X_s) \nabla \mathbf{K}(\tilde{\rho})(s, X_s)\right] \Phi_s \mathbf{1}_{\{s \geq \tau\}}\, ds \end{aligned}$$

is a local martingale issued from 0. Moreover, the assumptions (16) and (15) on $\tilde{\rho}$ and the fact that Φ is bounded imply that it is a true martingale. Consequently, as $h(\tau, X_0)\mathbf{1}_{\{\tau \leq t\}}$ is $\mathcal{F}_0$-measurable and $\mathbf{1}_{\{\tau \leq s\} \cap \{\tau \leq t\}} = \mathbf{1}_{\{\tau \leq s\}}$ for $s \leq t$, we see that

$$\begin{aligned} &E^P\big(\mathbf{f}(t, X_t)\Phi_t h(\tau, X_0)\mathbf{1}_{\{\tau \leq t\}}\big) - E^P\big(\mathbf{f}(\tau, X_0) h(\tau, X_0)\mathbf{1}_{\{\tau \leq t\}}\big) \\ &\quad - E^P\left(\int_0^t \left[\frac{\partial \mathbf{f}}{\partial s}(s, X_s) + \nu \triangle \mathbf{f}(s, X_s) \right.\right. \\ &\qquad\qquad + \nabla \mathbf{f}(s, X_s) \mathbf{K}(\tilde{\rho})(s, X_s) \\ &\qquad\qquad \left.\left. + \mathbf{f}(s, X_s) \nabla \mathbf{K}(\tilde{\rho})(s, X_s)\right] \Phi_s h(\tau, X_0) \mathbf{1}_{\{\tau \leq s\}}\, ds\right) = 0. \end{aligned} \tag{17}$$

Recalling that $\tilde{\rho}(t)$ is the density of the vector measure (13) for $Q = P$, the first term in the previous equation is seen to be equal to $\int \mathbf{f}(t, x)\tilde{\rho}(t, x)\, dx$. The second term is equal to the expression in Remark 2.4 with $\mathbf{f}(s, x)$ replaced by $\mathbf{f}(s, x)\mathbf{1}_{s \leq t}$, that is, $\int \mathbf{f}(0, y) w_0(y)\, dy + \int_0^t \int \mathbf{f}(s, y)\mathbf{g}(s, y)\, dy\, ds$. The third expectation can be interchanged with the time integral thanks to the assumptions and Fubini's theorem, and the result follows using again the definition of $\tilde{\rho}(s)$ in the resulting time integral. □

The proof of the well posedness of problem (MP) will be based on analytical results about the "mild form" of the vortex equation (2), which we state in next section. These will in particular provide a framework where the conditions required in Lemma 2.6 will hold.

3. The mild vortex equation in L^p with an external field. We shall next introduce the mild formulation of the forced vortex equation. We refer the reader to the book of Lemarié-Rieusset [17] for a comprehensive account on the mild-form approach to the Navier–Stokes equation in its velocity form. Our techniques are adapted from that framework.

We denote the heat kernel in $\mathbb{R}^3$ by

$$G_t^\nu(x) := (4\pi\nu t)^{-3/2}\exp\left(-\frac{|x|^2}{4\nu t}\right), \tag{18}$$

where $\nu > 0$. One has

LEMMA 3.1. *For all $p \in [1,\infty]$, $r \geq p$ and $w \in (L^p)^3$, there exist positive constants $\bar{C}_0(p;r)$ and $\bar{C}_1(p;r)$ such that for all $t > 0$:*

(i) $\|G_t^\nu * w\|_r \leq \bar{C}_0(p;r)t^{-3/2(1/p-1/r)}\|w\|_p$,
(ii) $\|\nabla G_t^\nu * w\|_r \leq \bar{C}_1(p;r)t^{-1/2-3/2(1/p-1/r)}\|w\|_p$.

PROOF. Use Young's inequality and the well-known estimates

$$\sup_{t\geq 0}\|G_t^\nu\|_m t^{3/2-3/(2m)} < \infty, \qquad \sup_{t\geq 0}\|\nabla G_t^\nu\|_m t^{2-3/(2m)} < \infty. \qquad \square$$

DEFINITION 3.2. Let w_0 and $\mathbf{g}$ be functions satisfying (H$_p$) for some $p \in [1,\infty]$. A function $\mathbf{w} \in L^\infty([0,T],(L^p(\mathbb{R}^3))^3)$ is a mild solution on $[0,T]$ of the vortex equation with initial condition w_0 and external field (or "mild solution") if:

(i) The functions $\mathbf{K}(\mathbf{w})_i(t,x) := \mathbf{K}(\mathbf{w}(t,\cdot))_i(x)$, $i = 1,2,3$ are defined a.e. on $[0,T]\times\mathbb{R}^3$ and satisfy the integrability conditions (8).

(ii) For dt-almost every t, the following identity holds in $(L^p)^3$:

$$\begin{aligned}\mathbf{w}(t,x) &= G_t^\nu * w_0(x) + \int_0^t G_{t-s}^\nu * \mathbf{g}(s,\cdot)(x)\,ds\\ &\quad + \sum_{j=1}^3\int_0^t\int_{\mathbb{R}^3}\frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y)[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{w}(s,y)\\ &\qquad - \mathbf{w}_j(s,y)\mathbf{K}(\mathbf{w})(s,y)]\,dy\,ds.\end{aligned} \tag{19}$$

We shall state in Theorems 3.6 and 3.8 below the analytical results we need about (19). As we shall see, that equation will admit an abstract formulation which is the same as in the case $\mathbf{g} = 0$, and so we will be able to adapt the techniques in [9] with no difficulties. We therefore provide an abbreviated account of these results.

We shall simultaneously deal with a family of "mollified" versions of (19). Consider a smooth function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ satisfying:

(i) $\int_{\mathbb{R}^3} \varphi(x)\, dx = 1$,
(ii) $\int_{\mathbb{R}^3} |x||\varphi(x)|\, dx < \infty$,

which is called a "cutoff function of order 1." For $\varepsilon > 0$, let $\varphi_\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ denote the regular approximation of the Dirac mass $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^3}\varphi(\frac{\varepsilon}{x})$. We define the convolution operators

$$\mathbf{K}^\varepsilon(w)(x) := \int_{\mathbb{R}^3} K_\varepsilon(x-y) \wedge w(y)\, dy, \tag{20}$$

where $K_\varepsilon := \varphi_\varepsilon * K = \mathbf{K}(\varphi_\varepsilon)$. The fact that K_ε is a regular function will follow from part (ii) in Lemma 3.3 below. To unify notation, we also write $K_0 = K$ and $\mathbf{K}^0(w)(x) := \mathbf{K}(w)(x)$.

We introduce the family $\{\mathbf{B}^\varepsilon\}_{\varepsilon \geq 0}$ of operators (formally) defined on functions $\mathbf{w}, \mathbf{v} : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\begin{aligned} &\mathbf{B}^\varepsilon(\mathbf{w}, \mathbf{v})(t, x) \\ &\quad = \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y) \\ &\qquad\qquad \times [\mathbf{K}^\varepsilon(\mathbf{w})_j(s, y)\mathbf{v}(s, y) - \mathbf{v}_j(s, y)\mathbf{K}^\varepsilon(\mathbf{w})(s, y)]\, dy\, ds. \end{aligned} \tag{21}$$

We are interested in the following family of "abstract" equations, for $\varepsilon \geq 0$:

$$\mathbf{v} = \mathbf{w}_0 + \mathbf{B}^\varepsilon(\mathbf{v}, \mathbf{v}), \tag{22}$$

where

$$\mathbf{w}_0(t, x) := G^\nu_t * w_0(x) + \int_0^t G^\nu_{t-s} * \mathbf{g}(s, \cdot)(x)\, ds.$$

For a given time interval $[0, T]$ we shall work in the Banach spaces

$$\mathbf{F}_{0,r,(T;p)}, \qquad \mathbf{F}_{1,r,(T;p)}, \qquad \mathbf{F}_{0,p,T} \quad \text{and} \quad \mathbf{F}_{1,p,T}$$

with norms, respectively, defined by:

- $|||\mathbf{w}|||_{0,r,(T;p)} := \sup_{0 \leq t \leq T} t^{3/2(1/p-1/r)} \|\mathbf{w}(t)\|_r$,
- $|||\mathbf{w}|||_{1,r,(T;p)} := \sup_{0 \leq t \leq T} \{t^{3/2(1/p-1/r)} \|\mathbf{w}(t)\|_r + t^{1/2+3/2(1/p-1/r)} \times \sum_{k=1}^3 \|\frac{\partial \mathbf{w}(t)}{\partial x_k}\|_r\}$,
- $|||\mathbf{w}|||_{0,p,T} := |||\mathbf{w}|||_{0,p,(T;p)}$ and
- $|||\mathbf{w}|||_{1,p,T} := |||\mathbf{w}|||_{1,p,(T;p)}$.

The following continuity property of the Biot–Savart kernel is crucial:

LEMMA 3.3. *Let* $1 < p < 3$ *be given and* $q \in (\frac{3}{2}, \infty)$ *be defined by* $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$.

(i) *For every* $w \in (L^3)^p$, *the integral* (20) *is absolutely convergent for almost every* x *and one has* $\mathbf{K}^{\varepsilon}(w) \in (L^q)^3$. *There exists further a positive constant* $\tilde{C}_{p,q}$ *such that*

$$\sup_{\varepsilon \geq 0} \|\mathbf{K}^{\varepsilon}(w)\|_q \leq \tilde{C}_{p,q} \|w\|_p \tag{23}$$

for all $w \in (L^p)^3$.

(ii) *If moreover* $w \in (W^{1,p})^3$, *then we have* $\mathbf{K}^{\varepsilon}(w) \in (W^{1,q})^3$, *with* $\frac{\partial}{\partial x_k}\mathbf{K}^{\varepsilon}(w) = \mathbf{K}^{\varepsilon}(\frac{\partial w}{\partial x_k})$, *and*

$$\sup_{\varepsilon \geq 0} \left\| \frac{\partial \mathbf{K}^{\varepsilon}(w)}{\partial x_k} \right\|_q \leq \tilde{C}_{p,q} \left\| \frac{\partial w}{\partial x_k} \right\|_p \tag{24}$$

for all $k = 1, 2, 3$.

PROOF. See Lemma 2.2 in [9] for the case $\varepsilon = 0$ and Remark 4.3 therein for the general case. □

LEMMA 3.4. (i) *Let* $p \in [1, 3)$ *and assume* (H_p). *Then, we have for all* $r \in [p, \frac{3p}{3-p})$ *that*

$$\mathbf{w}_0 \in F_{1,r,(T;p)} \qquad \textit{with } |||\mathbf{w}_0|||_{1,r,(T;p)} \leq C(r,p)(\|w_0\|_p + T\,|||\mathbf{g}|||_{0,p,T})$$

for some finite constant $C(r, p) > 0$.

(ii) *Let* $\frac{3}{2} < p < 3$, $p \leq l < \min\{\frac{6p}{6-p}, 3\}$ *and* $\frac{3l}{6-l} \leq l' < \frac{3l}{6-2l}$. *Then, there exists a finite constant* $C_1(l, l'; p)$ *not depending on* $T > 0$ *such that for all* $\mathbf{w}, \mathbf{v} \in \mathbf{F}_{1,l,(T;p)}$,

$$\sup_{\varepsilon \geq 0} |||\mathbf{B}^{\varepsilon}(\mathbf{w}, \mathbf{v})|||_{1,l',(T;p)} \leq C_1(l, l'; p) T^{1-3/(2p)} |||\mathbf{w}|||_{1,l,(T;p)} |||\mathbf{v}|||_{1,l,(T;p)},$$

where $1 - \frac{3}{2p} > 0$.

PROOF. Part (i) follows from Lemma 3.1. To bound the time integral we use, moreover, the fact that for all $r \geq p$, on has

$$\left\| \int_0^t G_{t-s}^{\nu} * \mathbf{g}(s, \cdot)\, ds \right\|_r \leq C t^{1+1/r-1/p} \Big(\sup_{t \in [0,T]} \|g_t\|_p \Big).$$

On the other hand, since $t \mapsto t^{-1/2+3/2(1/r-1/p)}$ is integrable in 0 if and only if $r < \frac{3p}{3-p}$, we have

$$\left\|\nabla\left(\int_0^t G^{\nu}_{t-s} * \mathbf{g}(s,\cdot)\,ds\right)\right\|_r \le C' t^{1/2+3/2(1/r-1/p)}\Bigl(\sup_{t\in[0,T]} \|\mathbf{g}(t,\cdot)\|_p\Bigr)$$

from where the statement follows. Part (ii) uses Lemma 3.3 and is proved in parts (ii) and (iv) of Proposition 3.1 in [9]. See also Remarks 4.3 and 6.3 therein for the uniformity (in $\varepsilon \ge 0$) of the bounds. □

REMARK 3.5. Observe that the previous lemma, in particular, implies (taking $p = r = l = l'$) that for $p \in (\frac{3}{2}, 3)$, the abstract equation (22) makes sense in $\mathbf{F}_{1,p,T}$ for each $\varepsilon \ge 0$.

Now we can state the extension of Theorem 3.1 in [9] to the 3d-vortex equation with external field.

THEOREM 3.6. *Assume that* (H$_p$) *for some* $\frac{3}{2} < p < 3$.

(a) *For each* $T > 0$ *and* $\varepsilon \ge 0$, *equation* (22) *has, at most, one solution in* $\mathbf{F}_{0,p,T}$.
(b) *There is a constant* $\Gamma_0(p) > 0$ *independent of* $\varepsilon \ge 0$ *such that for all* $T > 0$, w_0 *and* $\mathbf{g}$ *satisfying*

$$T^{1-3/(2p)}(\|w_0\|_p + T\,|||\mathbf{g}|||_{0,p,\theta}) < \Gamma_0(p),$$

each one of (22) *with* $\varepsilon \ge 0$, *has a solution* $\mathbf{w}^{\varepsilon} \in \mathbf{F}_{1,p,T}$. *Moreover, we have*

$$\sup_{\varepsilon\ge 0} |||\mathbf{w}^{\varepsilon}|||_{1,p,T} \le 2|||\mathbf{w}_0|||_{0,p,T}.$$

PROOF. For later purposes, we give, in detail, the argument of [9]. By Lemma 3.1(ii) (with p in the place of r and $\frac{3p}{6-p}$ in that of p) and Lemma 3.3(i), we have for all $\mathbf{v}, \mathbf{w} \in \mathbf{F}_{0,p,T}$ that

$$\|\mathbf{B}^{\varepsilon}(\mathbf{w},\mathbf{v})(t)\|_p \le C\int_0^t (t-s)^{-3/(2p)}\|\mathbf{w}(s)\|_p\|\mathbf{v}(s)\|_p\,ds.$$

It follows that if $\mathbf{w}$ and $\mathbf{v}$ are two solutions, one has

$$\|\mathbf{w}(t)-\mathbf{v}(t)\|_p \le C(|||\mathbf{w}|||_{0,p,T} + |||\mathbf{v}|||_{0,p,T})\int_0^t (t-s)^{-3/(2p)}\|\mathbf{w}(s)-\mathbf{v}(s)\|_p\,ds$$

and iterating the latter sufficiently many times [using the identity $\int_0^t s^{\varepsilon-1}(t-s)^{\theta-1}\,ds = Ct^{\varepsilon+\theta-1}$ for $\theta, \varepsilon > 0$] we get $\|\mathbf{w}(t)-\mathbf{v}(t)\|_p \le C\int_0^t \|\mathbf{w}(s)-\mathbf{v}(s)\|_p\,ds$. Gronwall's lemma concludes the proof.

(b) We notice that for $T > 0$ small enough, one has

$$4C(p,p)C_1(p,p;p)T^{1-3/(2p)}(\|w_0\|_p + T\,|||\mathbf{g}|||_{0,p,T}) < 1,$$

where $C(p,p)$ and $C_1(p,p;p)$ are, respectively, the constants in parts (i) and (ii) of Lemma 3.4 with all parameters equal to p. From this and Lemma 3.4(i), the same contraction argument used in Theorem 3.1(b) of [9] can be applied here in the space $\mathbf{F}_{1,p,T}$. $\square$

We observe that for $\mathbf{v} \in \mathbf{F}_{0,p,T}$, with $p \in (\frac{3}{2}, 3)$ we have $\mathbf{K}(\mathbf{v}) \in \mathbf{F}_{0,q,T}$ for $q \in (3, \infty)$. The previous global uniqueness and local existence result also holds in that space, and one can, moreover, show that the solution $\mathbf{w}(t) \in (L^p)$ is a continuous function of t. That type of result corresponds to a "vorticity version" of Kato's theorem for the mild Navier–Stokes equation in $(L^q)^3$, $q \in (3, \infty)$ (see [17], Theorem 15.3(A)).

We shall, later on, need additional regularity properties of the function $\mathbf{w}^\varepsilon$ and, more importantly, their uniformity in $\varepsilon \geq 0$. These results will rely on continuity properties of the "derivative" of the Biot–Savart operator.

LEMMA 3.7. *Let $1 < r < \infty$.*

(i) *For all $w \in (L^r)^3$ and $\varepsilon \geq 0$, we have $\frac{\partial}{\partial x_k}\mathbf{K}^\varepsilon(w) \in (L^r)^3$ for $k = 1, 2, 3$. There exists further a positive constant C_r depending only on r such that*

$$\sup_{\varepsilon \geq 0}\left\| \frac{\partial \mathbf{K}^\varepsilon(w)_j}{\partial x_k} \right\|_r \leq \tilde{C}_r \|w\|_r \tag{25}$$

for all $j = 1, 2, 3$, where $\mathbf{K}^\varepsilon(w)_j$ is the jth component of $\mathbf{K}^\varepsilon(w)$.

(ii) *If, moreover, $w \in (W^{1,r})^3$, we then have $\frac{\partial}{\partial x_k}\mathbf{K}^\varepsilon(w) \in (W^{1,r})^3$, with $\frac{\partial}{\partial x_i}(\frac{\partial}{\partial x_k}\mathbf{K}^\varepsilon(w)) = \frac{\partial}{\partial x_k}\mathbf{K}^\varepsilon(\frac{\partial}{\partial x_i} w)$ and*

$$\sup_{\varepsilon \geq 0}\left\| \frac{\partial^2 \mathbf{K}^\varepsilon(w)_j}{\partial x_i\, \partial x_k} \right\|_r \leq \tilde{C}_r \left\| \frac{\partial w}{\partial x_i} \right\|_r \tag{26}$$

for all $i, k = 1, 2, 3$.

PROOF. See Lemma 3.1 and Remark 4.3 in [9] for the proof, which relies on the fact that $\mathbf{w} \mapsto \frac{\partial \mathbf{K}(w)}{\partial x_k}$ is a singular integral operator. $\square$

THEOREM 3.8. *For $p \in (\frac{3}{2}, 3)$, let $\mathbf{w}^\varepsilon \in \mathbf{F}_{1,p,T}$, $\varepsilon \geq 0$ be the solution of (22) given by Theorem 3.6, and write $\mathbf{u}^\varepsilon(s,x) := \mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(s,x)$. Let $\mathcal{C}^\alpha$ denote the space of Hölder continuous functions $\mathbb{R}^3 \to \mathbb{R}^3$ of index $\alpha \in (0,1)$.*

(i) *For all $r \in [p, \frac{3p}{3-p})$, we have*

$$\sup_{\varepsilon \geq 0} |||\mathbf{w}^\varepsilon|||_{1,r,(T;p)} < \infty.$$

(ii) *We have*

$$\sup_{\varepsilon\geq 0}\sup_{t\in[0,T]} t^{1/2}\{\|\mathbf{u}^{\varepsilon}(t)\|_{\infty}+\|\mathbf{u}^{\varepsilon}(t)\|_{\mathcal{C}^{(2p-3)/p}}\}<\infty. \tag{27}$$

(iii) *For all* $r\in(3,\frac{3p}{3-p})$, $i=1,2,3$ *we have*

$$\sup_{\varepsilon\geq 0}\sup_{t\in[0,T]} t^{1/2+3/2(1/p-1/r)}\left\{\left\|\frac{\partial\mathbf{u}^{\varepsilon}(t)}{\partial x_i}\right\|_{\infty}+\left\|\frac{\partial\mathbf{u}^{\varepsilon}(t)}{\partial x_i}\right\|_{\mathcal{C}^{1-3/r}}\right\}<\infty. \tag{28}$$

In particular, the functions

$$t\mapsto\|\mathbf{u}(t)\|_{\infty}\quad\textit{and}\quad t\mapsto\left\|\frac{\partial\mathbf{u}(t)}{\partial x_i}\right\|_{\infty},\qquad i=1,2,3,$$

belong to $L^1([0,T],\mathbb{R})$.

PROOF. Observe that parts (i) and (ii) of Lemma 3.4 provide an estimate of the form

$$|||\mathbf{w}^{\varepsilon}|||_{1,l',(T;p)}\leq C(l',p)(\|w_0\|_p+T|||\mathbf{g}|||_{0,p,T})+\Lambda(T,l,l')A_l^2$$

for suitable l and l' and with $\Lambda(T,l,l')$ a uniform upper bound for the norms of the operators $\mathbf{B}^{\varepsilon}:(\mathbf{F}_{1,l,(T;p)})^2\to\mathbf{F}_{1,l',(T;p)}$ and A_l a given upper bound of $|||\mathbf{w}^{\varepsilon}|||_{1,l,(T;p)}$. Then, starting from the fact that the functions $\mathbf{w}^{\varepsilon}\in\mathbf{F}_{1,p,(T;p)}=\mathbf{F}_{1,p,T}$ are uniformly bounded in $\varepsilon\geq 0$, we can apply several times Lemma 3.4 and the previous inequality (using, also, the fact that $\mathbf{w}_0\in\mathbf{F}_{1,l',(T;p)}$ for all $l'\in[p,\frac{3p}{3-p}))$, and obtain an increasing sequence $l'=l_n$ such that $l_0=p$, $l_n\nearrow\frac{3p}{3-p}$, and $\mathbf{w}^{\varepsilon}\in\mathbf{F}_{1,l_n,(T;p)}$ with $|||\mathbf{w}^{\varepsilon}|||_{1,l_n,(T;p)}$ controlled in terms of $|||\mathbf{w}^{\varepsilon}|||_{1,l_{n-1},(T;p)}$ and $|||\mathbf{w}_0|||_{1,l_n,(T;p)}$. One can thus chose N large enough such that $l_N\geq r$ and conclude with an interpolation inequality in the spaces $\mathbf{F}_{1,l,(T;p)}$. We refer to the proof of Theorem 3.2(ii) in [9] for this and for an explicit construction of the sequence l_n.

Next, Lemma 3.3 and Theorem 3.6 imply that for $q=\frac{3p}{3-p}>3$,

$$\sup_{\varepsilon\geq 0}|||\mathbf{u}^{\varepsilon}|||_{1,q,T}\leq C\sup_{\varepsilon\geq 0}|||\mathbf{w}^{\varepsilon}|||_{1,p,T}\leq C'(\|w_0\|_p+T|||\mathbf{g}|||_{0,p,T}).$$

Using the continuous embedding of $(W^{1,m})^3$ into $(L^{\infty})^3\cap\mathcal{C}^{1-3/m}$ for all $m>3$, we deduce part (ii), taking $m=q$. To prove part (iii) we use part (i), Lemma 3.7 and the same embedding result as before but with $m=r$. See Corollary 3.1 in [9] for details. □

4. The nonlinear process. We shall, in this section, use the notation $F_{0,p,T}$, $F_{1,p,T}$, $F_{0,r,(T;p)}$ and $F_{1,r,(T;p)}$ for the scalar-function analogues of the spaces $\mathbf{F}$ defined in Section 3.

We also need the following definition.

DEFINITION 4.1. $\mathcal{P}^T_{b,3/2}$ is the space of probability measures $Q \in \mathcal{P}_b(\mathcal{C}_T)$ satisfying the following conditions:

- For each $t \in [0, T]$, $Q^\circ_t(dx)$ defined in Definition 2.3 is absolutely continuous with respect to Lebesgue's measure.
- The family of densities of $(Q^\circ_t(dx))_{t\in[0,T]}$, which we denote by $(t, x) \mapsto \rho^Q(t, x)$, has a version that belongs to $F_{0,p,T}$ for some $p > \frac{3}{2}$.
- The family of densities of the vectorial measures $(\tilde{Q}_t(dx))_{t\in[0,t]}$ [cf. (13)], which we denote by $(t, x) \mapsto \tilde{\rho}^Q(t, x)$, satisfies $\operatorname{div} \tilde{\rho}^Q_t = 0$ for dt-almost every $t \in [0, T]$.

We are ready to study the nonlinear process described in (MP).

THEOREM 4.2. *Assume that* (H$_1$) *and* (H$_p$) *are satisfied for some* $p \in (\frac{3}{2}, 3)$. *Then, the following hold*:

(a) *For every* $T > 0$, *the nonlinear martingale problem* (MP) *has, at most, one solution* P *in the class* $\mathcal{P}^T_{b,3/2}$. *Moreover, if such a solution* P *exists, then the function defined by*

$$\mathbf{w}(t, x) := \tilde{\rho}^P(t, x) = \rho^P(t, x) E^P\big(\Phi_t h(\tau, X_0)\mathbf{1}_{\{t\geq\tau\}}|X_t = x\big)$$

is the unique solution in $\mathbf{F}_{0,1,T} \cap \mathbf{F}_{0,p,T}$ *of the mild equation* (19).

(b) *In a given filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, *consider a standard three-dimensional Brownian motion* B, *and an* $\mathcal{F}_0$-*measurable random variable* (τ, X_0) *independent of* B *with law* P_0 [*defined as in* (11)]. *Then, on each interval* $[0, T]$, *the McKean nonlinear stochastic differential equation*

$$\begin{aligned} &\text{(i)} \quad X_t = X_0 + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{s\geq\tau\}}\,dB_s + \int_0^t \mathbf{K}(\tilde{\rho})(s, X_s)\mathbf{1}_{\{s\geq\tau\}}\,ds,\\ &\text{(ii)} \quad \Phi_t = I_3 + \int_0^t \nabla\mathbf{K}(\tilde{\rho})(s, X_s)\Phi_s\mathbf{1}_{\{s\geq\tau\}}\,ds, \\ &\text{(iii)} \quad \operatorname{law}(\tau, X, \Phi) \in \mathcal{P}^T_{b,3/2} \quad \textit{and} \quad \tilde{\rho}(t, x) = \tilde{\rho}^{\operatorname{law}(\tau,X,\Phi)}(t, x),\end{aligned} \tag{29}$$

has, at most, one pathwise solution. Moreover, if a solution exists, its law is a solution of (MP). *Thus, by* (a), *uniqueness in law for* (29) *holds.*

(c) *If the condition*

$$T^{1-3/(2p)}(\|w_0\|_p + T\,|||\mathbf{g}|||_{0,p,\theta}) < \Gamma_0(p)$$

is satisfied, where $\Gamma_0(p) > 0$ *is the constant provided by Theorem* 3.6, *then a unique solution* $P \in \mathcal{P}^T_{b,3/2}$ *to* (MP) *exists. Moreover, under the previous condition, strong existence holds for the nonlinear stochastic differential equation* (29) *in* $[0, T]$, *and by* (a) *and* (b), *one has* $P = \operatorname{law}(\tau, X, \Phi)$. *Finally,* ρ^P *is the unique solution in* $\mathbf{F}_{0,1,T} \cap \mathbf{F}_{0,p,T}$ *to the vortex equation* (19).

The proof of Theorem 4.2 requires some preliminary facts about a scalar problem implicitly included in the vectorial problem (MP).

4.1. *A nonlinear Fokker–Planck equation with external field associated with the* 3*d-vortex equation.* Recall that the notation $\tilde{Q}_t$ and Q_t° were, respectively, defined in Definition 2.3 and (13).

For any $Q \in \mathcal{P}(\mathcal{C}_T)$, we now denote by $\hat{Q}_t$ the sub-probability measure on $\mathbb{R}^3$ defined for scalar functions by

$$\hat{Q}_t(f) = \mathbb{E}^Q\big(f(X_t)\mathbf{1}_{\{\tau \le t\}}\big), \tag{30}$$

where (τ, X) are the two first marginal of the canonical process (τ, X, Φ) in $\mathcal{C}_T$. Obviously, for $Q \in \mathcal{P}_b(\mathcal{C}_T)$ we have

$$\tilde{Q}_t \ll \hat{Q}_t \ll Q_t^\circ,$$

and we shall denote

$$k_t^Q(x) := \frac{d\tilde{Q}_t}{d\hat{Q}_t}(x). \tag{31}$$

Notice that, indeed,

$$k_t^Q(x) = \frac{E^Q(\Phi_t h(\tau, X_0)\mathbf{1}_{\{\tau \le t\}} | X_t = x)}{Q(\tau \le t | X_t = x)} \mathbf{1}_{\{Q(\tau \le t | X_t = x) > 0\}}.$$

DEFINITION 4.3. If $Q_t^\circ(dx)$ has a density $\rho^Q(t, x)$ with respect to Lebesgue measure, we shall denote by $\hat{\rho}^Q(t, x)$ the family of densities of $\hat{Q}_t$.

Notice that one has

$$\tilde{\rho}^Q(t, x) = k_t^Q(x)\hat{\rho}^Q(t, x).$$

REMARK 4.4. If $Q \in \mathcal{P}_b(\mathcal{C}_T)$ is such that Q_t is absolutely continuous for all $t \in [0, T]$, the existence of a joint measurable version of $(t, x) \mapsto \rho^Q(t, x)$ is standard by continuity of X_t under Q_t°. We always work with such a version. Moreover, there exist measurable versions of $(t, x) \mapsto \hat{\rho}^Q(t, x)$ and $(t, x) \mapsto \tilde{\rho}^Q(t, x)$. This can be seen by Lebesgue derivation (see, e.g., Theorem 3.22 in [8]), taking $\delta \to 0$ in the quotients

$$\frac{Q(\tau \le t, X_t \in B(x, \delta))}{Q(X_t \in B(x, \delta))} \quad \text{and} \quad \frac{E^Q(\Phi_t h(\tau, X_0)\mathbf{1}_{\{\tau \le t\}}, X_t \in B(x, \delta))}{Q(X_t \in B(x, \delta))}$$

and using the previous relation between $\hat{\rho}^Q(t, x)$ and k^Q [here, $B(x, \delta)$ is the open ball of radius r centered at x].

LEMMA 4.5. *Assume that (MP) has a solution $P \in \mathcal{P}_b(\mathcal{C}_T)$ such that P_t° has a density for each $t \in [0,T]$. Let $\hat{\rho} := \hat{\rho}^P$ and $\tilde{\rho} := \tilde{\rho}^P$, respectively, denote the densities of $\hat{P}_t$ and $\tilde{P}_t$ and, moreover, assume that (15) holds. We have*:

(i) *The couple $(\hat{\rho}, \tilde{\rho})$ satisfies the weak evolution equation*

$$
\begin{aligned}
&\int_{\mathbb{R}^3} f(t,y)\hat{\rho}(t,y)\,dy \\
&\qquad = \int_{\mathbb{R}^3} f(0,y)\bar{w}_0(y)\,dy + \int_0^t \int_{\mathbb{R}^3} f(s,y)\bar{\mathbf{g}}(s,y)\,dy\,ds \\
&\qquad\quad + \int_0^t \int_{\mathbb{R}^3} \biggl[\frac{\partial f}{\partial s}(s,y) + \nu \triangle f(s,y) \\
&\qquad\qquad + \mathbf{K}(\tilde{\rho})(s,y)\nabla f(s,y)\biggr]\hat{\rho}(s,y)\,dy\,ds,
\end{aligned}
\tag{32}
$$

for all $f \in C_b^{1,2}$, where $\bar{w}_0$ and $\bar{\mathbf{g}}$ were defined in (10).

(ii) *$\hat{\rho}$ is, moreover, a solution of the mild equation in* $[0,T]$,

$$
\begin{aligned}
\hat{\rho}(t,x) &= G_t^\nu * \bar{w}_0(x) + \int_0^t G_{t-s}^\nu * \bar{\mathbf{g}}(s,\cdot)(x)\,ds \\
&\quad + \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y)\mathbf{K}(k\hat{\rho})_j(s,y)\hat{\rho}(s,y)\,dy\,ds,
\end{aligned}
\tag{33}
$$

with the multiple integral being absolutely convergent, and where $k := k^P$ is the function defined in (31).

PROOF. (i) By the definition of (MP) and the fact that $\mathbf{1}_{\{\tau \le t\}}$ is $\mathcal{F}_0$-measurable, we deduce that the expectation of the expression

$$
\begin{aligned}
&f(t,X_t)\mathbf{1}_{\{t\ge\tau\}} - f(\tau,X_0)\mathbf{1}_{\{t\ge\tau\}} \\
&\quad - \int_0^t \biggl[\frac{\partial f}{\partial s}(s,X_s) + \nu\triangle f(s,X_s)\,ds + \mathbf{K}(\tilde{\rho})(s,X_s)\nabla f(s,X_s)\biggr]\mathbf{1}_{\{s\ge\tau\}}\,ds
\end{aligned}
$$

vanishes (see also the beginning of the proof of Lemma 2.6). Taking expectation and recalling the definition of $\hat{\rho}$ and P_0 [cf. (30) and (11)], we obtain the desired result applying Fubini's theorem in the time integral, which is possible since

$$\int_{[0,T]\times\mathbb{R}^3} |\mathbf{K}(\tilde{\rho})(t,x)|\hat{\rho}(t,x)\,dx\,dt < \infty,$$

thanks to condition (15).

(ii) Fix $\psi \in \mathcal{D}$ and $t \in [0,T]$ and take in (32) the $C_b^{1,2}$-function $f_t : [0,t]\times\mathbb{R}^3 \to \mathbb{R}^3$ given by $f_t(s,y) = G_{t-s}^\nu * \psi(y)$ (which solves the backward heat equation on $[0,t]\times\mathbb{R}^3$ with final condition $f_t(t,y)=\psi(y)$). By Lemma 3.1 and condition (15), it is not hard to check that

$$\int_0^t\int_{(\mathbb{R}^3)^2}\sum_{j=1}^3\left|\frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y)\right||\mathbf{K}(\tilde\rho)_j(s,y)||\psi(x)|\rho(s,y)\,dx\,dy\,ds<\infty.$$

By Fubini's theorem we easily conclude. □

Consider now a fixed but arbitrary function $k:[0,T]\times\mathbb{R}^3\to\mathbb{R}^3$ of class $L^\infty([0,T],(L^\infty)^3)$, and formally define an operator $\mathbf{b}^k$ on functions $\eta,\nu\in\mathcal{M}eas^T$ by

$$\mathbf{b}^k(\eta,\nu)(t,x)=\int_0^t\sum_{j=1}^3\int_{\mathbb{R}^3}\frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y)\mathbf{K}(k\nu)_j(s,y)\eta(s,y)\,dy\,ds.$$

REMARK 4.6. For each $p\in[1,\infty]$ (resp., each $p\in[1,\infty]$ and $r\geq p$), the mapping $\eta\mapsto k\eta$ is continuous from $F_{0,p,T}$ to $\mathbf{F}_{0,p,T}$ (resp., from $F_{0,r,(T;p)}$ to $\mathbf{F}_{0,r,(T;p)}$).

Write now

$$\gamma_0(t,x):=G^\nu_t*\bar w_0(x)+\int_0^t G^\nu_{t-s}*\bar{\mathbf{g}}(s,\cdot)(x)\,ds,$$

where $\bar w_0$ and $\bar{\mathbf{g}}$ were defined in (10). We can state the following properties of the scalar equation (33).

PROPOSITION 4.7. *Assume* (H$_1$) *and* (H$_p$) *with* $p\in(\frac{3}{2},3)$, *and let* $k\in L^\infty([0,T],(L^\infty)^3)$ *be a fixed but arbitrary function.*

(i) *For each* $r\in[p,\infty)$, *we have*

$$\gamma_0\in F_{0,r,(T;p)}\qquad\text{with } |||\gamma_0|||_{0,r,(T;p)}\leq C(r,p)\|\bar w_0\|_p+T|||\bar{\mathbf{g}}|||_{0,p,T}$$

for some finite constant $C(r,p)>0$.

(ii) *Suppose that* $\frac{3}{2}<p<3$, $p\leq l<\min\{\frac{6p}{6-p},3\}$ *and* $\frac{3l}{6-l}\leq l'<\frac{3l}{6-2l}$. *Then, there exists a finite constant* $C_0(l,l';p)$ *not depending on* $T>0$ *such that for all* $\eta,\nu\in F_{0,l,(T;p)}$,

$$|||\mathbf{b}^k(\eta,\nu)|||_{0,l',(T;p)}\leq C_0(l,l';p)T^{1-3/(2p)}|||\eta|||_{0,l,(T;p)}|||\nu|||_{0,l,(T;p)}.$$

(iii) *The mild Fokker–Planck equation with external field* (33) *has, at most, one solution* $\hat\rho\in F_{0,p,T}$ *for each* $T>0$.

(iv) *If* $\hat\rho\in F_{0,p,T}$ *is a solution of* (33), *then* $\hat\rho\in F_{0,r,(T;p)}$ *for all* $r\in[p,\infty)$ *with* $|||\hat\rho|||_{0,r,(T;p)}\leq C(T,p,r,|||\hat\rho|||_{0,p,T})<\infty$.

(v) *We deduce that for all* $l\in[\frac{3p}{3-p},\infty)$, $\mathbf{K}(k\hat\rho)\in\mathbf{F}_{1,l,(T;3p/(3-p))}$.

PROOF. Part (i) follows from Lemma 3.1 in a similar way as part (i) of Lemma 3.4. We notice that the restriction on r in the latter was needed only to ensure that the derivative of time integral was convergent, and so it is not needed here. Thanks to Remark 4.6, part (ii) is similar to part (ii) of Proposition 3.1 in [9].

From the previous parts, equation (33) admits the abstract formulation in $F_{0,p,T}$

$$\hat{\rho} = \gamma_0 + \mathbf{b}^k(\hat{\rho}, \hat{\rho}).$$

Then, the arguments yielding parts (i) of Theorems 3.6 and 3.8 also provide the assertions of parts (iii) and (iv), respectively. For part (v), we notice that from (iv), $k\hat{\rho} \in \mathbf{F}_{0,r,(T;p)}$ holds for all $r \in [p, \infty[$. Thus, if we take $l \geq q := \frac{3p}{3-p}$ and set $r := (\frac{1}{l} + \frac{1}{3})^{-1}$, then one has $r \geq p$, and so Lemma 3.3(i) implies that

$$\sup_{t\in[0,T]} t^{3/2(1/p-1/r)} \|\mathbf{K}(k\hat{\rho})(t,\cdot)\|_l = \sup_{t\in[0,T]} t^{3/2(1/q-1/l)} \|\mathbf{K}(k\hat{\rho})(t,\cdot)\|_l < \infty.$$

This shows that $\mathbf{K}(k\hat{\rho}) \in \mathbf{F}_{0,l,(T;q)}$. We conclude that $\mathbf{K}(k\hat{\rho}) \in \mathbf{F}_{1,l,(T;q)}$, noting that since $k\hat{\rho} \in \mathbf{F}_{0,l,(T;p)}$ for all $l \geq q$, Lemma 3.7(i) implies that $\frac{\partial \mathbf{K}(k\hat{\rho})}{\partial x_k} \in \mathbf{F}_{0,l,(T;p)}$ for all $k = 1, 2, 3$. In other words,

$$\begin{aligned} &\sup_{t\in[0,T]} t^{3/2(1/p-1/l)} \left\| \frac{\partial \mathbf{K}(k\hat{\rho})(t,\cdot)}{\partial x_k} \right\|_l \\ &\quad = \sup_{t\in[0,T]} t^{1/2+3/2(1/q-1/l)} \left\| \frac{\partial \mathbf{K}(k\hat{\rho})(t,\cdot)}{\partial x_k} \right\|_l < \infty, \end{aligned}$$

which is the required estimate. $\square$

4.2. *Uniqueness in law and pathwise uniqueness*. We need the following version of Gronwall's lemma:

LEMMA 4.8. *Let g and k be positive functions on $[0, T]$, such that $\int_0^T k(s)\,ds < \infty$, g is bounded, and*

$$g(t) \leq C + \int_0^t g(s)k(s)\,ds \qquad \text{for all } t \in [0, T].$$

Then, we have

$$g(t) \leq C \exp \int_0^T k(s)\,ds \qquad \text{for all } t \in [0, T].$$

We are ready to prove parts (a) and (b) in Theorem 4.2.

PROOF OF THEOREM 4.2. Let $P \in \mathcal{P}^T_{b,3/2}$ be a solution of (MP). Since $\rho \in F_{0,1,T} \cap F_{0,p,T}$, by interpolation we have $\rho \in F_{0,3/2,T}$. By Lemma 3.3(i) we deduce that (15) holds. Moreover, by Lemma 4.5(ii), Proposition 4.7(iv) and

Lemma 3.7(i), we have that $\nabla \mathbf{K}(\tilde{\rho}) \in F_{0,3,(T;p)}$, and, consequently, condition (16) also holds. By Lemma 2.6 we deduce that $\tilde{\rho}$ is a weak solution of the vortex equation, and, since k_t^P is bounded, we have $\tilde{\rho} \in \mathbf{F}_{0,p,T}$.

We now need to prove that the latter implies that $\tilde{\rho} \in \mathbf{F}_{0,p,T}$ is uniquely determined. By Theorem 3.6(a) this will follow by checking that $\tilde{\rho}$ is also mild solution. For fixed $\psi \in (\mathcal{D})^3$ and $t \in [0, T]$, define $\mathbf{f}_t : [0, t] \times \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{f}_t(s, y) = G_{t-s}^{\nu} * \psi(y)$, which is a function of class $(C_b^{1,2})^3$ that solves the backward heat equation on $[0, t] \times \mathbb{R}^3$ with final condition $\mathbf{f}(t, y) = \psi(y)$. One can thus take $\mathbf{f}_t$ in the weak vortex equation and, thanks to conditions (15) and (16), apply Fubini's theorem to deduce [since $\psi \in (\mathcal{D})^3$ is arbitrary] that

$$\tilde{\rho}(t, x) = \mathbf{w}_0(t, x) + \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \left[\frac{\partial G_{t-s}^{\nu}}{\partial y_j}(x - y)[\mathbf{K}(\tilde{\rho})_j(s, y)\tilde{\rho}(s, y)] \right.$$
$$\left. + G_{t-s}^{\nu}(x - y)\left[\tilde{\rho}_j(s, y)\, \frac{\partial \mathbf{K}(\tilde{\rho})}{\partial y_j}(s, y)\right]\right] dy\, ds.$$

Since $\tilde{\rho}$ is divergence-free, to see that $\tilde{\rho}$ solves the mild equation it is enough to justify an integration by parts of the last term in the previous equation. We cannot do that at this point since we cannot ensure enough (Sobolev) regularity of $\tilde{\rho}$. But noting that for $q = \frac{3p}{3-p}$ one has $1 < q^* < \frac{3}{2}$, we see that the function $\tilde{\rho} = k^P \hat{\rho}$ belongs to $\mathbf{F}_{0,q^*,T}$ by interpolation. On the other hand, one has $G_{t-s}^{\nu}(x - \cdot)\mathbf{K}(\tilde{\rho})(s, \cdot) \in (W^{1,q})^3$ thanks to Proposition 4.7(v). Since by hypothesis, $\operatorname{div} \tilde{\rho}(s) = 0$ in the distribution sense, the fact that $\tilde{\rho}(s) \in (L^q)^3$ and a density argument allow us to check that

$$\sum_{j=1}^3 \int_{\mathbb{R}^3} \tilde{\rho}_j(s, y)\, \frac{\partial}{\partial y_j}[G_{t-s}^{\nu}(x - y)\mathbf{K}(\tilde{\rho})(s, y)]\, dy = 0$$

for all $s \in\,]0, T]$. Thus, $\mathbf{w} := \tilde{\rho}$ is the unique solution of (19) in $\mathbf{F}_{0,p,T}$.

Now, by a standard argument using the semi-martingale decomposition of the coordinate processes X^i and their products $X^i X^j$, we obtain that the martingale part of $f(t, X_t)$ in (MP) is given by the stochastic integral $\sqrt{2\nu}\int_0^t \nabla f(s, X_s) \times \mathbf{1}_{\{s \geq \tau\}}\, dB_s$, with respect to a Brownian motion B defined on some extension of the canonical space. From this and the previously established uniqueness of $\tilde{\rho}$, P is the law of a weak solution of the stochastic differential equation

$$\text{(34)} \quad \begin{aligned} &\text{(i)} \quad X_t = X_0 + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{s \geq \tau\}}\, dB_s + \int_0^t \mathbf{K}(\mathbf{w})(s, X_s)\mathbf{1}_{\{s \geq \tau\}}\, ds, \\ &\text{(ii)} \quad \Phi_t = I_3 + \int_0^t \nabla \mathbf{K}(\mathbf{w})(s, X_s)\Phi_s \mathbf{1}_{\{s \geq \tau\}}\, ds. \end{aligned}$$

Since (34) is *linear* in the sense of McKean, to conclude uniqueness in law it is enough to prove pathwise uniqueness for it. This is done first for X and then for Φ, both with help of the estimate on $\|\nabla\mathbf{K}(\mathbf{w})(t)\|_\infty$ in Theorem 3.8 and Gronwall's lemma. $\square$

4.3. *Pathwise convergence of the mollified processes and strong existence for small time*. To prove part (c) of Theorem 4.2, we shall construct a strong solution to the nonlinear SDE of part (b) therein. We shall do so via approximation by solutions to nonlinear SDEs with regular drift terms $\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)$ and $\nabla\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)$, where for each $\varepsilon > 0$, $\mathbf{w}^\varepsilon \in F_{1,p,T} \cap F_{0,1,T}$ is given by Theorem 3.6. Thus, our arguments improve the ones developed in [9] by providing a pathwise approximation result at an explicit rate. This will be the key to carry out the additional improvements on that work in the forthcoming sections.

If $q = \frac{3p}{3-p}$, Hölder's inequality and the properties of $\mathbf{K}$ imply that that for all $t \in [0,T]$,

$$\begin{aligned}\|\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(t,\cdot)\|_\infty &\le C\|\varphi_\varepsilon\|_{q^*}|||\mathbf{K}(\mathbf{w}^\varepsilon)|||_{0,q,T}\\ &\le C\|\varphi_\varepsilon\|_{q^*}|||\mathbf{w}^\varepsilon|||_{0,p,T}.\end{aligned}$$

Similarly, one has $\|\nabla\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(t)\|_\infty \le C\|\nabla\varphi_\varepsilon\|_{q^*}|||\mathbf{w}^\varepsilon|||_{0,p,T}$ and analogous estimates hold for all derivatives. Thus, for each $\varepsilon > 0$, the function $(s,y) \mapsto \mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(s,y)$ is bounded and continuous in $y \in \mathbb{R}^3$, and has infinitely many derivatives in $y \in \mathbb{R}^3$, which are uniformly bounded in $[0,T] \times \mathbb{R}^3$.

We fix now the time interval $[0,T]$ given by Theorem 4.2. It will be useful to consider in what follows the stochastic flow

$$\begin{aligned}\xi^\varepsilon_{s,t}(x) &= x + \sqrt{2\nu}(B_t - B_s)\\ &\quad + \int_s^t \mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(\theta, \xi^\varepsilon_{s,\theta}(x))\,d\theta \qquad \text{for all } t \in [s,T],\end{aligned}\tag{35}$$

which has a version that is continuously differentiable in x for all (s,t) thanks to the previously mentioned regularity properties of $\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)$ (cf. Kunita [15]).

We also consider the strong solution of the stochastic differential equation in $[0,T]$,

$$\begin{aligned}X^\varepsilon_t &= X_0 + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{s\ge\tau\}}\,dB_s + \int_0^t \mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(s, X^\varepsilon_s)\mathbf{1}_{\{s\ge\tau\}}\,ds,\\ \Phi^\varepsilon_t &= I_3 + \int_0^t \nabla\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(s, X^\varepsilon_s)\Phi^\varepsilon_s\mathbf{1}_{\{s\ge\tau\}}\,ds,\end{aligned}\tag{36}$$

where (τ, X_0) is independent of B. We denote by P^ε the joint law of $(\tau, X^\varepsilon, \Phi^\varepsilon)$ and observe that $P^\varepsilon \in \mathcal{P}^T_b$. Since $X^\varepsilon_t = X_0$ for all $t \le \tau$, we have that

$$X^\varepsilon_t = \xi^\varepsilon_{\tau,t}(X_0)\mathbf{1}_{\{t\ge\tau\}} + X_0\mathbf{1}_{\{t<\tau\}}.$$

Denoting by $G^\varepsilon(s,x;t,y), (s,x,t,y) \in (\mathbb{R}_+ \times \mathbb{R}^2)^2, s < t$, the density of $\xi^\varepsilon_{s,t}(x)$ (which is a continuous function of (s,x,t,y), see [12]), and conditioning with respect to (τ, X_0), we obtain for bounded and measurable functions f that

$$\begin{aligned} E(f(X_t^\varepsilon)) &= \int_0^t \int_{(\mathbb{R}^3)^2} f(y) G^\varepsilon(s,x;y,t)\, dy\, P_0(ds,dx) \\ &\quad + \int_t^T \int_{\mathbb{R}^3} f(x) P_0(ds,dx) \\ &= \int_{\mathbb{R}^3} f(x) \bar{w}_0(x)\, dx \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} f(y) G^\varepsilon(s,x;t,y)\, dy \right] \bar{\mathbf{g}}(s,x)\, dx\, ds \\ &\quad + \int_t^T \int_{\mathbb{R}^3} f(x) \bar{\mathbf{g}}(s,x)\, dx\, ds. \end{aligned}$$

Consequently, X_t^ε has a (bi-measurable) family of densities that we denote by ρ^ε. Observe that one has $\rho^\varepsilon(t) \in L^p$ for all $t \in [0,T]$ from the assumption on w_0 and $\mathbf{g}$ and standard Gaussian bounds for $G^\varepsilon(s,x;t,y)$.

The functions $\hat{\rho}^\varepsilon$ and $\tilde{\rho}^\varepsilon$ correspond to the densities of, respectively, the sub-probability measure and the vectorial measure

$$f \mapsto E\big[f(\xi^\varepsilon_{\tau,t}(X_\tau))\mathbf{1}_{\{t \geq \tau\}}\big]$$

and

$$\mathbf{f} \mapsto E\big[\mathbf{f}(\xi^\varepsilon_{\tau,t}(X_\tau))\nabla_x \xi^\varepsilon_{\tau,t}(X_\tau) h(\tau, X_0)\mathbf{1}_{\{t \geq \tau\}}\big].$$

They are bi-measurable by similar arguments as in Remark 4.4, and we have $\hat{\rho}^\varepsilon(t) \in L^p$ and $\tilde{\rho}^\varepsilon(t) \in L^p_3$.

The assumptions on φ ensure the following estimate concerning the approximations φ_ε of the Dirac mass (see Lemma 4.4 in Raviart [23]):

LEMMA 4.9. *Let φ be a cutoff function of order* 1. *Then, for all $v \in W^{1,r}$ and $r \in [1,\infty]$, one has*

$$\|v - \varphi_\varepsilon * v\|_r \leq C\varepsilon \sum_{i=1}^3 \left\| \frac{\partial v}{\partial x_i} \right\|_r.$$

We deduce the following result:

LEMMA 4.10. (i) *We have $\tilde{\rho}^\varepsilon = \mathbf{w}^\varepsilon$ and, consequently,*

$$\sup_{\varepsilon>0} |||\tilde{\rho}^\varepsilon|||_{0,p,T} < \infty \quad \textit{and} \quad \sup_{\varepsilon>0} |||\hat{\rho}^\varepsilon|||_{0,p,T} < \infty. \tag{37}$$

(ii) *If* φ *is a cutoff function of order* 1, *then we have that*

$$\sup_{t\in[0,T]} t^{3/(2p)-1/2}\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|_p \le C(T)\varepsilon$$

for some finite constant $C(T)$.

PROOF. (i) Since $E(\int_0^T |\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(t,X_t^\varepsilon)|\,dt)$ and $E(\int_0^T |\nabla\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(t,X_t^\varepsilon)|\,dt)$ are finite, we can follow the lines of Lemma 2.6 and use Remark 2.4 to see that for all $\mathbf{f}\in(C_b^{1,2})^3$,

$$\begin{aligned}
&\int_{\mathbb{R}^3}\mathbf{f}(t,y)\tilde\rho^\varepsilon(t,y)\,dy\\
&\qquad=\int_{\mathbb{R}^3}\mathbf{f}(0,y)w_0(y)\,dy+\int_0^t\int_{\mathbb{R}^3}\mathbf{f}(s,y)\mathbf{g}(s,y)\,dy\,ds\\
&\qquad\quad+\int_0^t\int_{\mathbb{R}^3}\biggl[\frac{\partial\mathbf{f}}{\partial s}(s,y)+\nu\triangle\mathbf{f}(s,y)\\
&\qquad\qquad\qquad+\nabla\mathbf{f}(s,y)\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(s,y)\\
&\qquad\qquad\qquad+\mathbf{f}(s,y)\nabla\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(s,y)\biggr]\tilde\rho^\varepsilon(s,y)\,dy\,ds.
\end{aligned}\tag{38}$$

On the other hand, the regularity properties of the stochastic flow (35) imply that for all $\phi\in\mathcal{D}$ and $\theta\in\,]0,T]$, the Cauchy problem

$$\begin{aligned}
&\frac{\partial}{\partial s}f(s,y)+\nu\Delta f(s,y)\\
&\qquad+\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(s,y)\nabla f(s,y)=0,\qquad (s,y)\in[0,\theta[\,\times\mathbb{R}^3,\\
&f(\theta,y)=\phi(y)
\end{aligned}\tag{39}$$

has a unique solution f that belongs to $C_b^{1,3}([0,\theta]\times\mathbb{R}^3)$ (see Lemma 4.3 in [9]). One can thus use the function $\mathbf{f}=\nabla f$ in (38), and after simple computations obtain, thanks to the null divergence of w_0 and $\mathbf{g}(s,\cdot)$, that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\nabla\phi(y)\tilde\rho^{(n)}(t,y)\,dy\\
&\qquad=\int_0^t\int_{\mathbb{R}^3}\nabla\biggl[\frac{\partial f}{\partial s}(s,y)+\nu\triangle f(s,y)+\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(s,y)\nabla f(s,y)\biggr]\\
&\qquad\qquad\qquad\times\tilde\rho^{(n)}(s,y)\,dy\,ds=0
\end{aligned}$$

for all $\phi\in\mathcal{D}$. Thus, $\operatorname{div}\tilde\rho^\varepsilon(t)=0$, and we can adapt the arguments of Section 4.2 to conclude that $\tilde\rho^\varepsilon$ solves the linear mild equation

$$\mathbf{v}=\mathbf{w}_0+\mathbf{B}^\varepsilon(\mathbf{v},\mathbf{w}^\varepsilon),\qquad \mathbf{v}\in\mathbf{F}_{0,p,T}.\tag{40}$$

Since uniqueness for (40) holds (by similar arguments as for the nonlinear version), and $\mathbf{w}^\varepsilon$ also solves the equation, we conclude that $\tilde{\rho}^\varepsilon = \mathbf{w}^\varepsilon$. The asserted uniform bound for $\tilde{\rho}^\varepsilon$ is thus granted by Theorem 3.6. To obtain the uniform bound for $\hat{\rho}^\varepsilon$, we take L^p norm to (40), and follow the arguments of the proof of Theorem 3.6(i), to get that

$$\|\tilde{\rho}^\varepsilon(t)\|_p \le \||\mathbf{w}_0\||_{0,p,T} + C\||\mathbf{w}^\varepsilon\||_{0,p,T} \int_0^t (t-s)^{-3/(2p)} \|\tilde{\rho}^\varepsilon(s)\|_p \, ds.$$

The conclusion follows by a similar application of Gronwall's lemma as therein.

(ii) By an iterative argument as in the proof of Theorem 3.6(i), we get that

$$\begin{aligned} \|\tilde{\rho}^\varepsilon(t) - \mathbf{w}(t)\|_p \le C \int_0^t \alpha(t-s) \|\mathbf{K}^\varepsilon(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q \, ds \\ + C(T) \int_0^t \|\tilde{\rho}^\varepsilon(s) - \mathbf{w}(s)\|_q \, ds, \end{aligned} \tag{41}$$

where $\alpha(s) = \sum_{k=1}^{\tilde{N}(p)} s^{k\theta_0 - 1}$, $\theta_0 = 1 - \frac{3}{2p}$ and $\tilde{N}(p) = \lfloor \theta_0^{-1} \rfloor + 1$. Integrating in time and using Gronwall's lemma, Theorem 3.8(i) and Lemma 4.9, we obtain that for all $\theta \in [0, T]$,

$$\begin{aligned} \int_0^\theta \|\tilde{\rho}^\varepsilon(t) - \mathbf{w}(t)\|_p \, dt &\le C \int_0^T \int_0^t \alpha(t-s) \|\mathbf{K}^\varepsilon(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q \, ds \, dt \\ &\le C\varepsilon \int_0^T \sum_{k=1}^{\tilde{N}(p)} t^{k(1-3/(2p))-1/2} \, dt = \varepsilon C(T). \end{aligned}$$

Substituting the latter in (41), we obtain

$$\begin{aligned} \|\tilde{\rho}^\varepsilon(t) - \mathbf{w}(t)\|_p &\le \varepsilon C(T) + C \int_0^t \alpha(t-s) \|\mathbf{K}^\varepsilon(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q \, ds \\ &\le \varepsilon C(T) + C t^{1/2 - 3/(2p)} \varepsilon, \end{aligned}$$

and the conclusion follows. □

The proof of Theorem 4.2(c) will be completed by the following result, which, moreover, establishes the strong pathwise convergence of the nonlinear processes $(X^\varepsilon, \Phi^\varepsilon)$ as $\varepsilon \to 0$. We are inspired here by ideas introduced in [11], but we need a finer use of analytical properties, as we shall improve the rate of ε^δ with $\delta \in (0, 1)$, that was obtained therein for a particular choice of kernel. Further difficulties also will arise because of the additional (and more singular) drift term of the "vortex stretching processes" Φ, proper to dimension 3.

PROPOSITION 4.11. *Let φ be a cutoff of order* 1 *and K^ε be defined in terms of φ as before. Then*, *as ε goes to* 0, *the family of processes* $(X^\varepsilon - X_0, \Phi^\varepsilon)$, $\varepsilon > 0$

is Cauchy in the Banach space of continuous processes (Y, Ψ) *with values in* $\mathbb{R}^3 \times \mathbb{R}^{3\otimes 3}$ *with finite norm* $E(\sup_{t\in[0,T]}|Y_t| + |\Psi_t|)$. *Moreover, one has*

$$E\Big(\sup_{t\in[0,T]}|X_t - X_t^{\varepsilon}| + |\Phi_t - \Phi_t^{\varepsilon}|\Big) \leq C(T)\varepsilon,$$

where (X, Φ) *is a solution of the nonlinear s.d.e.* (29).

PROOF. We observe that the substraction of X_0 is only needed to avoid a moment-type assumption on X_0. Let $\varepsilon > \varepsilon' > 0$. We have

$$\begin{aligned}
&E\Big(\sup_{s\leq t}|X_s^{\varepsilon} - X_s^{\varepsilon'}|\Big)\\
&\quad\leq \int_0^t E\big|\big(\mathbf{K}^{\varepsilon}(\mathbf{w}^{\varepsilon})(s, X_s^{\varepsilon}) - \mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon})(s, X_s^{\varepsilon})\big)\mathbf{1}_{\{s\geq\tau\}}\big|\,ds\\
&\qquad+ \int_0^t E\big|\big(\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon})(s, X_s^{\varepsilon}) - \mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon'})(s, X_s^{\varepsilon})\big)\mathbf{1}_{\{s\geq\tau\}}\big|\,ds\\
&\qquad+ \int_0^t E\big|\big(\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon'})(s, X_s^{\varepsilon}) - \mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon'})(s, X_s^{\varepsilon'})\big)\mathbf{1}_{\{s\geq\tau\}}\big|\,ds.
\end{aligned} \tag{42}$$

The third term on the right-hand side of (42) is bounded thanks to Theorem 3.8(iii) by

$$C\int_0^t s^{-1/2-3/2(1/p-1/r)}E\Big(\sup_{\theta\leq s}|X_\theta^{\varepsilon} - X_\theta^{\varepsilon'}|\Big)\,ds$$

for any fixed $r \in (3, \frac{3p}{3-p})$. Writing $q = \frac{3p}{3-p}$ and q^* for its Hölder conjugate, and using Lemmas 3.3 and 4.10(ii), we bound the second term by

$$\int_0^T \|\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon})(s) - \mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon'})(s)\|_q \|\hat{\rho}^{\varepsilon}(s)\|_{q^*}\,ds \leq C(T)\varepsilon.$$

We have used the fact that $\sup_{\varepsilon>0}|||\hat{\rho}^{\varepsilon}|||_{0,q^*,T} < \infty$ by interpolation since $q^* < \frac{3}{2} < p$. By similar arguments, the first term on the right-hand side of (42) can be bounded above by

$$\int_0^T \|\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon})(s) - \mathbf{K}^{\varepsilon}(\mathbf{w}^{\varepsilon})(s)\|_q \|\hat{\rho}^{\varepsilon}(s)\|_{q^*}\,ds \leq C(T)\varepsilon.$$

Bringing all together and using Gronwall's lemma we deduce that

$$E\Big(\sup_{s\leq T}|X_t^{\varepsilon} - X_t^{\varepsilon'}|\Big) \leq C(T)\varepsilon. \tag{43}$$

Now, notice that Gronwall's lemma and Theorem 3.8(iii) imply that the processes Φ_t^{ε} are bounded in $L^{\infty}([0, T] \times \Omega, dt \otimes \mathbb{P})$ uniformly in ε. Therefore, we have

$$
\begin{aligned}
&E\Big(\sup_{s\le t}|\Phi^{\varepsilon}_s-\Phi^{\varepsilon'}_s|\Big)\\
&\quad\le C\int_0^t E\big|\big(\nabla\mathbf{K}^{\varepsilon}(\mathbf{w}^{\varepsilon})(s,X^{\varepsilon}_s)-\nabla\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon})(s,X^{\varepsilon}_s)\big)\mathbf{1}_{\{s\ge\tau\}}\big|\,ds\\
&\qquad+C\int_0^t E\big|\big(\nabla\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon})(s,X^{\varepsilon}_s)-\nabla\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon'})(s,X^{\varepsilon}_s)\big)\mathbf{1}_{\{s\ge\tau\}}\big|\,ds\\
&\qquad+C\int_0^t E\big|\big(\nabla\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon'})(s,X^{\varepsilon}_s)-\nabla\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon'})(s,X^{\varepsilon'}_s)\big)\mathbf{1}_{\{s\ge\tau\}}\big|\,ds\\
&\qquad+C\int_0^t E\Big(|\nabla\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon'})(s,X^{\varepsilon'}_s)|\sup_{\theta\le s}|\Phi^{\varepsilon}_\theta-\Phi^{\varepsilon'}_\theta|\Big)\,ds.
\end{aligned}\tag{44}
$$

By Theorem 3.8(iii), for fixed $r\in(3,q)$ the last term in the right-hand side of (44) is bounded by

$$
C\int_0^t s^{-1/2-3/2(1/p-1/r)}E\Big(\sup_{\theta\le s}|\Phi^{\varepsilon}_\theta-\Phi^{\varepsilon'}_\theta|\Big)\,ds,
$$

and the third one is by

$$
C\int_0^t s^{-1/2-3/2(1/p-1/r)}E|X^{\varepsilon}_s-X^{\varepsilon'}_s|\,ds\le C(T)\varepsilon,
$$

using also the previous estimates on $E|X^{\varepsilon}_s-X^{\varepsilon'}_s|$. The first term in (44) is upper bounded by

$$
C\int_0^T\|\hat\rho^{\varepsilon}(s)\|_{p^*}\|\nabla\mathbf{K}^{\varepsilon}(\mathbf{w}^{\varepsilon})(s)-\nabla\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon})(s)\|_p\,ds.\tag{45}
$$

If $p\ge 2$, then we have $p^*\le 2$ and so by (37) and interpolation, we deduce that (45) is bounded by

$$
\begin{aligned}
C|||\hat\rho^{\varepsilon}|||_{0,p^*,T}\int_0^T &\|\nabla\mathbf{K}(\varphi_\varepsilon*\mathbf{w}^{\varepsilon})(s)-\nabla\mathbf{K}(\mathbf{w}^{\varepsilon})\|_p\\
&+\|\nabla\mathbf{K}(\mathbf{w}^{\varepsilon})-\nabla\mathbf{K}(\varphi_{\varepsilon'}*\mathbf{w}^{\varepsilon})(s)\|_p\,ds\le CT\varepsilon.
\end{aligned}
$$

This last inequality is obtained by Lemmas 3.7(i), 4.9, 4.10(i) and the uniform boundedness of $(\mathbf{w}^{\varepsilon})_{\varepsilon\ge 0}$ in $\mathbf{F}_{1,p,T}$. If now $\frac{3}{2}<p<2$, then we have $3>p^*>2>p$ and by similar steps as in the previous case $p\ge 2$, we can upper bound (45) by

$$
\begin{aligned}
C|||\hat\rho^{\varepsilon}|||_{0,p^*,(T;p)}&\int_0^T s^{-3/2(1/p-1/p^*)}\|\nabla\mathbf{K}^{\varepsilon}(\mathbf{w}^{\varepsilon})(s)-\nabla\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon})(s)\|_p\,ds\\
&\le\varepsilon\sup_{\delta\ge 0}|||\hat\rho^{\delta}|||_{0,p^*,(T;p)}\int_0^T s^{-3/2(1/p-1/p^*)}s^{-1/2}\,ds\\
&\le\varepsilon C(T)\sup_{\delta\ge 0}|||\hat\rho^{\delta}|||_{0,p^*,(T;p)}.
\end{aligned}
$$

We have used here Lemma 4.9, the fact that $(\mathbf{w}^{\varepsilon})_{\varepsilon\geq 0}$ is uniformly bounded in $\mathbf{F}_{1,p,T}$ and that $-\frac{3}{2}(\frac{1}{p}-\frac{1}{p^*})-\frac{1}{2}>-1$ since $p>\frac{3}{2}$. The fact that the supremum in the previous estimate is finite, is seen in the same way as part (vi) of Proposition 4.7, namely by an iterative argument using the mild equation (similar as therein) satisfied by $\hat{\rho}^{\varepsilon}$, starting from the uniform bound in Lemma 4.10(i).

Thus, we have shown that the first term in the right-hand side of (44) is bounded by a constant times ε. Let us now tackle the second term in the right-hand side of (44). This is bounded by

$$\begin{aligned}&C\int_0^T \|\hat{\rho}^{\varepsilon}(s)\|_{p^*}\|\nabla\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon})(s)-\nabla\mathbf{K}^{\varepsilon'}(\mathbf{w}^{\varepsilon'})(s)\|_p\,ds\\&\qquad\leq C\int_0^T \|\hat{\rho}^{\varepsilon}(s)\|_{p^*}\|\mathbf{w}^{\varepsilon}(s)-\mathbf{w}^{\varepsilon'}(s)\|_p\,ds\end{aligned}\tag{46}$$

thanks to Lemma 3.7. By Lemma 4.10(ii) we can upper bound (46), respectively, by

$$C\varepsilon\int_0^T s^{1/2-3/(2p)}\,ds=\varepsilon C(T)$$

in the case $p\geq 2$, or by

$$C\varepsilon\int_0^T s^{-3/2(1/p-1/p^*)}s^{1/2-3/(2p)}\,ds=C'(T)\varepsilon$$

in the case $p<2$, where the constants are finite since $p>\frac{3}{2}$.

Consequently, we have an estimate of the form

$$E\Big(\sup_{s\leq t}|\Phi_s^{\varepsilon}-\Phi_s^{\varepsilon'}|\Big)\leq C\varepsilon+C\int_0^t s^{-1/2-3/2(1/p-1/r)}E\Big(\sup_{\theta\leq s}|\Phi_\theta^{\varepsilon}-\Phi_\theta^{\varepsilon'}|\Big)\,ds$$

for each fixed $r\in(3,q)$, and Gronwall's lemma yields

$$E\Big(\sup_{s\leq t}|\Phi_s^{\varepsilon}-\Phi_s^{\varepsilon'}|\Big)\leq C(T)\varepsilon\tag{47}$$

for all $\varepsilon\geq\varepsilon'>0$.

Estimates (43) and (47) thus show that $(X^{\varepsilon}-X_0,\Phi^{\varepsilon})$ is a Cauchy sequence in the Banach space of continuous processes (Y,Ψ) with values in $\mathbb{R}^3\times\mathbb{R}^{3\otimes 3}$ and finite norm $E(\sup_{t\in[0,T]}|Y_t|+|\Psi_t|)$. Write the limit in the form $(X-X_0,\Phi)$, for a continuous process (X,Φ) and define $\mathcal{E}_t^1$ and $\mathcal{E}_t^2$ by the relations

$$\begin{aligned}&X_t=X_0+\sqrt{2\nu}\int_0^t\mathbf{1}_{\{s\geq\tau\}}\,dB_s+\int_0^t\mathbf{K}(\mathbf{w})(s,X_s)\mathbf{1}_{\{s\geq\tau\}}\,ds+\mathcal{E}_t^1,\\&\Phi_t=I_3+\int_0^t\nabla\mathbf{K}(\mathbf{w})(s,X_s)\Phi_s\mathbf{1}_{\{s\geq\tau\}}\,ds+\mathcal{E}_t^2.\end{aligned}\tag{48}$$

Comparing (X,Φ) and $(X^{\varepsilon},\Phi^{\varepsilon})$, and using similar estimates as so far in this proof, but with 0 instead of ε' (and $\mathbf{w}$ instead of $\mathbf{w}^{\varepsilon'}$), we get that (X,Φ) satisfies (48) with $\mathcal{E}_t^i = 0$, $i = 1, 2$. Since that is a linear s.d.e. (in McKean's sense), the proof that (X, Φ) is the asserted nonlinear process will be achieved by checking that for all bounded Lipschitz function $\mathbf{f}: \mathbb{R}^3 \to \mathbb{R}^3$, one has

$$E\big(\mathbf{f}(X_t)\Phi_t h(\tau, X_0)\mathbf{1}_{\{s\geq\tau\}}\big) = \int_{\mathbb{R}^3} \mathbf{f}(x)\mathbf{w}(t,x)\,dx.$$

The latter follows from the facts that

$$E\big(\mathbf{f}(X_t^\varepsilon)\Phi_t^\varepsilon h(\tau, X_0)\mathbf{1}_{\{s\geq\tau\}}\big) = \int_{\mathbb{R}^3} \mathbf{f}(x)\mathbf{w}^\varepsilon(t,x)\,dx$$

and

$$\begin{aligned}
&\big|E\big(\mathbf{f}(X_t)\Phi_t h(\tau, X_0)\mathbf{1}_{\{s\geq\tau\}}\big) - E\big(\mathbf{f}(X_t^\varepsilon)\Phi_t^\varepsilon h(\tau, X_0)\mathbf{1}_{\{s\geq\tau\}}\big)\big| \\
&\quad\leq \big(\|\Phi\|_{L^\infty([0,T]\times\Omega)} + 1\big)\|h\|_\infty \|\mathbf{f}\|_{\mathrm{Lip}} E(|X_t - X_t^\varepsilon| + |\Phi_t - \Phi_t^\varepsilon|) \\
&\quad\leq C\|\mathbf{f}\|_{\mathrm{Lip}}\varepsilon.
\end{aligned} \tag{49}$$

□

REMARK 4.12. (a) By Lemma 4.10(i), the process $(X^\varepsilon, \Phi^\varepsilon)$ defined in (36) is a solution in $[0, T]$ of the nonlinear s.d.e.:

$$\begin{aligned}
&\text{(i)}\quad X_t^\varepsilon = X_0 + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{s\geq\tau\}}\,dB_s + \int_0^t \mathbf{K}^\varepsilon(\tilde\rho^\varepsilon)(s, X_s^\varepsilon)\mathbf{1}_{\{s\geq\tau\}}\,ds, \\
&\text{(ii)}\quad \Phi_t^\varepsilon = I_3 + \int_0^t \nabla\mathbf{K}^\varepsilon(\tilde\rho^\varepsilon)(s, X_s^\varepsilon)\Phi_s^\varepsilon\mathbf{1}_{\{s\geq\tau\}}\,ds \quad \text{and} \\
&\text{(iii)}\quad \text{the law } P^\varepsilon \text{ of } (\tau, X^\varepsilon, \Phi^\varepsilon) \text{ belongs to } \mathcal{P}_{b,3/2}^T \quad \text{and} \\
&\qquad \tilde P_t^\varepsilon(dx) = \tilde\rho^\varepsilon(t,x)\,dx.
\end{aligned} \tag{50}$$

(b) It is also possible to associate a unique pathwise solution of (29) with any solution $\mathbf{w} \in \mathbf{F}_{0,p,T} \cap \mathbf{F}_{0,1,T}$ of the mild vortex equation (i.e., not necessarily the one given by Theorem 3.6). This can be done by an approximation argument similar to the previous one, but considering linear processes in the sense of McKean [with drift terms $\mathbf{K}^\varepsilon(\mathbf{w})$ and $\nabla\mathbf{K}^\varepsilon(\mathbf{w})$] instead of the processes (36).

(c) Denote now by $\mathcal{W}_T$ the Wasserstein distance in $\mathcal{P}(\mathcal{C}_T)$ associated with the metric in $\mathcal{C}_T := [0, T] \times C([0, T], \mathbb{R}^3 \times \mathbb{R}^{3\otimes 3})$

$$\begin{aligned}
&d((\theta, y, \psi), (\eta, x, \phi)) \\
&\quad = |\theta - \eta| + \sup_{t\in[0,T]}\big(\min\{|x(t) - y(t)|, 1\} + \min\{|\psi(t) - \phi(t)|, 1\}\big).
\end{aligned}$$

Then, the previous proof states that

$$\mathcal{W}_T(P^\varepsilon, P) \leq C(T)\varepsilon,$$

where P is the law of the nonlinear process (29).

(d) By the regularity results of Section 3, one can prove in a similar way as in Corollary 4.3 of [9] that the stochastic flow (6) is of class C^1, in spite of the fact that $\mathbf{u}$ and $\nabla\mathbf{u}$ are singular at $t = 0$. Thus, identity (7) holds.

5. The stochastic vortex method. We first consider a McKean–Vlasov model with mollified interaction and cutoff. This extends the model studied in [9] to the present situation involving random space–time births.

Denote by M_ε the sup-norm of K_ε on $\mathbb{R}^3$ and by L_ε a Lipschitz constant for it, which, respectively, behave like $\frac{1}{\varepsilon^3}$ and $\frac{1}{\varepsilon^4}$ when $\varepsilon \ll 1$. Notice that $\operatorname{div} K_\varepsilon = (\operatorname{div} K) * \varphi_\varepsilon = 0$.

For $R > 0$, we denote by $\chi_R : \mathbb{R}^{3\otimes 3} \to \mathbb{R}^{3\otimes 3}$ a Lipschitz continuous truncation function such that $|\chi_R(\phi)| \leq R$. We may and shall assume that χ_R has Lipschitz constant less than or equal to 1.

Consider now a filtered probability space endowed with an adapted standard three-dimensional Brownian motion B and with a $[0, T] \times \mathbb{R}^3$-valued random variable (τ, X_0) independent of B and with law P_0.

THEOREM 5.1. *There is existence and uniqueness* (*pathwise and in law*) *for the nonlinear process with random space–time births*, *nonlinear in the sense of McKean*

$$
\begin{aligned}
X_t^{\varepsilon,R} &= X_0 + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{s\geq\tau\}}\,dB_s + \int_0^t \mathbf{u}^{\varepsilon,R}(s, X_s^{\varepsilon,R})\mathbf{1}_{\{s\geq\tau\}}\,ds \\
\Phi_t^{\varepsilon,R} &= I_3 + \int_0^t \nabla\mathbf{u}^{\varepsilon,R}(s, X_s^{\varepsilon,R})\chi_R(\Phi_s^{\varepsilon,R})\mathbf{1}_{\{s\geq\tau\}}\,ds
\end{aligned}
\tag{51}
$$

with

$$
\mathbf{u}^{\varepsilon,R}(s,x) = E\big[K_\varepsilon(x - X_s^{\varepsilon,R}) \wedge \chi_R(\Phi_s^{\varepsilon,R})h(\tau, X_0)\mathbf{1}_{\{s\geq\tau\}}\big]. \tag{52}
$$

The proof is based in the classic contraction argument of Sznitmann [26] and is not hard to obtain by combining elements of Theorems 5.1 in [9] and Theorem 3.1 in [11].

Consider next a probability space endowed with a sequence $(B^i)_{i\in\mathbb{N}}$ of independent three-dimensional Brownian motions, and a sequence of independent random variables $(\tau^i, X_0^i)_{i\in\mathbb{N}}$ with law P_0 and independent of the Brownian motions. For each $n \in \mathbb{N}$ and $R, \varepsilon > 0$, we define the following system of interacting particles:

$$
\begin{aligned}
X_t^{i,\varepsilon,R,n} &= X_0^i + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{s\geq\tau^i\}}\,dB_s^i \\
&\quad + \int_0^t \frac{1}{n}\sum_{j\neq i} K_\varepsilon(X_s^{i,\varepsilon,R,n} - X_s^{j,\varepsilon,R,n}) \\
&\qquad\qquad \wedge \chi_R(\Phi_s^{j,\varepsilon,R,n})h(\tau^j, X_0^j)\mathbf{1}_{\{s\geq\tau^i,\tau^j\}}\,ds, \\
\Phi_t^{i,\varepsilon,R,n} &= I_3 + \int_0^t \frac{1}{n}\sum_{j\neq i}[\nabla K_\varepsilon(X_s^{i,\varepsilon,R,n} - X_s^{j,\varepsilon,R,n}) \\
&\qquad\qquad \wedge \chi_R(\Phi_s^{j,\varepsilon,R,n})h(\tau^j, X_0^j)] \\
&\qquad\qquad \times \chi_R(\Phi_s^{i,\varepsilon,R,n})\mathbf{1}_{\{s\geq\tau^i,\tau^j\}}\,ds,
\end{aligned}
\tag{53}
$$

for $i = 1, \dots, n$, and with $\nabla K(y) \wedge z = \nabla_y(K(y) \wedge z)$ for $y, z \in \mathbb{R}^3$, $y \neq 0$. Pathwise existence and uniqueness can be proved by adapting standard arguments, thanks to the Lipschitz continuity of the coefficients.

In the same probability space, we also consider the sequence

$$\begin{aligned} X_t^{i,\varepsilon,R} &= X_0^i + \sqrt{2\nu} \int_0 \mathbf{1}_{\{s \geq \tau^i\}} \, dB_s^i + \int_0^t \mathbf{u}^{\varepsilon,R}(s, X_s^{i,\varepsilon,R}) \mathbf{1}_{\{s \geq \tau^i\}} \, ds, \\ \Phi_t^{i,\varepsilon,R} &= I_3 + \int_0^t \nabla \mathbf{u}^{\varepsilon,R}(s, X_s^{i,\varepsilon,R}) \chi_R(\Phi_s^{i,\varepsilon,R}) \mathbf{1}_{\{s \geq \tau^i\}} \, ds, \qquad i \in \mathbb{N}, \end{aligned} \tag{54}$$

of independent copies of (51). Their common law in $\mathcal{C}_T$ is denoted by $P^{\varepsilon,R}$, and we write $\bar{h} := \|w_0\|_1 + \|\mathbf{g}\|_{1,T}$. Recall that χ_R is a Lipschitz-continuous function, bounded by $R > 0$ and with Lipschitz constant less than or equal to 1. It is not hard to adapt the proof of Theorem 5.2 in [9] to get the following:

THEOREM 5.2. *For $\varepsilon > 0$ sufficiently small and all $R > 0$, we have*

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \{|X_t^{i,\varepsilon,R,n} - X_t^{i,\varepsilon,R}| + |\Phi_t^{i,\varepsilon,R,n} - \Phi_t^{i,\varepsilon,R}|\}\Big] \leq \frac{1}{\sqrt{n}} C(\varepsilon, R, \bar{h}, T) \tag{55}$$

for all $i \leq n$, where

$$C(\varepsilon, R, \bar{h}, T) = C_1 \varepsilon (1 + R\bar{h}T)(R\bar{h}T) \exp\{C_2 \varepsilon^{-9} \bar{h} T (R+1)(\bar{h} + RT)\}$$

for some positive constants C_1, C_2 independent of R, ε, T and $\bar{h}$.

Let us now make the assumptions of Theorem 3.6, and consider, in the corresponding time interval $[0, T]$, independent copies $(X^{i,\varepsilon}, \Phi^{i,\varepsilon})$ and (X, Φ^i) of the processes (29) and (50) constructed on the given data (X_0^i, τ^i, B^i), $i \in \mathbb{N}$.

Recall again that the uniform bound of Theorem 3.8(iii) and Gronwall's lemma imply that the processes Φ^ε are uniformly bounded, say

$$\sup_{t \in [0,T], \varepsilon \geq 0, \omega \in \Omega} |\Phi_t^\varepsilon(\omega)| \leq R_\circ(T, \mathbf{w}_0)$$

for some finite positive constant $R_\circ(T, \mathbf{w}_0)$. Thus, for any $R \geq R_\circ$, one has for all $t \in [0, T]$ that

$$(X_t^{i,\varepsilon}, \Phi_t^{i,\varepsilon}) = (X_t^{i,\varepsilon}, \chi_R(\Phi_t^{i,\varepsilon})).$$

Consequently, $(X^{i,\varepsilon}, \Phi^{i,\varepsilon})$ is a pathwise solution in $[0, T]$ of (54), and so we conclude that

$$(X^{i,\varepsilon}, \Phi^{i,\varepsilon}) = (X^{i,\varepsilon,R}, \Phi^{i,\varepsilon,R})$$

almost surely. Bringing it all together, we obtain the following pathwise approximation result:

THEOREM 5.3. *Assume that* (H$_1$) *and* (H$_p$) *hold with* $p \in (\frac{3}{2}, 3)$ *and that the hypothesis of Theorem* 3.6(i) *is satisfied. Let* K_ε *be defined as in* (20), *with* φ *a cutoff function of order* 1 *and write* $\bar{h} = \|w_0\|_1 + \|\mathbf{g}\|_{1,T}$. *Let, furthermore,* $R \geq R_\circ(T, \mathbf{w}_0)$ *and*

$$\varepsilon_n = (c_\alpha \ln n)^{-1/9}$$

with

$$0 < c_\alpha < \alpha\big(C_2 \bar{h} T(R+1)(\bar{h} + RT)\big)^{-1}$$

for some alpha $\alpha \in (0, \frac{1}{2})$. *Then, we have for all* $i \leq n$,

$$\begin{aligned} &\mathbb{E}\Big[\sup_{t\in[0,T]} \{|X_t^{i,\varepsilon_n,R,n} - X_t^i| + |\Phi_t^{i,\varepsilon_n,R,n} - \Phi_t^i|\}\Big] \\ &\qquad \leq C(T, w_0, \mathbf{g}, \alpha)\bigg[\frac{1}{n^{1/2-\alpha}(\ln n)^{1/9}} + \frac{1}{(\ln n)^{1/9}}\bigg], \end{aligned} \tag{56}$$

where (X, Φ) *is the unique pathwise solution of* (29), *and the constant* $C(T, w_0, \mathbf{g}, \alpha)$ *depends on the data* w_0 *and* $\mathbf{g}$ *only through the quantities* $\|w_0\|_p$, $|||\mathbf{g}|||_{0,p,T}$ *and* $\|w_0\|_1 + \|\mathbf{g}\|_{1,T}$.

REMARK 5.4. (i) The rate at which the second term in the right-hand side of (56) goes to 0 is exactly that of $\varepsilon = \varepsilon_n$. The logarithmic order of latter was needed to make the upper bound in Theorem 5.2 go to 0 with n, which then happens at an algebraic rate. The global rate is, therefore, conditioned by the techniques used in the proof of Theorem 5.2 (see [9] for details). Under additional regularity assumptions, it is possible by analytic arguments to slightly improve the convergence rate (see the discussion at the end). An attempt for a more substantial improvement should, however, exploit specific features of the interaction at the level of the particle systems.

(ii) The previous result implies as usual that $\mathcal{W}_T(\mathrm{law}(X^{i,\varepsilon,R,n}, \Phi^{i,\varepsilon,R,n}), P)$ goes to 0 at least that fast, and that (with the obviously extended meaning of $\mathcal{W}_T$)

$$\mathcal{W}_T(\mathrm{law}((X^{1,\varepsilon,R,n}, \Phi^{1,\varepsilon,R,n}), \ldots, (X^{k,\varepsilon,R,n}, \Phi^{k,\varepsilon,R,n})), P^{\otimes k}) \leq k\delta_n,$$

where δ_n stands for the quantity in the right-hand side of (56).

We deduce the convergence at the level of empirical processes:

COROLLARY 5.5. *Under the assumptions of Theorem* 5.3, *the family* $(\tilde{\mu}_t^{n,\varepsilon_n,R})_{0\leq t\leq T}$ *of* $\mathbb{R}^3$-*weighted empirical measures on* $\mathbb{R}^3$

$$\tilde{\mu}_t^{n,\varepsilon_n,R} := \frac{1}{n}\sum_{i=1}^n \delta_{X_t^{i,\varepsilon_n,R,n}} \cdot (\chi_R(\Phi_t^{i,\varepsilon_n,R,n}) h_0(\tau, X_0^i))\mathbf{1}_{\{t\geq r\}}$$

converges in probability to $(\mathbf{w}(t,x)\,dx)_{0\le t\le T}$ *in the space* $C([0,T],\mathcal{M}_3(\mathbb{R}^3))$, *where* $\mathcal{M}_3(\mathbb{R}^3)$ *denotes the space of finite* $\mathbb{R}^3$*-valued measures on* $\mathbb{R}^3$ *endowed with the weak topology. Moreover, we have*

$$\sup_{t\in[0,T],\|\mathbf{f}\|_{\mathrm{Lip}}\le 1} E|\langle\tilde{\mu}_t^{n,\varepsilon_n,R}-\mathbf{w}(t),\mathbf{f}\rangle| \le C\left[\frac{1}{\sqrt{n}}+\frac{1}{n^{1/2-\alpha}(\ln n)^{1/9}}+\frac{1}{(\ln n)^{1/9}}\right],$$

where $\|\mathbf{f}\|_{\mathrm{Lip}}$ *is the usual norm in the space of bounded Lipshitz continuous functions* $\mathbf{f}:\mathbb{R}^3\to\mathbb{R}^3$.

PROOF. It is enough to prove the bound for Lipshitz bounded functions. For such a function $\mathbf{f}:\mathbb{R}^3\to\mathbb{R}^3$, it holds that

$$\begin{aligned}
&|\langle\tilde{\mu}_t^{n,\varepsilon_n,R},\mathbf{f}\rangle-\langle\mathbf{w}(t),\mathbf{f}\rangle|\\
&\quad\le\left|\langle\tilde{\mu}_t^{n,\varepsilon_n,R},\mathbf{f}\rangle-\frac{1}{n}\sum_{i=1}^n\mathbf{f}(X_t^{i,\varepsilon_n,R})\wedge(\chi_R(\Phi_t^{i,\varepsilon_n,R}))h(\tau,X_0^i)\mathbf{1}_{\{\tau\ge t\}}\right|\\
&\qquad+\left|\frac{1}{n}\sum_{i=1}^n\mathbf{f}(X_t^{i,\varepsilon_n,R})\wedge(\chi_R(\Phi_t^{i,\varepsilon_n,R}))h(\tau,X_0^i)\mathbf{1}_{\{\tau\ge t\}}\right.\\
&\qquad\left.-\int_{\mathcal{C}_T}\mathbf{f}(y(t))\wedge\chi_R(\phi(t))h(\theta,x(0))P^{\varepsilon_n,R}(d\theta,dy,d\phi)\right|\\
&\qquad+|\langle\mathbf{w}^{\varepsilon_n}(t)-\mathbf{w}(t),\mathbf{f}\rangle|
\end{aligned}\tag{57}$$

with $P^{\varepsilon_n,R}=P^{\varepsilon_n}=\mathrm{law}(\tau,X^{i,\varepsilon_n,R},\Phi^{i,\varepsilon_n,R})$. The independence of the processes $(\tau^i,X^{i,\varepsilon_n,R},\Phi^{i,\varepsilon_n,R})$, $i\in\mathbb{N}$, and the definition of h imply that the expectation of the second term in the right-hand side of (57) is bounded by $\frac{1}{\sqrt{n}}2\|\mathbf{f}\|_{\mathrm{Lip}}R\bar{h}$, where $\bar{h}=(\|w_0\|_1+\|\mathbf{g}\|_{1,T})$. We use the latter and estimate in Theorem 5.2 to bound the first term, and get that

$$\begin{aligned}
&E|\langle\tilde{\mu}_t^{n,\varepsilon_n,R}-\mathbf{w}(t),\mathbf{f}\rangle|\\
&\quad\le\|\mathbf{f}\|_{\mathrm{Lip}}(R+1)\bar{h}\frac{1}{\sqrt{n}}C(\varepsilon_n,R,\bar{h},T)\\
&\qquad+\frac{2\|\mathbf{f}\|_{\mathrm{Lip}}R\bar{h}}{\sqrt{n}}+|\langle\mathbf{w}^{\varepsilon_n}-\mathbf{w}(t),\mathbf{f}\rangle|.
\end{aligned}$$

The last term being equal to the first term in (49), the conclusion follows. □

REMARK 5.6. In the case $\mathbf{g}=0$, Philipowski [22] obtained a similar approximation result of the vorticity field, for a simpler particle system, under the additional assumption that the test function $\mathbf{f}$ belongs to L^{p^*}.

Finally, we establish an approximation result with convergence rate for the velocity field. To that end, we need to strengthen the already shown convergence of $\mathbf{w}^\varepsilon$ to $\mathbf{w}$. We will need the following:

LEMMA 5.7. *For each $\tilde{p} \in (\frac{3}{2}, p)$, there is a constant $C(T, \tilde{p})$ such that*

$$\sup_{t\in[0,T]} t^{3/(2\tilde{p})} \|\nabla \mathbf{w}^\varepsilon(t) - \nabla \mathbf{w}(t)\|_{\tilde{p}} \le C(T, \tilde{p})\varepsilon.$$

PROOF. We need $\tilde{p} \in (\frac{3}{2}, 3)$ in order to dispose from a integrable (in time) bound for $\|D^2\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(t)\|_{3\tilde{p}/(3-\tilde{p})}$, which we do not have for $\tilde{p} = p$. Indeed, for any $\tilde{p}$ in that interval we have $\tilde{q} := \frac{3\tilde{p}}{3-\tilde{p}} \in (3, \frac{3p}{3-p})$, and so by Theorem 3.8(i) and Lemma 3.7 we have for $k, j, i = 1, 2, 3$ that

$$\sup_{t\in[0,T],\varepsilon\ge 0} t^{3/2(1/p-1/\tilde{q})} \left\| \frac{\partial \mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)_j}{\partial x_i} \right\|_{\tilde{q}} + \sup_{t\in[0,T],\varepsilon\ge 0} t^{1/2+3/2(1/p-1/\tilde{q})} \left\| \frac{\partial^2 \mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)_j}{\partial x_i\,\partial x_k} \right\|_{\tilde{q}} < \infty \tag{58}$$

with $-\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{\tilde{q}}) = -1 + \frac{3}{2}(\frac{1}{\tilde{p}} - \frac{1}{p}) > -1$. Let us now check that one has

$$\sup_{\varepsilon\ge 0} |||\mathbf{w}^\varepsilon|||_{1,\tilde{p},T} < \infty. \tag{59}$$

This is not immediate, since $T > 0$ given by Theorem 3.6 was determined by the norm of $\mathbf{w}_0$ and of the operator $\mathbf{B}^\varepsilon$ in the spaces corresponding to the parameter $p > \tilde{p}$. We will prove (59) using continuity properties of the operators $\mathbf{B}^\varepsilon$. It follows from Proposition 3.1(iii) in [9] that for $\frac{3}{2} \le r < 3$ and $\frac{3r}{6-r} \le r' \le r$, one has

$$\sup_{\varepsilon\ge 0} |||\mathbf{B}^\varepsilon(\mathbf{v}, \mathbf{v})|||_{1,r',T} \le C_{r,r'}(T)(|||\mathbf{v}|||_{1,r,T})^2 \tag{60}$$

for some finite constant $C_{r,r'}(T)$. From this, we deduce that $\mathbf{w}^\varepsilon \in \mathbf{F}_{1,\tilde{p},T}$, with a uniform (in ε) bound, by the following iterative procedure. Define a real sequence by $r_0 = \tilde{p}$, $r_{n+1} = \frac{6r_n}{3+r_n}$, and notice that it is increasingly convergent to 3. We can thus take $N \in \mathbb{N}$ such that $r_N < p \le r_{N+1}$. The function $s \mapsto \frac{3s}{6-s}$ being increasing on $[0, 6]$, we then have $\frac{3p}{6-p} \le \frac{3r_{N+1}}{6-r_{N+1}} = r_N$. By (60) with $r = p$ and $r' = r_N$, we see that $\mathbf{B}^\varepsilon(\mathbf{w}^\varepsilon, \mathbf{w}^\varepsilon) \in \mathbf{F}_{1,r_N,T}$, and since also $\mathbf{w}_0 \in \mathbf{F}_{1,r_N,T}$ holds by Lemma 3.4(i) (taking r_N in the place of p and r therein), we get that $\mathbf{w}^\varepsilon \in \mathbf{F}_{1,r_N,T}$, with a bound in that space that is uniform in ε. We repeat the previous arguments with $r = r_N$ and $r' = \frac{3r_N}{6-r_N} = r_{N-1}$ and get that $\mathbf{w}^\varepsilon \in \mathbf{F}_{1,r_{N-1},T}$, with a bound that is a uniform in ε. Continuing $N - 1$ times this scheme we get (59).

We now take derivatives in the mild vortex equation with $\varepsilon \geq 0$ (as justified in the proof of Proposition 3.1 in [9]),

$$\frac{\partial (\mathbf{w}^\varepsilon)_k}{\partial x_i}(t,x) = \int_{\mathbb{R}^3} \frac{\partial G_t^\nu}{\partial x_i}(x-y)(w_0)_k(y)\,dy + \int_0^t \int_{\mathbb{R}^3} \frac{\partial G_t^\nu}{\partial x_i}(x-y)\mathbf{g}(0,y)\,dy\,ds$$

$$- \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^\nu}{\partial x_i}(x-y)\bigg[\mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)_j(s,y)\,\frac{\partial \mathbf{w}_k^\varepsilon(s,y)}{\partial y_j}$$

$$- \mathbf{w}_j^\varepsilon(s,y)\,\frac{\partial \mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)_k(s,y)}{\partial y_j}\bigg]\,dy\,ds$$

for $k = 1, 2, 3$. Notice now that, thanks to the estimates (59), Lemma 4.10(ii) also holds with p replaced by $\tilde{p}$. By estimates as those in the proof of Theorem 3.6(i) and using Lemma 4.10(ii) and estimates (58) and (59), we then have

$$\begin{aligned}
&\|\nabla \mathbf{w}^\varepsilon(t) - \nabla \mathbf{w}(t)\|_{\tilde{p}} \\
&\quad \leq C\int_0^t (t-s)^{-3/(2\tilde{p})} s^{-1/2}[\|\mathbf{w}^\varepsilon(s) - \mathbf{w}(s)\|_{\tilde{p}} \\
&\qquad\qquad + \|\mathbf{K}^\varepsilon(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_{\tilde{q}}]\,ds \\
&\qquad + C\int_0^t (t-s)^{-3/(2\tilde{p})}[\|\nabla \mathbf{w}^\varepsilon(s) + \nabla \mathbf{w}(s)\|_{\tilde{p}} \\
&\qquad\qquad + \|\nabla \mathbf{K}^\varepsilon(\mathbf{w})(s) - \nabla \mathbf{K}(\mathbf{w})(s)\|_{\tilde{q}}]\,ds \\
&\quad \leq C\varepsilon t^{1-3/\tilde{p}} + C\varepsilon \int_0^t (t-s)^{-3/(2\tilde{p})} s^{-1+3/(2\tilde{p})-3/(2p)}\,ds \\
&\qquad + C\int_0^t (t-s)^{-3/(2\tilde{p})}\|\nabla \mathbf{w}^\varepsilon(s) - \nabla \mathbf{w}(s)\|_{\tilde{p}}\,ds \\
&\quad \leq C\varepsilon t^{-3/(2\tilde{p})} + C\int_0^t (t-s)^{-3/(2\tilde{p})}\|\nabla \mathbf{w}^\varepsilon(s) - \nabla \mathbf{w}(s)\|_{\tilde{p}}\,ds.
\end{aligned}$$

Iterating the latter sufficiently many times (using the identity quoted in the proof of Theorem 3.6) (i), we obtain that

$$\begin{aligned}
\|\nabla \mathbf{w}^\varepsilon(t) - \nabla \mathbf{w}(t)\|_{\tilde{p}} &\leq C\varepsilon\big(t^{-3/(2\tilde{p})} + 1\big) \\
&\quad + C(T)\int_0^t \|\nabla \mathbf{w}^\varepsilon(s) - \nabla \mathbf{w}(s)\|_{\tilde{p}}\,ds.
\end{aligned} \tag{61}$$

Integrating (61) in time and using Gronwall's lemma, and then inserting the obtained bound in the right-hand side of (61), we obtain

$$\|\nabla \mathbf{w}^\varepsilon(t) - \nabla \mathbf{w}(t)\|_{\tilde{p}} \leq C\varepsilon\big(t^{-3/(2\tilde{p})} + 1\big), \tag{62}$$

and the convergence statement for $\nabla \mathbf{w}^\varepsilon$ follows. $\square$

COROLLARY 5.8. *Consider fixed real numbers* $\tilde{p} \in (\frac{3}{2}, 3)$ *and* $\alpha \in (0, \frac{1}{2})$. *Under the assumptions of Theorem 5.3, there exists a constant* $\mathbf{C}$ *depending on* $\tilde{p}, T, \|w_0\|_p, |||\mathbf{g}|||_{0,p,T}, \|w_0\|_1 + \|\mathbf{g}\|_{1,T}$ *and* α, *such that for all* $n \in \mathbb{N}$,

$$\sup_{t\in[0,T]} \gamma(t) E\big(|\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) - \mathbf{u}(t,x)|\big) \\ \leq \mathbf{C}\bigg(\frac{(\ln n)^{1/3}}{n^{1/2-\alpha}} + \frac{(\ln n)^{1/3}}{\sqrt{n}} + \frac{1}{(\ln n)^{1/9}}\bigg),$$

where $\gamma(t) = (t^{3/(2\tilde{p})} + t^{1-3/2(1/\tilde{p}-1/p)})$.

PROOF. For all $(t,x) \in [0,T] \times \mathbb{R}^3$, it holds that

$$\begin{aligned} &|\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) - \mathbf{u}(t,x)| \\ &\quad \leq \Bigg|\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) \\ &\qquad - \frac{1}{n}\sum_{i=1}^n K_{\varepsilon_n}(x - X_t^{i,\varepsilon_n,R}) \wedge (\chi_R(\Phi_t^{i,\varepsilon_n,R}))h(\tau, X_0^i)\mathbf{1}_{\{\tau\geq t\}}\Bigg| \\ &\qquad + \Bigg|\frac{1}{n}\sum_{i=1}^n K_{\varepsilon_n}(x - X_t^{i,\varepsilon_n,R}) \wedge (\chi_R(\Phi_t^{i,\varepsilon_n,R}))h(\tau, X_0^i)\mathbf{1}_{\{\tau\geq t\}} \\ &\qquad - \int_{\mathcal{C}_T} K_{\varepsilon_n}\big(x - y(t)\big) \wedge \chi_R(\phi(t))h(\theta, x(0))P^{\varepsilon_n,R}(d\theta, dy, d\phi)\Bigg| \\ &\qquad + |\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(t,x) - \mathbf{u}(t,x)| \end{aligned} \tag{63}$$

with $P^{\varepsilon_n,R}$ as in Corollary 5.5. By similar reasons as in (57), the expectation of the second term is now bounded by $\frac{1}{\sqrt{n}} 2M_{\varepsilon_n} R\bar{h}$. With the estimate in Theorem 5.2 we get that

$$\begin{aligned} &E|\mathbf{K}_{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) - \mathbf{u}(t,x)| \\ &\quad \leq (L_{\varepsilon_n} R + M_{\varepsilon_n})\bar{h}\frac{1}{\sqrt{n}} C(\varepsilon_n, R, \bar{h}, T) \\ &\qquad + \frac{2M_{\varepsilon_n} R\bar{h}}{\sqrt{n}} + \|\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(t) - \mathbf{K}(\mathbf{w})(t)\|_\infty. \end{aligned}$$

Thus, from the estimates for L_ε and M_ε we deduce that for fixed $\tilde{p} \in (\frac{3}{2}, 3)$,

$$\begin{aligned} &E|\mathbf{K}_{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) - \mathbf{u}(t,x)| \\ &\quad \leq C(1 + R\bar{h}T)(R\bar{h}T)\frac{(c\ln n)^{1/3}}{n^{1/2-\alpha}} + CR\bar{h}\frac{(c\ln n)^{1/3}}{\sqrt{n}} \\ &\qquad + \|\mathbf{w}^{\varepsilon_n}(t) - \mathbf{w}(t)\|_{W^{1,\tilde{p}}} + \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(t) - \mathbf{K}(\mathbf{w})(t)\|_{W^{1,\tilde{q}}}, \end{aligned}$$

where $\tilde{q} = \frac{3\tilde{p}}{3-\tilde{p}} < \frac{3p}{3-p}$. We have used here again the Sobolev inclusions quoted in the proof of Theorem 3.8, and Lemma 3.3. Now, by Lemmas 3.7 and 4.9, one has

$$\|\nabla \mathbf{K}^{\varepsilon_n}(\mathbf{w})(t) - \nabla \mathbf{K}(\mathbf{w})(t)\|_{\tilde{q}} \leq C\|\varphi_{\varepsilon_n} * \mathbf{w}(t) - \mathbf{w}(t)\|_{\tilde{q}} \leq C\varepsilon_n \|\nabla \mathbf{w}(t)\|_{\tilde{q}} \\ \leq Ct^{-1+3/2(1/\tilde{p}-1/p)}\varepsilon_n,$$

where we have also used part (i) of Theorem 3.8 in the last inequality. On the other hand,

$$\|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(t) - \mathbf{K}(\mathbf{w})(t)\|_{\tilde{q}} \leq C\|\varphi_{\varepsilon_n} * \mathbf{w}(t) - \mathbf{w}(t)\|_{\tilde{p}} \leq C\varepsilon_n \|\nabla \mathbf{w}(t)\|_{\tilde{p}} \leq Ct^{-1/2}\varepsilon_n$$

thanks to the estimate (59). From the previous estimates and Lemmas 4.10 and 5.7, we deduce that

$$\begin{aligned} &E|\mathbf{K}_{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) - \mathbf{u}(t,x)| \\ &\quad \leq C\frac{(\ln n)^{1/3}}{n^{1/2-\alpha}} + C\frac{(\ln n)^{1/3}}{\sqrt{n}} \\ &\qquad + C\varepsilon_n\big(t^{-3/(2\tilde{p})} + t^{-1/2} + t^{-1+3/2(1/\tilde{p}-1/p)}\big), \end{aligned}$$

and the statement follows. □

6. Convergence rate under additional regularity assumptions. Let us finally explain how the convergence rate can be slightly improved by assuming further regularity of the data w_0 and $\mathbf{g}$. Since it is an adaptation of the developments in the previous sections, we only sketch the main arguments.

First, it is possible to show that if the data w_0 and $\mathbf{g}$ are such that

$$\|w_0\|_{W^{m,p}}, \qquad \sup_{t\in[0,T]} \|\mathbf{g}(t)\|_{W^{m,p}} < \infty \tag{64}$$

for some integer $m \geq 1$, then the mild solutions $\mathbf{w}^\varepsilon$, $\varepsilon \geq 0$, given by Theorem 3.6 belong to the space $\mathbf{F}_{m+1,p,T}$ of functions $\mathbf{v}(t)$ such that

$$\sum_{i=1}^{m-1} |||D^i\mathbf{v}|||_{0,p,T} + |||D^m\mathbf{v}|||_{1,p,T} < \infty,$$

where D^i stands for the ith order space derivative. To prove this, one easily first checks that $\mathbf{w}_0$ belongs to that space, since the successive derivatives in the convolutions the heat kernel can be applied to the data w_0 and $\mathbf{g}$. On the other hand, on can show by induction that the bilinear operators $\mathbf{B}^\varepsilon$ are continuous in $\mathbf{F}_{m+1,p,T}$, and more generally, in the naturally generalized versions $\mathbf{F}_{m+1,r,(T;p)}$ of the space $\mathbf{F}_{1,r,(T;p)}$. That is, the spaces of functions $\mathbf{v}$ such that

$$\sum_{i=1}^{m-1} |||D^i\mathbf{v}|||_{0,r,(T;p)} + |||D^m\mathbf{v}|||_{1,r,(T;p)}$$

is finite. From this, one gets a local existence result in the space $\mathbf{F}_{m+1,p,T}$, from which a regularity result can be obtained by arguments that can be adapted from those in the proof Theorem 3.2 in [9]. Moreover, one also checks that the norms $|||\mathbf{w}^{\varepsilon}|||_{m+1,r,(T;p)}$ are bounded uniformly in $\varepsilon \geq 0$.

Now, we impose additional conditions on the regularizing kernel φ, namely:

(i) $\int_{\mathbb{R}^3} \varphi(x)\,dx = 1$.
(ii) $\int_{\mathbb{R}^3} |x|^{m+1}|\varphi(x)|\,dx < \infty$.
(iii) $\int_{\mathbb{R}^3} x_{i_1}\cdots x_{i_r}\varphi(x)\,dx = 0$ for all $i_1,\dots,i_r \in \{1,2,3\}$ and $r \leq m$.

Such function is called a cutoff function of order $m+1$. Then, one has the following approximation result (see Lemma 4.4 in [23]):

$$\|\varphi_\varepsilon * w - w\|_r \leq C\varepsilon^{m+1}\|D^{m+1}w\|_r$$

for all $w \in W^{m+1,r}$. Therefore, without any modification, for such function φ, the proofs of Lemmas 4.10 and 5.7 yield the same convergence results but at rate ε^{m+1}.

By following exactly the same steps as in the previous section, we finally deduce:

THEOREM 6.1. *Assume the hypotheses of Theorems 5.3 and, moreover, that (64) holds for some integer $m \geq 1$ and that φ is a cutoff of order $m+1$. Then, we have for all $i \leq n$,*

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}\{|X_t^{i,\varepsilon_n,R,n} - X_t^i| + |\Phi_t^{i,\varepsilon_n,R,n} - \Phi_t^i|\}\Big]\\ \leq C(T,w_0,\mathbf{g},\alpha)\left[\frac{1}{n^{1/2-\alpha}(\ln n)^{1/9}} + \frac{1}{(\ln n)^{(m+1)/9}}\right]\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in[0,T],x\in\mathbb{R}^3} \gamma(t)E\big(|\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) - \mathbf{u}(t,x)|\big)\\ \leq \mathbf{C}\left(\frac{(\ln n)^{1/3}}{n^{1/2-\alpha}} + \frac{(\ln n)^{1/3}}{\sqrt{n}} + \frac{1}{(\ln n)^{(m+1)/9}}\right),\end{aligned}$$

where $\gamma(t)$ was defined in Corollary 5.8, where the constants now, moreover, depend on m.

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