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THE RANDOM CONDUCTANCE MODEL WITH CAUCHY TAILS[1]

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We consider a random walk in an i.i.d. Cauchy-tailed conductances environment. We obtain a quenched functional CLT for the suitably rescaled random walk, and, as a key step in the arguments, we improve the local limit theorem for $p^{\omega}_{n^2t}(0, y)$ in [*Ann. Probab.* (2009). To appear], Theorem 5.14, to a result which gives uniform convergence for $p^{\omega}_{n^2t}(x, y)$ for all x, y in a ball.

0. Introduction. In this paper we will establish the convergence to Brownian motion of a random walk in a symmetric random environment in a critical case that has not been covered by the papers [1, 3]. We begin by recalling the "random conductance model" (RCM). We consider the Euclidean lattice $\mathbb{Z}^d$ with $d \geq 2$. Let E_d be the set of nonoriented nearest neighbour bonds, and, writing $e = \{x, y\} \in E_d$, let $(\mu_e, e \in E_d)$ be nonnegative i.i.d. r.v. on $[1, \infty)$ defined on a probability space $(\Omega, \mathbb{P})$. We write $\mu_{xy} = \mu_{\{x,y\}} = \mu_{yx}$; let $\mu_{xy} = 0$ if $x \not\sim y$, and set $\mu_x = \sum_y \mu_{xy}$.

We consider two continuous time random walks on $\mathbb{Z}^d$ which jump from x to $y \sim x$ with probability μ_{xy}/μ_x. These are called in [1] the *constant speed random walk* (CSRW) and *variable speed random walk* (VSRW), and have generators

$$\mathcal{L}_C(\omega) f(x) = \mu_x(\omega)^{-1} \sum_y \mu_{xy}(\omega)\big(f(y) - f(x)\big), \tag{0.1}$$

$$\mathcal{L}_V(\omega) f(x) = \sum_y \mu_{xy}(\omega)\big(f(y) - f(x)\big). \tag{0.2}$$

We write X for the CSRW and Y for the VSRW. Thus X jumps out of a state x at rate 1 while Y jumps out at rate μ_x. We will abuse notation slightly by writing P^x_ω for the laws of both X and Y started at $x \in \mathbb{Z}^d$ in the random environment $[\mu_e(\omega)]$. Since the generators of these processes differ by a multiple, X and Y are time changes of each other. More explicitly, as in [3], define the *clock process*

$$S_t = \int_0^t \mu_{Y_s}\, ds, \tag{0.3}$$

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and let A_t be its inverse. Then the CSRW can be defined by

$$X_t = Y_{A_t}, \qquad t \ge 0. \tag{0.4}$$

In the case when $\mu_e \in [0,1]$, and $\mathbb{P}(\mu_e > 0) > p_c(d)$, the critical probability for bond percolation in $\mathbb{Z}^d$, the papers [7, 11] prove that both X and Y satisfy a quenched functional central limit theorem (QFCLT), and that the limiting process is nondegenerate. The paper [1] studies the case when $\mu_e \in [1,\infty)$, and proves that for $\mathbb{P}$-a.a. ω the rescaled VSRW, defined by

$$Y_t^{(n)} = n^{-1} Y_{n^2 t}, \qquad t \ge 0, \tag{0.5}$$

converges to $(\sigma_V W_t, t \ge 0)$ where W is a standard Brownian motion, and $\sigma_V > 0$. It is also proved there that $S_t/t \to \mathbb{E}\mu_0 \in [1,\infty]$. It follows from (0.4) that the CSRW with the standard rescaling,

$$X_t^{(n,1)} = n^{-1} X_{n^2 t}, \qquad t \ge 0,$$

converges to $\sigma_C W$ where

$$\sigma_C = \begin{cases} \sigma_V / \sqrt{2d\mathbb{E}\mu_e}, & \text{if } \mathbb{E}\mu_e < \infty, \\ 0, & \text{if } \mathbb{E}\mu_e = \infty. \end{cases}$$

If $\mathbb{E}\mu_e = \infty$ it is natural to ask if a different rescaling of X will give a nontrivial limit. In the case when $d \ge 3$, $\mu_e \in [1,\infty)$ and there exists $\alpha \in (0,1)$ such that

$$\mathbb{P}(\mu_e > u) \sim \frac{c}{u^\alpha} \qquad \text{as } u \to \infty, \tag{0.6}$$

then [3] proves that the process

$$X_t^{(n,\alpha)} = n^{-1} X_{n^{2/\alpha} t}, \qquad t \ge 0,$$

converges to the "fractional kinetic motion" with index α. (For details of this process, and its connection with aging see [4–6].) These papers leave open the case when $\alpha = 1$. In this paper we assume that (μ_e) satisfies (0.6) with $\alpha = 1$; for simplicity we take $c = 1/(2d)$, so that μ_e satisfies

$$\mathbb{P}(\mu_e \ge 1) = 1, \tag{0.7}$$

$$\mathbb{P}(\mu_e \ge u) \sim \frac{1}{2du} \qquad \text{as } u \to \infty. \tag{0.8}$$

We define the process

$$X_t^{(n)} = n^{-1} X_{n^2 (\log n) t}, \qquad t \ge 0. \tag{0.9}$$

Our main theorem follows:

THEOREM 1. *Let $d \ge 3$, and assume that μ_e satisfies (0.7) and (0.8). Then for $\mathbb{P}$-a.a. ω, $(X^{(n)}, P_\omega^0)$ converges in $D([0,\infty); \mathbb{R}^d)$ to $\sigma_1 W$ where $\sigma_1 = \sigma_V/\sqrt{2} > 0$, and W is a standard d-dimensional Brownian motion.*

As in [3] we prove this theorem by using (0.4) and proving convergence of a rescaled clock process. Let

$$S_t^{(n)} = \frac{1}{n^2 \log n} \int_0^{n^2 t} \mu_{Y_s}\, ds; \tag{0.10}$$

then it is easy to check that if $A^{(n)}$ is the inverse of $S^{(n)}$, then

$$X_t^{(n)} = Y^{(n)}_{A_t^{(n)}}, \qquad t \geq 0. \tag{0.11}$$

It follows that to prove Theorem 1 it is enough to prove.

THEOREM 2. *Let $d \geq 3$, and assume that μ_e satisfies* (0.7) *and* (0.8). *For $\mathbb{P}$-a.a. ω, under the law P_ω^0,*

$$\big(S_t^{(n)}, t \geq 0\big) \Rightarrow (2t, t \geq 0) \qquad on\ C([0,\infty); \mathbb{R}). \tag{0.12}$$

REMARK 1. For $\lambda \in [1,\infty)$, let $S_t^{(\lambda)} = \frac{1}{\lambda^2 \log \lambda} \int_0^{\lambda^2 t} \mu_{Y_s}\, ds$. Then if $n \leq \lambda \leq (n+1)$,

$$\frac{n^2 \log n}{(n+1)^2 \log(n+1)} \cdot S_t^{(n)} \leq S_t^{(\lambda)} \leq \frac{(n+1)^2 \log(n+1)}{n^2 \log n} \cdot S_t^{(n+1)}.$$

It follows that the convergence (0.12) holds for $(S_t^{(\lambda)}, t \geq 0)_{\lambda \geq 1}$, and hence Theorem 1 extends to $(X_t^{(\lambda)})_{\lambda \geq 1} := (\lambda^{-1} X_{\lambda^2 (\log \lambda) t})_{\lambda \geq 1}$.

As in [3], the result is proved by estimating the growth of the clock process S_t, $0 \leq t \leq n^2 T$. Since the limit of the processes $S^{(n)}$ is deterministic, overall this case is much easier than when $\alpha \in (0,1)$: after suitable truncation it is enough to use a mean–variance calculation. There is, however, one respect in which this case is more delicate than when $\alpha < 1$. When $\alpha < 1$ it turns out that the main contribution to $S_{n^2 T}$ is from visits by Y to x such that $\varepsilon n^{2/\alpha} \leq \mu_x \leq \varepsilon^{-1} n^{2/\alpha}$ (see Sections 5 and 7 of [3]). When $\alpha = 1$ one finds that each set of edges of the form $E_i = \{e : 2^{i-1} n \leq \mu_e < 2^i n\}$, $i = 1, \ldots, \log n$, has a roughly comparable contribution to $S_{n^2 T}$, so a much greater range of values of μ_e need to be considered.

To motivate the proof, consider the classical case of a sum of i.i.d. r.v. ξ_i, with $\mathbb{P}(\xi_i > t) \sim t^{-1}$. We have that if

$$U_t^{(n)} = (n \log n)^{-1} \sum_{i=1}^{[nt]} \xi_i, \tag{0.13}$$

then $\sup_{0 \leq t \leq T} |U_t^{(n)} - t| \to 0$ in probability. Let $a_i = i (\log i)^\beta$ where $\beta \in (1,2)$, and $\xi_i' = \xi_i \mathbf{1}_{(\xi_i > a_i)}$. Then $\sum P(\xi_i \neq \xi_i')$ converges, so it is enough to consider the convergence of

$$V_t^{(n)} = (n \log n)^{-1} \sum_{i=1}^{[nt]} \xi_i'. \tag{0.14}$$

A straightforward argument calculating the mean and variance of

$$M_t^{(n)} = (n \log n)^{-1} \sum_{i=1}^{[nt]} (\xi_i' - E\xi_i') \tag{0.15}$$

then gives convergence of $U^{(n)}$. [Note that one does not have a.s. convergence, since $P(\max_{2^{n-1} \le i \le 2^n} \xi_i > 2^n \log 2^n) \sim c/n$.]

The equivalent arguments in our case rely on good control of the process Y. Define the heat kernel and Green's functions for Y by

$$p_t^\omega(x, y) = P_\omega^x(Y_t = y), \qquad g^\omega(x, y) = \int_0^\infty p_t^\omega(x, y)\, dt. \tag{0.16}$$

We extend these functions from $\mathbb{Z}^d \times \mathbb{Z}^d$ to $\mathbb{R}^d \times \mathbb{R}^d$ by linear interpolation on each cube in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$. Let W be a standard Brownian motion on $\mathbb{R}^d$, and let $W_t^* = \sigma_V W_t$, so that W^* is the weak limit of the processes $Y^{(n)}$. Let

$$k_t(x) = (2\pi\sigma_V^2)^{-d/2} \exp(-|x|^2/2\sigma_V^2) \tag{0.17}$$

be the density of the W^*.

A key element of the arguments is the following strengthening of the local limit theorem for $p_{n^2 t}^\omega(0, y)$ in [1], Theorem 5.14, to a result which gives uniform convergence for $p_{n^2 t}^\omega(x, y)$ for all x, y in a ball.

THEOREM 3. *Let $d \ge 2$, and assume μ_e satisfies* (0.7). *For any $\varepsilon > 0$, $0 < \delta < T < \infty$ and $K > 0$, we have the following $\mathbb{P}$-almost sure uniform convergence*:

$$\begin{aligned} \frac{1}{1+\varepsilon} &< \liminf_{n\to\infty} \inf_{\delta \le t \le T} \inf_{|x|,|y| \le K} \frac{n^d p_{n^2 t}^\omega(nx, ny)}{k_t(x, y)} \\ &\le \limsup_{n\to\infty} \sup_{\delta \le t \le T} \sup_{|x|,|y| \le K} \frac{n^d p_{n^2 t}^\omega(nx, ny)}{k_t(x, y)} < 1 + \varepsilon. \end{aligned} \tag{0.18}$$

This result is proved in Section 1.1.

NOTATION. We write

$$B(x, r) = \{y \in \mathbb{Z}^d : |x - y| \le r\} \quad \text{and} \quad B_{\mathbb{R}}(x, r) = \{y \in \mathbb{R}^d : |x - y| \le r\}.$$

If $e = \{x_e, y_e\} \in E_d$, we write $e \in B(x, r)$ if $\{x_e, y_e\} \subset B(x, r)$. We will follow the custom of writing $f \sim g$ to mean that the ratio f/g converges to 1, and $f \asymp g$ to mean that the ratio f/g remains bounded away from 0 and ∞. For any $a, b \in \mathbb{R}$, $a \wedge b := \min(a, b)$, and $a \vee b := \max(a, b)$. Throughout the paper, c, C, C_1, C', et cetera, denote generic constants whose values may change from line to line.

REMARK 2. One can also consider the more general case when the tail of μ_e satisfies

$$\mathbb{P}(\mu_e \geq u) \sim c\frac{(\log u)^\rho}{u} \qquad \text{as } u \to \infty,$$

where $\rho \geq -1$ (so that $\mathbb{E}\mu_e = \infty$). Define for $t \geq 0$

$$X_t^{(n)} = \begin{cases} n^{-1} X_{n^2(\log n)^{1+\rho} t}, & \text{when } \rho > -1, \\ n^{-1} X_{n^2(\log\log n) t}, & \text{when } \rho = -1. \end{cases}$$

Then using the same strategy as in this article one can show that for $\mathbb{P}$-a.a. ω, $(X^{(n)}, P_\omega^0)$ converges to a (multiple of a) Brownian motion.

1. Preliminaries.

1.1. *Heat kernel*: *Proof of Theorem* 3. We collect some known estimates for $p_t^\omega(x, y)$ and $g^\omega(x, y)$ which will be used in our arguments.

LEMMA 4. *Let* $\eta \in (0, 1)$. *There exist random variables* U_x $(x \in \mathbb{Z}^d)$ *and constants* c_i *such that*

$$\mathbb{P}(U_x \geq n) \leq c_1 \exp(-c_2 n^\eta), \qquad \textit{for all } n \geq 1.$$

(a) [1], *Theorem* 1.2(a). *There exists* $c_3 > 0$ *such that for all* x, y *and* t,

$$p_t^\omega(x, y) \leq c_3 t^{-d/2}.$$

(b) [1], *Theorem* 1.2(b). *If* $|x - y| \vee \sqrt{t} \geq U_x$, *then*

$$\begin{aligned} &p_t^\omega(x, y) \\ &\quad \leq \begin{cases} c_4 t^{-d/2} \exp(-c_5 |x - y|^2/t), & \textit{when } t \geq |x - y|, \\ c_4 \exp\bigl(-c_5 |x - y| \bigl(1 \vee \log(|x - y|/t)\bigr)\bigr), & \textit{when } t \leq |x - y|. \end{cases} \end{aligned} \tag{1.1}$$

(c) [1], *Theorem* 1.2(c). *If* $t \geq U_x^2 \vee |x - y|^{1+\eta}$, *then*

$$p_t^\omega(x, y) \geq c_6 t^{-d/2} \exp(-c_7 |x - y|^2/t).$$

(d) *Let* $\tau(x, R) = \inf\{t \geq 0 : |Y_t - x| > R\}$. *If* $R \geq U_x$, *then*

$$P_\omega^x\bigl(\tau(x, R) \leq t\bigr) \leq c_8 \exp(-c_9 R^2/t).$$

(e) [3], *Lemma* 3.4. *When* $d \geq 3$,

$$c_{10} U_x^{2-d} \leq g^\omega(x, x) \leq c_{11}. \tag{1.2}$$

(f) [3], *Proposition* 3.2(b). *When* $d \geq 3$, *if* $|x| \geq U_0$, *then*

$$g^\omega(0, x) \leq \frac{c_{12}}{|x|^{d-2}}. \tag{1.3}$$

(g) [3], *Lemma* 3.3. *There exists* $c_{13} > 0$ *such that for each* $K > 0$, *if*

$$b_n = c_{13}(\log n)^{1/\eta}, \tag{1.4}$$

then with $\mathbb{P}$*-probability no less than* $1 - c_{14}K^d n^{-2}$ *the following holds*:

$$\max_{|x| \le Kn} U_x \le b_n. \tag{1.5}$$

In particular, (1.5) *holds for all n large enough* $\mathbb{P}$*-a.s.*

(h) [1], *Theorem* 5.14. *For any* $\delta > 0$, $\mathbb{P}$*-a.s.*,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{Z}^d} \sup_{t\ge\delta} |n^d p^{\omega}_{n^2t}(0, x) - k_t(x/n)| = 0. \tag{1.6}$$

(i) *There exists* $\theta > 0$ *such that for* $x, y, y' \in \mathbb{Z}^d$,

$$n^d |p^{\omega}_{n^2t}(x, y) - p^{\omega}_{n^2t}(x, y')| \le c_{15} t^{-(d+\theta)/2} \cdot \left(\frac{|y - y'| \vee U_y}{n}\right)^{\theta}. \tag{1.7}$$

PROOF. (d) The tail bound on $\tau(x, R)$ in (d) follows from Proposition 2.18 and Theorem 4.3 of [1]. (i) This follows from [1], Theorem 3.7 and [2], Proposition 3.2. $\square$

We begin by improving the local limit theorem in (1.6).

LEMMA 5. *For any* $\varepsilon > 0$, $K > 0$ *and* $0 < \delta < T < \infty$, *there exists* $\varepsilon_b > 0$ *such that* $\mathbb{P}$*-a.s.*, *for all but finitely many n*,

$$\begin{aligned} &\sup_{\delta\le t\le T} \sup\left\{\frac{p^{\omega}_{n^2t}(nx_1, ny_1)}{p^{\omega}_{n^2t}(nx_2, ny_2)} : |x_i|, |y_i| \le K, |x_1 - x_2| \le \varepsilon_b, |y_1 - y_2| \le \varepsilon_b\right\} \\ &\quad < 1 + \varepsilon. \end{aligned} \tag{1.8}$$

PROOF. By Lemma 4(g), we can assume that the event $\{\max_{|x|\le Kn} U_x \le b_n\}$ holds. So, by Lemma 4(i) we get that for all $t \ge \delta$,

$$n^d |p^{\omega}_{n^2t}(nx_1, ny_1) - p^{\omega}_{n^2t}(nx_1, ny_2)| \le C\delta^{-(d+\theta)/2} \cdot |y_1 - y_2|^{\theta} \vee \left|\frac{b_n}{n}\right|^{\theta}.$$

On the other hand, by Lemma 4(c), there exists $\varepsilon_1 > 0$ such that for all n large such that $n^2\delta \ge b_n^2 \vee n^{1+\eta}(2K)^{1+\eta}$, all $\delta \le t \le T$ and $|x_1|, |y_1| \le K$,

$$n^d p^{\omega}_{n^2t}(nx_1, ny_1) \ge \varepsilon_1.$$

Hence

$$\left|1 - \frac{p^{\omega}_{n^2t}(nx_1, ny_2)}{p^{\omega}_{n^2t}(nx_1, ny_1)}\right| \le \frac{C\delta^{-(d+\theta)/2}}{\varepsilon_1} \cdot |y_1 - y_2|^{\theta} \vee \left|\frac{b_n}{n}\right|^{\theta}.$$

The conclusion follows by taking ε_b small enough so that

$$\frac{C\delta^{-(d+\theta)/2}}{\varepsilon_1}\cdot\varepsilon_b^{\theta}<\sqrt{1+\varepsilon}-1,$$

and then interchanging the roles of x and y in the argument above. □

PROOF OF THEOREM 3. Let $\varepsilon_0>0$, to be chosen later. We first show that for any fixed $|x|,|y|\le K$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\frac{1}{(1+\varepsilon_0)^4}&\le\liminf_{n\to\infty}\inf_{\delta\le t\le T}\frac{n^d p^{\omega}_{n^2t}(nx,ny)}{k_t(x,y)}\\&\le\limsup_{n\to\infty}\sup_{\delta\le t\le T}\frac{n^d p^{\omega}_{n^2t}(nx,ny)}{k_t(x,y)}\le(1+\varepsilon_0)^4.\end{aligned}\tag{1.9}$$

The proof is similar to that in Lemma 4.2 in [3]. First fix an ε_b so that the LHS in (1.8) in Lemma 5 is bounded by $1+\varepsilon_0$. For any path $\gamma\in D([0,\infty);\mathbb{R}^d)$, define the hitting time $\sigma(\gamma)=\inf\{t:\gamma_t\in B(x,\varepsilon_b)\}$. Then by the QFCLT for the VSRW $Y^{(n)}$ we get that $\mathbb{P}$-a.s.,

$$\begin{aligned}&\lim_n E_0^{\omega}\mathbf{1}\{Y^{(n)}_{\sigma(Y^{(n)})+t}\in B(y,\varepsilon_b)\}\\&\quad=E_0\biggl(\mathbf{1}\{\sigma(W^*)<\infty\}\int_{z\in B(y,\varepsilon_b)}k_t\bigl(W^*_{\sigma(W^*)},z\bigr)\,dz\biggr),\end{aligned}$$

where W^* is the limit of the VSRW $Y^{(n)}$. So, writing $\sigma=\sigma(Y^{(n)})$, for all large n,

$$\begin{aligned}P^0_{\omega}\bigl(Y^{(n)}_{\sigma+t}\in B(y,\varepsilon_b)|Y^{(n)}_{\sigma},\sigma<\infty\bigr)&=\sum_{z\in B(ny,n\varepsilon_b)}p^{\omega}_{n^2t}\bigl(nY^{(n)}_{\sigma},z\bigr)\\&\ge(1+\varepsilon_0)^{-1}|B(ny,n\varepsilon_b)|\cdot p^{\omega}_{n^2t}\bigl(nY^{(n)}_{\sigma},ny\bigr)\\&\ge(1+\varepsilon_0)^{-2}|B(ny,n\varepsilon_b)|\cdot p^{\omega}_{n^2t}(nx,ny).\end{aligned}$$

Note that $|B(ny,n\varepsilon_b)|\sim n^d\cdot\mathrm{Vol}(B_{\mathbb{R}}(y,\varepsilon_b))$; using this and the analogous result for $k_t(x,y)$, we get that

$$\limsup_n n^d p^{\omega}_{n^2t}(nx,ny)\cdot P^0_{\omega}\bigl(\sigma\bigl(Y^{(n)}\bigr)<\infty\bigr)\le(1+\varepsilon_0)^4P_0\bigl(\sigma(W^*)<\infty\bigr)k_t(x,y).$$

But by the QFCLT for the VSRW $Y^{(n)}$ again, $\lim_n P^0_{\omega}(\sigma(Y^{(n)})<\infty)=P_0(\sigma(W^*)<\infty)$, hence we get the desired upper bound. The lower bound in (1.9) can be proved similarly.

We now let x,y vary over $B_{\mathbb{R}}(0,K)$. Find a finite set $\{z_1,\dots,z_\ell\}$ such that $B_{\mathbb{R}}(0,K)$ is covered by the balls $B_{\mathbb{R}}(z_i,\varepsilon_b)$. By the previous argument, $\mathbb{P}$-a.s., for all $i,j=1,\dots,\ell$, $n^d p^{\omega}_{n^2t}(nz_i,nz_j)/k_t(z_i,z_j)$ is bounded above by $(1+\varepsilon_0)^4$

for all large n. Given $x, y \in B_{\mathbb{R}}(0, K)$, choose z_i, z_j so that $x \in B_{\mathbb{R}}(z_i, \varepsilon_b)$, $y \in B_{\mathbb{R}}(z_j, \varepsilon_b)$. Then using (1.8),

$$\frac{n^d p^{\omega}_{n^2t}(nx, ny)}{k_t(x, y)} = \frac{n^d p^{\omega}_{n^2t}(nz_i, nz_j)}{k_t(z_i, z_j)} \cdot \frac{n^d p^{\omega}_{n^2t}(nx, ny)}{n^d p^{\omega}_{n^2t}(nz_i, nz_j)} \cdot \frac{k_t(z_i, z_j)}{k_t(x, y)} < (1 + \varepsilon_0)^6$$

for all large n. Taking $(1 + \varepsilon_0)^6 < 1 + \varepsilon$ gives the upper bound in (0.18), and the lower bound can be proved similarly. $\square$

1.2. *Convergences after truncation.* For any given $a > 0$, we introduce the following truncation of μ_x:

$$\widetilde{\mu}_e = \widetilde{\mu}_e^{(n)} = \mu_e \cdot \mathbf{1}_{\{\mu_e \le an^2\}}, \qquad \widetilde{\mu}_x = \widetilde{\mu}_x^{(n)} = \sum_{y \sim x} \widetilde{\mu}_{xy}. \tag{1.10}$$

Then we have

$$\mathbb{E}\widetilde{\mu}_x \sim \log(an^2), \qquad \mathbb{E}\widetilde{\mu}_x^2 \le Can^2, \tag{1.11}$$

where C is a constant independent of a and n. Note that $\widetilde{\mu}_x$ and $\widetilde{\mu}_y$ are independent if $|x - y| > 1$.

LEMMA 6. *Let $K > 0$ and $d \ge 3$.*

(a) *If $f : B_{\mathbb{R}}(0, K) \to \mathbb{R}$ is continuous, then $\mathbb{P}$-a.s.,*

$$\frac{1}{n^d \log n} \sum_{|x| \le Kn} \widetilde{\mu}_x f(x/n) \to 2 \int_{B_{\mathbb{R}}(0,K)} f(x)\, dx. \tag{1.12}$$

(b) *If $g : (B_{\mathbb{R}}(0, K))^2 \to \mathbb{R}$ is continuous, then $\mathbb{P}$-a.s.,*

$$\frac{1}{n^{2d} (\log n)^2} \sum_{|x|, |y| \le Kn} \widetilde{\mu}_x \widetilde{\mu}_y g(x/n, y/n) \to 4 \int_{(B_{\mathbb{R}}(0,K))^2} g(x, y)\, dx\, dy. \tag{1.13}$$

PROOF. In both cases we use a straightforward mean–variance calculation.
(a) Write I_n for the LHS of (1.12). Then as $\mathbb{E}\widetilde{\mu}_x \sim \log(an^2) \sim 2 \log n$,

$$\mathbb{E} I_n = \frac{\mathbb{E}\widetilde{\mu}_0}{\log n} \sum_{|x| \le Kn} f(x/n) n^{-d} \to 2 \int_{|x| \le K} f(x)\, dx \qquad \text{as } n \to \infty. \tag{1.14}$$

If $|x - y| \le 1$, then $|\operatorname{Cov}(\widetilde{\mu}_x, \widetilde{\mu}_y)| \le \operatorname{Var}(\widetilde{\mu}_0)$ by Cauchy–Schwarz. So

$$\begin{aligned} \operatorname{Var}_{\mathbb{P}}(I_n) &\le \frac{c \|f\|_\infty^2}{n^{2d} (\log n)^2} \sum_{|x| \le Kn} \operatorname{Var}(\widetilde{\mu}_0) \\ &\le \frac{C}{n^d (\log n)^2} an^2 \le \frac{C'}{n^{d-2} (\log n)^2}. \end{aligned}$$

So, for any $\varepsilon > 0$ we deduce

$$\mathbb{P}(|I_n - \mathbb{E}I_n| > \varepsilon) \le \frac{\mathrm{Var}_{\mathbb{P}}(I_n)}{\varepsilon^2} \le \frac{c(\varepsilon)}{n^{d-2}(\log n)^2},$$

and so by Borel–Cantelli, we have that $|I_n - \mathbb{E}I_n| < \varepsilon$ for all large n.

(b) Let J_n be the left-hand side of (1.13). Write $B = B(0, Kn)$ and

$$J_n' = \frac{1}{n^{2d}(\log n)^2} \sum_{x,y \in B, |x-y| \le 3} \widetilde{\mu}_x \widetilde{\mu}_y g(x/n, y/n),$$

$$J_n'' = \frac{1}{n^{2d}(\log n)^2} \sum_{x,y \in B, |x-y| > 3} \widetilde{\mu}_x \widetilde{\mu}_y g(x/n, y/n).$$

Then since $\widetilde{\mu}_x \widetilde{\mu}_y \le \widetilde{\mu}_x^2 + \widetilde{\mu}_y^2$,

$$\mathbb{E}|J_n'| \le \frac{c}{n^{2d}(\log n)^2} \sum_{x \in B} \mathbb{E}\widetilde{\mu}_x^2 \|g\|_\infty \le \frac{c\|g\|_\infty}{n^{d-2}(\log n)^2}.$$

As this sum converges, by Borel–Cantelli $J_n' \to 0$ $\mathbb{P}$-a.s.

For J_n'' we have

$$\mathbb{E}J_n'' = \frac{(\mathbb{E}\widetilde{\mu}_x)^2}{n^{2d}(\log n)^2} \sum_{x,y \in B, |x-y| > 3} g(x/n, y/n) \to 4 \int_{|x|,|y| \le K} g(x,y)\,dx\,dy.$$

Furthermore,

$$\begin{aligned} \mathrm{Var}_{\mathbb{P}}(J_n'') \le {}& \frac{C}{n^{4d}(\log n)^4} \\ & \times \sum_{x,y \in B, |x-y| > 3} \left(\sum_{x',y' \in B, |x'-y'| > 3} |\mathrm{Cov}(\widetilde{\mu}_x \widetilde{\mu}_y, \widetilde{\mu}_{x'} \widetilde{\mu}_{y'})| \right). \end{aligned} \tag{1.15}$$

If all of x, y, x', y' are at a distance greater than 1 apart in the sum in (1.15), then $\mathrm{Cov}(\widetilde{\mu}_x \widetilde{\mu}_y, \widetilde{\mu}_{x'} \widetilde{\mu}_{y'}) = 0$. So, after relabelling, we only have to handle two cases: when $|x - x'| \le 1$ and $|y - y'| \le 1$, and when $|x - x'| \le 1$ and $|y - y'| > 1$. Write K_n' and K_n'' for these two sums. Observe that in both cases, since $|x - y| > 3$ and $|x' - y'| > 3$, we have $|y' - x| > 1$ and $|y - x'| > 1$.

In the first case,

$$|\mathrm{Cov}(\widetilde{\mu}_x \widetilde{\mu}_y, \widetilde{\mu}_{x'} \widetilde{\mu}_{y'})| \le \mathbb{E}\widetilde{\mu}_x \widetilde{\mu}_{x'} \cdot \mathbb{E}\widetilde{\mu}_y \widetilde{\mu}_{y'} \le cn^4, \tag{1.16}$$

and so

$$K_n' \le \frac{cn^{2d} n^4}{n^{4d}(\log n)^4} \le \frac{c}{n^{2d-4}(\log n)^4}.$$

In the second case,

$$|\mathrm{Cov}(\widetilde{\mu}_x \widetilde{\mu}_y, \widetilde{\mu}_{x'} \widetilde{\mu}_{y'})| \le \mathbb{E}\widetilde{\mu}_x \widetilde{\mu}_{x'} \cdot \mathbb{E}\widetilde{\mu}_y \widetilde{\mu}_{y'} \le cn^2(\log n)^2,$$

and so as the sum in K_n'' contains $O(n^{3d})$ terms

$$K_n'' \le \frac{cn^{3d}n^2(\log n)^2}{n^{4d}(\log n)^4} \le \frac{c}{n^{d-2}(\log n)^2}.$$

Hence $\sum_n \mathrm{Var}_{\mathbb{P}}(J_n'') < \infty$, proving (1.13). $\square$

Finally we state a simple lemma which can be proved by direct computations.

LEMMA 7. *For any $K > 0$,*

(a)

$$\sum_{1\le|x|\le Kn} |x|^{2-d} = O(n^2).$$

(b)

$$\sum_{1\le|x|\le Kn} |x|^{4-2d} = \begin{cases} O(n), & \text{when } d=3, \\ O(\log n), & \text{when } d=4, \\ O(1), & \text{when } d\ge 5. \end{cases}$$

2. Estimates involving Green's functions. For the usual simple random walk on $\mathbb{Z}^d$, $d \ge 3$, Green's function $g(x,x)$ is a positive constant for all x. In our case, the best available lower bound [see Lemma 4(e)] gives that $\mathbb{P}$-a.s., for all large n, and for all $|x| \le Kn$, $g^\omega(x,x) \ge C/(\log n)^{(d-2)/\eta}$. As this is not quite strong enough for the truncation arguments in the next section, we now derive some more precise bounds on sums of Green's functions in a ball.

Recall that E_d denotes the set of edges in $\mathbb{Z}^d$, and in Lemma 4(g) we defined $b_n = c_{13}(\log n)^{1/\eta}$. For $e = \{x_e, y_e\} \in E_d$, let $B(e,r) = B(x_e,r) \cap B(y_e,r)$. For $e = \{x_e, y_e\} \in E_d$ and $z \in \mathbb{Z}^d$, let

$$\gamma_n(e) = C_{\mathrm{eff}}[\{x_e, y_e\}, B(e,b_n)^c], \tag{2.1}$$

$$\gamma_n(z) = C_{\mathrm{eff}}[z, B(z,b_n+1)^c], \tag{2.2}$$

where $C_{\mathrm{eff}}[A,B]$ denotes the effective conductivity between the sets A and B (see (3.8) in [3] or [10], Section 9.4). Note that both $\gamma_n(e)$ and $\gamma_n(x)$ are decreasing in n, and $\gamma_\infty(e) := \lim_n \gamma_n(e)$ is the effective conductivity between e and infinity while $\gamma_\infty(x) := \lim_n \gamma_n(x)$ is equal to $1/g^\omega(x,x)$. By [3], Lemma 6.2, for any $k \ge 1$, $\lim_n \mathbb{E}\gamma_n(e)^k < \infty$. Note further that μ_e and $\gamma_n(e)$ are independent, and also that $\gamma_n(e)$ and $\gamma_n(e')$ are independent if $|e-e'| \ge 2b_n + 1$. When $d \ge 3$, by Lemma 4(e), $g^\omega(x,x) < C < \infty$, and hence

$$\gamma_n(e) \ge \gamma_n(x) \ge \gamma_\infty(x) = 1/g^\omega(x,x) \ge 1/C > 0. \tag{2.3}$$

Let a_p be large enough so that $\mathbb{P}(\mu_e > a_p) < p_c(d)$ where $p_c(d)$ is the critical probability for bond percolation in $\mathbb{Z}^d$. Let $\mathcal{C}(e)$ denote the cluster containing e

in the bond percolation process for which $\{e \text{ is open}\} = \{\mu_e > a_p\}$. Then we have (see [8], Theorems 6.75 and 5.4)

$$\begin{aligned} &\mathbb{P}(|\mathcal{C}(e)| > m) \le \exp(-c_1 m), \\ &\mathbb{P}\big(\operatorname{diam}(\mathcal{C}(e)) > m\big) \le \exp(-c_2 m), \qquad \text{for all } m \ge 1 \end{aligned} \tag{2.4}$$

Let

$$F_n(e) = \{\operatorname{diam}(\mathcal{C}(e)) \ge \tfrac{1}{2} b_n\}, \qquad \gamma_n'(e) = \gamma_n(e) \cdot \mathbf{1}_{F_n(e)^c}.$$

LEMMA 8. (a) *For any $K > 0$, $\mathbb{P}$-a.s., for all sufficiently large n, $\gamma_n(e) = \gamma_n'(e)$ for all $e \in B(0, 2Kn)$.*

(b) *There exists $\theta > 0$ and $\Gamma = \Gamma(\theta) < \infty$ such that for all n,*

$$\mathbb{E} e^{\theta \gamma_n'(e)} < \Gamma.$$

(c) *There exists $C = C(d) > 0$ such that for any $K > 0$, $\mathbb{P}$-a.s., for all large n,*

$$\inf_{|x| \le Kn} g^\omega(x, x) \ge C / \log n.$$

PROOF. (a) First note that

$$\mathbb{P}\Bigg(\bigcup_{e \in B(0,2Kn)} F_n(e)\Bigg) \le c n^d \exp(-c_2 b_n / 2) = c \exp\big(d \log n - c'(\log n)^{1/\eta}\big). \tag{2.5}$$

Since $\eta < 1$ the RHS in (2.5) is summable, so that, for all but finitely many n, $\gamma_n(e) = \gamma_n'(e)$ for all $e \in B(0, 2Kn)$.

(b) On $F_n(e)^c$ the cluster $\mathcal{C}(e)$ is contained in $B(e, b_n)$, and each bond from $\mathcal{C}(e)$ to $\mathcal{C}(e)^c$ has conductivity less than a_p. Since there are at most $2d|\mathcal{C}(e)|$ such bonds, we deduce that $\gamma_n(e) \le d a_p |\mathcal{C}(e)|$. So,

$$\mathbb{P}\big(\gamma_n'(e) > \lambda\big) \le \mathbb{P}\big(d a_p |\mathcal{C}(e)| > \lambda\big) \le \exp(-c\lambda). \tag{2.6}$$

(c) Using (2.6) it is enough to consider

$$\mathbb{P}\Big(\max_{e \in B(0,Kn)} \gamma_n'(e) > \lambda \log n\Big) \le c' n^d e^{-c\lambda \log n}$$

which is summable when λ is large enough. □

For any $0 < a < b \le \infty$, define the sets

$$E_n(a, b) = \{e : a n^2 \le \mu_e < b n^2\}. \tag{2.7}$$

Let m_n be chosen later with $m_n \ge 3 b_n$. We tile $\mathbb{Z}^d$ with cubes of the form $Q = [0, m_n - 1]^d + m_n \mathbb{Z}^d$ so that each cube contains m_n^d vertices. Let z_i, $1 \le i \le d$, be the unit vectors in $\mathbb{Z}^d$, and given a cube Q in the tiling let

$$E(Q) = \big\{\{x, x + z_i\}, x \in Q, 1 \le i \le d\big\};$$

it is clear that $E(Q)$ gives a tiling of E_d, and that $|E(Q)| = dm_n^d$ for each Q. Let $K > 0$ be fixed, and let $\mathcal{Q}_n$ be the set of Q such that $Q \cap B(0, Kn + 1) \neq \varnothing$. We have $|\mathcal{Q}_n| \asymp (Kn/m_n)^d$.

LEMMA 9 (See [3], Lemma 6.3). *Let $a, K, \delta > 0$ be fixed.*

(a) *Suppose that $Kn/\sqrt{d} \geq m_n \geq n^{\theta_1}$ for some $\theta_1 > 2/d$. Then there exists $\lambda > 0$ such that $\mathbb{P}$-a.s., for all but finitely many n,*

$$\max_{Q \in \mathcal{Q}_n} \sum_{e \in E(Q) \cap E_n(a,\infty)} \gamma_n(e) \leq \lambda m_n^d (an^2)^{-1} \mathbb{E}\gamma_n(e). \tag{2.8}$$

(b) *Let $\theta_2 < 1/d$. Then $\mathbb{P}$-a.s., $B(0, n^{\theta_2}) \cap E_n(a, \infty) = \varnothing$ for all but finitely many n.*

PROOF. (a) By Lemma 8(a) it is enough to bound the sum (2.8) with $\gamma_n'(e)$ instead of $\gamma_n(e)$. Let $Q \in \mathcal{Q}_n$. We divide $E(Q)$ into disjoint sets $(E(Q, j), j \in J)$ such that if e and e' are distinct edges in $E(Q, j)$, then $|e - e'| \geq 3b_n - 2$, each $|E(Q, j)| = (m_n/3b_n)^d := N_n$, and $|J| \sim d(3b_n)^d$.

Let $\eta_e = \mathbf{1}_{(\mu_e > an^2)}$, $p_n = \mathbb{E}\eta_e \sim 1/(2d) \cdot 1/(an^2)$, and

$$\xi_j = \sum_{e \in E(Q,j)} \gamma_n'(e)\eta_e.$$

Then the r.v. $(\gamma_n'(e), \eta_e, e \in E(Q, j))$ are independent, and so if θ and Γ are as in Lemma 8,

$$\mathbb{E}e^{\theta\xi_j} \leq \big(1 + p_n(\Gamma - 1)\big)^{N_n} \leq e^{N_n p_n(\Gamma-1)}.$$

Hence for any $\lambda > 0$, writing $\mathbb{E}\xi_j = N_n p_n \mathbb{E}\gamma_n'(e)$,

$$\begin{aligned}\mathbb{P}(\xi_j > \lambda\mathbb{E}\xi_j) &\leq \exp\big(-\lambda\theta N_n p_n \mathbb{E}\gamma_n'(e) + N_n p_n(\Gamma - 1)\big) \\ &= \exp\big(-N_n p_n\big(\lambda\theta\mathbb{E}\gamma_n'(e) - \Gamma + 1\big)\big).\end{aligned}$$

By (2.3),

$$\mathbb{E}\gamma_n'(e) \geq 1/C \cdot \mathbb{P}(F_n(e)^c) \to 1/C,$$

hence there exists $\lambda > 0$ such that for all n large, $\lambda\theta\mathbb{E}\gamma_n'(e) - \Gamma + 1 \geq 1$, and so

$$\mathbb{P}(\xi_j > \lambda\mathbb{E}\xi_j) \leq e^{-N_n p_n}.$$

Thus

$$\mathbb{P}\bigg(\sum_{j \in J} \xi_j > \lambda m_n^d p_n \mathbb{E}\gamma_n'(e)\bigg) \leq d(3b_n)^d e^{-N_n p_n},$$

and so since $|\mathcal{Q}_n| \leq cn^d$ and $N_n p_n \geq n^\varepsilon$ for some $\varepsilon > 0$, (2.8) follows by Borel–Cantelli.

(b) We have

$$\mathbb{P}\big(B(0, n^{\theta_2}) \cap E_n(a, \infty) \neq \varnothing\big) \leq cn^{d\theta_2}(an^2)^{-1} \leq cn^{d\theta_2 - 2};$$

so again the result follows using Borel–Cantelli. □

3. Proof of Theorem 2.

LEMMA 10. *Let $\omega \in \Omega$. If for each $t \geq 0$,*

$$S_t^{(n)} \to 2t \qquad \text{in } P_\omega^0\text{-probability}, \tag{3.1}$$

then (0.12) *holds.*

PROOF. Note that the LHS and RHS are both increasing processes, and the RHS is continuous and deterministic. The conclusion then follows from Theorem VI.3.37 in [9]. $\square$

LEMMA 11. *For each $\varepsilon > 0$ and $T > 0$, there exist $K > 0$ and $a > 0$ such that for $\mathbb{P}$-a.a. ω, for all $t \leq T$, the following two inequalities hold:*

$$\limsup_n P_\omega^0\left(\frac{1}{n^2 \log n} \sum_{|x| \geq Kn} \int_0^{n^2 t} \mu_x \cdot \mathbf{1}_{\{Y_s = x\}} \, ds > 0\right) \leq \varepsilon; \tag{3.2}$$

$$\limsup_n P_\omega^0\left(\frac{1}{n^2 \log n} \sum_{|x| \leq Kn} \int_0^{n^2 t} \mu_x \cdot \mathbf{1}_{\{\mu_x \geq a n^2\}} \mathbf{1}_{\{Y_s = x\}} \, ds > 0\right) \leq \varepsilon. \tag{3.3}$$

PROOF. Write F_K for the event in (3.2). Then by Lemma 4(d),

$$P_\omega^0(F_K) \leq P_\omega^0(\tau(0, Kn) < n^2 t) \leq c_8 \exp(-c_9 K^2 / t),$$

provided that $Kn > U_0$. So, taking K sufficiently large, (3.2) holds for all sufficiently large n.

Choose $\theta_1 = (2 + \varepsilon_1)/d$, $\theta_2 = (1 - \varepsilon_2)/(d - 2)$ where $\varepsilon_1 > 0$, $\varepsilon_2 > 2/d$ (so that $\theta_2 < 1/d$) and $\varepsilon_1 + \varepsilon_2 < 1$. Let $m_n = n^{\theta_1}$, and $\mathcal{Q}_n$ be as in Lemma 9. Let n be large enough so that (2.8) holds, and also that

$$B(0, n^{\theta_2}) \cap E_n(a, \infty) = \varnothing. \tag{3.4}$$

Then

$$P_\omega^0\big(Y \text{ hits } E_n(a, \infty) \cap B(0, Kn)\big) \leq \sum_{Q \in \mathcal{Q}_n} \sum_{x \in E_n(a, \infty) \cap Q} \frac{g^\omega(0, x)}{g^\omega(x, x)}. \tag{3.5}$$

For $x \in E_n(a, \infty)$, if e_x is an edge containing x, then by (2.3) $1/g^\omega(x, x) \leq \gamma_n(e_x)$. By (3.4) and (1.3) we can bound $g^\omega(0, x)$ by $c|x|^{2-d}$.

Let $\mathcal{Q}_n'$ be the set of $Q \in \mathcal{Q}_n$ such that $|x| \geq m_n/2$ for all $x \in Q$. Let first $Q \in \mathcal{Q}_n \backslash \mathcal{Q}_n'$. Then by Lemma 9 and (3.4),

$$\begin{aligned}\sum_{x \in E_n(a, \infty) \cap Q} \frac{g^\omega(0, x)}{g^\omega(x, x)} &\leq \max_{x \in E_n(a, \infty) \cap Q} c|x|^{2-d} \sum_{x \in E_n(a, \infty) \cap Q} \gamma_n(e_x) \\ &\leq C n^{\theta_2(2-d)} \cdot \lambda m_n^d (a n^2)^{-1} \leq C' n^{\varepsilon_1 + \varepsilon_2 - 1}.\end{aligned}$$

So, since there are only 2^d cubes in $\mathcal{Q}_n - \mathcal{Q}_n'$ and $\varepsilon_1 + \varepsilon_2 < 1$ by the choices of ε_1 and ε_2,

$$\lim_n \sum_{Q \in \mathcal{Q}_n - \mathcal{Q}_n'} \sum_{x \in E_n(a,\infty) \cap Q} \frac{g^\omega(0,x)}{g^\omega(x,x)} = 0. \tag{3.6}$$

Now let $Q \in \mathcal{Q}_n'$, and let x_Q be the point in Q closest to 0. Then if $Q \in \mathcal{Q}_n'$,

$$\begin{aligned} \sum_{x \in E_n(a,\infty) \cap Q} \frac{g^\omega(0,x)}{g^\omega(x,x)} &\le c \sum_{x \in E_n(a,\infty) \cap Q} |x|^{2-d} \gamma_n(e_x) \\ &\le c|x_Q|^{2-d} \cdot \lambda m_n^d (an^2)^{-1} \\ &\le c' \lambda a^{-1} n^{-2} \sum_{x \in Q} |x|^{2-d}. \end{aligned} \tag{3.7}$$

So, summing over $Q \in \mathcal{Q}_n'$,

$$\begin{aligned} P_\omega^0\Bigl(Y \text{ hits } E_n(a,\infty) \cap \Bigl(\bigcup_{Q \in \mathcal{Q}_n'} Q\Bigr)\Bigr) &\le c\lambda a^{-1} n^{-2} \sum_{x \in B(0,(K+1)n)} (1 \vee |x|)^{2-d} \\ &\le c' \lambda (K+1)^2 a^{-1}, \end{aligned}$$

and so taking a large enough and noting (3.6), (3.3) follows. $\square$

By Lemma 11 to prove (0.12) it suffices to consider the convergence of

$$\begin{aligned} \widetilde{S}_t^{(n)} &= \frac{1}{n^2 \log n} \sum_{|x| \le Kn} \widetilde{\mu}_x \cdot \int_0^{n^2 t} \mathbf{1}_{\{Y_s = x\}} \, ds \\ &= \frac{1}{\log n} \sum_{|x| \le Kn} \widetilde{\mu}_x \cdot \int_0^t \mathbf{1}_{\{Y_{n^2 s} = x\}} \, ds, \end{aligned} \tag{3.8}$$

where $\widetilde{\mu}_x$ is as in (1.10). Taking expectations with respect to P_ω^0 we have

$$\begin{aligned} E_\omega^0 \widetilde{S}_t^{(n)} &= \frac{1}{n^2 \log n} \sum_{|x| \le Kn} \widetilde{\mu}_x \cdot \int_0^{n^2 t} p_s^\omega(0,x) \, ds \\ &= \frac{1}{\log n} \sum_{|x| \le Kn} \widetilde{\mu}_x \cdot \int_0^t p_{n^2 r}^\omega(0,x) \, dr. \end{aligned} \tag{3.9}$$

LEMMA 12. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that, $\mathbb{P}$-a.s. for all sufficiently large n,*

$$E_\omega^0 \widetilde{S}_\delta^{(n)} \le \varepsilon. \tag{3.10}$$

PROOF. By Lemma 4(g), we can assume n is large enough so that $\{\max_{|x|\le Kn} U_x \le b_n\}$. Hence, by Lemma 4(b), if $|x| \vee \sqrt{t} \ge b_n$, then

$$p_t^\omega(0,x) \le \begin{cases} c_4 t^{-d/2}\exp(-c_5|x|^2/t), & \text{when } t \ge |x|, \\ c_4\exp(-c_5|x|), & \text{when } t \le |x|. \end{cases}$$

Hence, by decomposing according to whether $|x| < b_n$ or $|x| \ge b_n$, we obtain

$$E_\omega^0 \widetilde{S}_\delta^{(n)} = \frac{1}{n^2\log n}\sum_{|x|\le Kn}\widetilde{\mu}_x \cdot \int_0^{n^2\delta} p_s^\omega(0,x)\,ds$$

$$\le \frac{1}{n^2\log n}\sum_{|x|\le b_n}\widetilde{\mu}_x \cdot \int_0^{n^2\delta} c(1\vee s)^{-d/2}\,ds \tag{3.11}$$

$$+ \frac{1}{n^2\log n}\sum_{b_n\le|x|\le Kn}\widetilde{\mu}_x \int_0^{|x|} c_4 e^{-c_5|x|}\,ds \tag{3.12}$$

$$+ \frac{1}{n^2\log n}\sum_{b_n\le|x|\le Kn}\widetilde{\mu}_x \cdot \int_{|x|}^{n^2\delta} c_4 s^{-d/2} e^{-c_5|x|^2/s}\,ds. \tag{3.13}$$

Write $\xi_n^{(i)}$, $i = 1, 2, 3$, for the terms in (3.11)–(3.13). Since the integral in (3.11) is bounded by $\int_0^\infty c(1\vee s)^{-d/2}\,ds < \infty$, we have

$$\mathbb{E}\xi_n^{(1)} \le c\frac{b_n^d}{n^2\log n}\mathbb{E}\widetilde{\mu}_x \le cn^{-2}(\log n)^{d/\eta}.$$

Similarly for (3.12) we have

$$\mathbb{E}\xi^{(2)} \le cn^{-2}\sum_{|x|\le Kn} c_4|x|e^{-c_5|x|} \le c'n^{-2}.$$

As these sums converge it follows from Borel–Cantelli that $\xi_n^{(i)} \le \varepsilon/3$ for all large n, for $i = 1, 2$.

It remains to control (3.13). First note that when $s \ge 1$,

$$\sum_{x\in\mathbb{Z}^d} s^{-d/2}e^{-\kappa|x|^2/s} \le C(\kappa). \tag{3.14}$$

So, interchanging the order of the sum and integral in (3.13),

$$\mathbb{E}\xi_n^{(3)} \le \frac{C}{n^2\log n}\mathbb{E}\widetilde{\mu}_0 \cdot n^2\delta \le C'\delta.$$

Setting $t = s/|x|^2$ we have

$$\int_{|x|}^{n^2\delta} c_4 s^{-d/2}e^{-c_5|x|^2/s}\,ds \le C|x|^{2-d}\int_0^\infty t^{-d/2}e^{-c_5/t}\,dt \le C|x|^{2-d}. \tag{3.15}$$

Hence, applying Lemma 7 we get

$$\mathrm{Var}_{\mathbb{P}}(\xi_n^{(3)}) \le \frac{C}{n^4(\log n)^2} \cdot \sum_{b_n \le |x| \le Kn} an^2|x|^{4-2d} \le \frac{C}{n(\log n)^2}.$$

By Chebyshev's inequality and Borel–Cantelli we then get that for δ small enough, $\mathbb{P}$-a.s. for all sufficiently large n, $\xi_n^{(3)} \le \varepsilon/3$. $\square$

PROPOSITION 13. *Let*

$$A_1(K,t,\delta) = \int_{|y|\le K} \int_\delta^t k_s(x)\,dx\,ds. \tag{3.16}$$

When $d \ge 3$, for any $K > 0$, $0 < \delta < T < \infty$, and $t \in (\delta, T]$, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} E_\omega^0(\widetilde{S}_t^{(n)} - \widetilde{S}_\delta^{(n)}) = 2A_1(K,t,\delta). \tag{3.17}$$

PROOF. By Lemma 6(a), it suffices to show that $\mathbb{P}$-a.s.,

$$\frac{1}{\log n} \sum_{|x|\le Kn} \widetilde{\mu}_x \cdot \int_\delta^t \big(p_{n^2 s}^\omega(0,x) - n^{-d} k_s(x/n)\big)\,ds \to 0.$$

The LHS is bounded in absolute value by

$$\frac{1}{n^d \log n} \sum_{|x|\le Kn} \widetilde{\mu}_x \cdot T \sup_{x\in\mathbb{Z}^d} \sup_{s\ge\delta} |n^d p_{n^2 s}^\omega(0,x) - k_s(x/n)|.$$

This converges to 0 $\mathbb{P}$-a.s. by Lemmas 6(a) and 4(h). $\square$

PROPOSITION 14. *When $d \ge 3$, for any $\varepsilon > 0$, $K > 0$, $0 < \delta < T < \infty$, and $t \in (\delta, T]$, $\mathbb{P}$-a.s.,*

$$\begin{aligned} &\limsup_n E_\omega^0(\widetilde{S}_t^{(n)} - \widetilde{S}_\delta^{(n)})^2 \\ &\quad \le \varepsilon + 8(1+\varepsilon) \int_{|x|,|y|\le K} \int_\delta^t k_s(x) \int_0^{t-s} k_r(x,y)\,dr\,ds\,dx\,dy. \end{aligned} \tag{3.18}$$

PROOF. Using the Markov property and the symmetry of Y,

$$\begin{aligned} &E_\omega^0(S_t^{(n)} - S_\delta^{(n)})^2 \\ &\quad = \frac{2}{(\log n)^2} \bigg(\sum_{|x|,|y|\le Kn} \widetilde{\mu}_x \widetilde{\mu}_y \cdot \int_\delta^t p_{n^2 s}^\omega(0,x) \int_0^{t-s} p_{n^2 r}^\omega(x,y)\,dr\,ds \bigg). \end{aligned}$$

We begin by proving that, given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $\mathbb{P}$-a.s., for all large n,

$$\frac{2}{(\log n)^2} \sum_{|x|,|y|\le Kn} \widetilde{\mu}_x \widetilde{\mu}_y \cdot \int_\delta^t p_{n^2 s}^\omega(0,x) \int_0^{\delta_1} p_{n^2 r}^\omega(x,y)\,dr\,ds \le \varepsilon. \tag{3.19}$$

By Lemma 4(a) we have $p^{\omega}_{n^2 s}(0,x) \le cn^{-d}$ for all $s \ge \delta$ and so the LHS of (3.19) is bounded by

$$\frac{C}{n^d(\log n)^2} \sum_{|x|,|y| \le Kn} \widetilde{\mu}_x \widetilde{\mu}_y \int_0^{\delta_1} p^{\omega}_{n^2 r}(x,y)\,dr \tag{3.20}$$

$$= \frac{C}{n^{d+2}(\log n)^2} \sum_{|x|,|y| \le Kn, |x-y|>1} \widetilde{\mu}_x \widetilde{\mu}_y \int_0^{n^2\delta_1} p^{\omega}_{r}(x,y)\,dr \tag{3.21}$$

$$+ \frac{C}{n^{d+2}(\log n)^2} \sum_{|x|,|y| \le Kn, |x-y|\le 1} \widetilde{\mu}_x \widetilde{\mu}_y \int_0^{n^2\delta_1} p^{\omega}_{r}(x,y)\,dr. \tag{3.22}$$

Write A_n and B_n for the terms in (3.21) and (3.22).

The first term can be handled in the same way as in Lemma 12. Let $B = B(0, Kn)$, and write $A_n = A_n^{(1)} + A_n^{(2)} + A_n^{(3)}$ where

$$A_n^{(1)} = \frac{C}{n^{d+2}(\log n)^2} \sum_{x,y \in B, 1<|x-y|\le b_n} \widetilde{\mu}_x \widetilde{\mu}_y \int_0^{n^2\delta_1} p^{\omega}_{r}(x,y)\,dr, \tag{3.23}$$

$$A_n^{(2)} = \frac{C}{n^{d+2}(\log n)^2} \sum_{x,y \in B, |x-y|\ge b_n} \widetilde{\mu}_x \widetilde{\mu}_y \int_0^{|x-y|} p^{\omega}_{r}(x,y)\,dr, \tag{3.24}$$

$$A_n^{(3)} = \frac{C}{n^{d+2}(\log n)^2} \sum_{x,y \in B, |x-y|\ge b_n} \widetilde{\mu}_x \widetilde{\mu}_y \int_{|x-y|}^{n^2\delta_1} p^{\omega}_{r}(x,y)\,dr. \tag{3.25}$$

For (3.23) we have

$$\begin{aligned}
\mathbb{E}A_n^{(1)} &\le \frac{C}{n^{d+2}(\log n)^2} \sum_{x,y \in B, 1<|x-y|<b_n} \mathbb{E}(\widetilde{\mu}_x \widetilde{\mu}_y) \int_0^{\infty} c_4 (1 \vee s)^{-d/2}\,ds \\
&\le \frac{C}{n^{d+2}} K^d n^d b_n^d \\
&\le c \frac{(\log n)^{d/\eta}}{n^2},
\end{aligned}$$

and since this sum converges, we have $A_n^{(1)} \le \varepsilon/4$ for all large n, $\mathbb{P}$-a.s. The term $\mathbb{E}A_n^{(2)}$ is bounded in the same way as was the term $\xi_n^{(2)}$ in Lemma 12.

For (3.25),

$$\begin{aligned}
A_n^{(3)} &\le \frac{C}{n^{d+2}(\log n)^2} \\
&\quad \times \sum_{x,y \in B, |x-y|>b_n} \widetilde{\mu}_x \widetilde{\mu}_y \int_{|x-y|}^{n^2\delta_1} c_4 s^{-d/2} \exp(-c_5 |x-y|^2/s)\,ds.
\end{aligned} \tag{3.26}$$

Using (3.14) we have

$$\mathbb{E}A_n^{(3)} \le \frac{C}{n^{d+2}(\log n)^2} \cdot n^d (\mathbb{E}\widetilde{\mu}_0)^2 \cdot n^2 \delta_1 = O(\delta_1).$$

We now bound $\mathrm{Var}_{\mathbb{P}}(A_n^{(3)})$. By (3.15), the integral in (3.26) is bounded by $c|x-y|^{2-d}$, so

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}}(A_n^{(3)}) \le{}& \frac{C}{n^{2d+4}(\log n)^4} \\ &\times \sum_{x_1,y_1\in B,|x_1-y_1|>b_n} \sum_{x_2,y_2\in B,|x_2-y_2|>b_n} |x_1-y_1|^{2-d}|x_2-y_2|^{2-d} \\ &\qquad \times |\mathrm{Cov}(\widetilde{\mu}_{x_1}\widetilde{\mu}_{y_1}, \widetilde{\mu}_{x_2}\widetilde{\mu}_{y_2})|.\end{aligned}$$

We now bound this sum in the same way as was done for the variance in Lemma 6(b). Let

$$\mathcal{C}_1 = \{(x_1,x_2,y_1,y_2) \in B^4 : |x_i - y_i| > b_n, i = 1,2, |x_1-x_2| \le 1, |y_1-y_2| \le 1\},$$

$$\mathcal{C}_2 = \{(x_1,x_2,y_1,y_2) \in B^4 : |x_i - y_i| > b_n, i = 1,2, |x_1-x_2| \le 1, |y_1-y_2| > 1\}.$$

Note that if $|x_1-x_2| \le 1$, then since $|x_i - y_i| > b_n$, none of the y_i can be within distance 1 of x_j. If $(x_1,\dots,y_2) \in \mathcal{C}_1$, then $|\mathrm{Cov}(\widetilde{\mu}_{x_1}\widetilde{\mu}_{y_1}, \widetilde{\mu}_{x_2}\widetilde{\mu}_{y_2})| \le cn^4$, while if $(x_1,\dots,y_2) \in \mathcal{C}_2$, then $|\mathrm{Cov}(\widetilde{\mu}_{x_1}\widetilde{\mu}_{y_1}, \widetilde{\mu}_{x_2}\widetilde{\mu}_{y_2})| \le c(\log n)^2 n^2$. So,

$$\begin{aligned}&\frac{C}{n^{2d+4}(\log n)^4} \sum_{(x_1,\dots,y_2)\in\mathcal{C}_1} |x_1-y_1|^{2-d}|x_2-y_2|^{2-d} \cdot |\mathrm{Cov}(\widetilde{\mu}_{x_1}\widetilde{\mu}_{y_1}, \widetilde{\mu}_{x_2}\widetilde{\mu}_{y_2})| \\ &\quad\le \frac{C}{n^{2d+4}(\log n)^4} \sum_{x_1,y_1\in B} (1\vee|x_1-y_1|)^{4-2d} cn^4 \\ &\quad\le \frac{C}{n^{2d}(\log n)^4} n^d \max_{x_1\in B} \sum_{y_1\in B(x,2Kn)} (1\vee|x_1-y_1|)^{4-2d} \\ &\quad\le \frac{Cn}{n^d(\log n)^4},\end{aligned}$$

where in the last inequality we used Lemma 7(b).

Also,

$$\begin{aligned}&\frac{C}{n^{2d+4}(\log n)^4} \sum_{(x_1,\dots,y_2)\in\mathcal{C}_2} |x_1-y_1|^{2-d}|x_2-y_2|^{2-d} |\mathrm{Cov}(\widetilde{\mu}_{x_1}\widetilde{\mu}_{y_1}, \widetilde{\mu}_{x_2}\widetilde{\mu}_{y_2})| \\ &\quad\le \frac{C}{n^{2d+2}(\log n)^2} \sum_{(x_1,\dots,y_2)\in\mathcal{C}_2} |x_1-y_1|^{2-d}|x_2-y_2|^{2-d} \\ &\quad\le \frac{C}{n^{2d+2}(\log n)^2} \sum_{x_1\in B} \sum_{y_1,y_2\in B(x,2Kn)} (1\vee|x_1-y_1|)^{2-d}(1\vee|x_1-y_2|)^{2-d}\end{aligned}$$

$$\le \frac{C}{n^{d+2}(\log n)^2}\biggl(\sum_{y_1\in B(0,2Kn)}(1\vee|y_1|)^{2-d}\biggr)^2$$
$$\le \frac{Cn^4}{n^{d+2}(\log n)^2}=\frac{C}{n^{d-2}(\log n)^2}.$$

Thus $\sum_n \mathrm{Var}_{\mathbb{P}}(A_n^{(3)})<\infty$, and so if δ_1 is small enough then by Chebyshev's inequality and Borel–Cantelli, $\mathbb{P}$-a.s. for all sufficiently large n, $A_n^{(3)}\le \varepsilon/4$.

To finish the proof of (3.19), it remains to bound the term (3.22). By Lemma 4(a), $\int_0^{n^2\delta_1} p_r^\omega(x,y)\,dr\le C$. Therefore by Cauchy–Schwarz,

$$B_n=\frac{C}{n^{d+2}(\log n)^2}\sum_{|x|\le Kn,|y-x|\le 1}\widetilde{\mu}_x\widetilde{\mu}_y\int_0^{n^2\delta_1}p^\omega_{n^2r}(x,y)\,dr$$
$$\le \frac{C}{n^{d+2}(\log n)^2}\sum_{|x|\le Kn}\widetilde{\mu}_x^2.$$

Hence

$$\mathbb{E}B_n\le \frac{C}{n^{d+2}(\log n)^2}\cdot n^d\cdot n^2\to 0,$$

and since $\mathrm{Var}_{\mathbb{P}}(\widetilde{\mu}_x^2)\le cn^6$,

$$\mathrm{Var}_{\mathbb{P}}(B_n)\le \frac{C}{n^{2d+4}(\log n)^4}\cdot n^d\cdot n^6\le \frac{C}{n^{d-2}(\log n)^4}.$$

Since this bound is summable, (3.19) follows.

It remains to show that for any $\delta_1>0$, $\mathbb{P}$-a.s.,

$$\limsup_n \frac{2}{(\log n)^2}\sum_{|x|,|y|\le Kn}\widetilde{\mu}_x\widetilde{\mu}_y\cdot\int_\delta^t p^\omega_{n^2s}(0,x)\int_{\delta_1}^{t-s}p^\omega_{n^2r}(x,y)\,dr\,ds$$
$$\le 8(1+\varepsilon)\int_{|x|,|y|\le K}\biggl(\int_\delta^t k_s(0,x)\int_0^{t-s}k_r(x,y)\,dr\,ds\biggr)\,dx\,dy.$$

This follows easily from Theorem 3 and Lemma 6. $\square$

PROOF OF THEOREM 2. By Lemma 10, it suffices to show that for any $t>0$ and $0<\varepsilon<t/2$, for $\mathbb{P}$-a.a. ω,

$$\lim_n P^0_\omega\bigl(|S_t^{(n)}-2t|\ge\epsilon\bigr)\le\epsilon. \tag{3.27}$$

Write

$$\begin{aligned}S_t^{(n)}-2t&=\bigl(S_t^{(n)}-\widetilde{S}_t^{(n)}\bigr)+\widetilde{S}_\delta^{(n)}+\bigl(\widetilde{S}_t^{(n)}-\widetilde{S}_\delta^{(n)}-E^0_\omega(\widetilde{S}_t^{(n)}-\widetilde{S}_\delta^{(n)})\bigr)\\&\quad+\bigl(E^0_\omega(\widetilde{S}_t^{(n)}-\widetilde{S}_\delta^{(n)})-2A_1(K,t,\delta)\bigr)+\bigl(2A_1(K,t,\delta)-2t\bigr).\end{aligned} \tag{3.28}$$

By Proposition 13, $\mathbb{P}$-a.s., $(E^0_\omega(\widetilde{S}^{(n)}_t - \widetilde{S}^{(n)}_\delta) - 2A_1(K,t,\delta)) \to 0$. Let $0 < \varepsilon_0 < \varepsilon/16$, to be chosen later. Choose K large enough so that the LHS in (3.2) is bounded by ε_0, and also

$$\sup_{0<\delta\le t} |A_1(K,t,\delta) - (t-\delta)| \le \varepsilon_0 < \varepsilon/16. \tag{3.29}$$

Now choose $a > 0$ large enough so that the LHS in (3.3) is also bounded by ε_0. Hence, for all large n,

$$P^0_\omega\big(|S^{(n)}_t - \widetilde{S}^{(n)}_t| > 0\big) \le 2\varepsilon_0 < \varepsilon/4.$$

Next choose $0 < \delta < t/2$ so that by Lemma 12 for all sufficiently large n, $E^0_\omega \widetilde{S}^{(n)}_\delta < \varepsilon^2/16$, and hence $P^0_\omega(\widetilde{S}^{(n)}_\delta > \varepsilon/4) \le \varepsilon/4$. Furthermore, by Propositions 13 and 14 and (3.29),

$$\begin{aligned}\limsup_n \operatorname{Var}_{\mathbb{P}}\big(\widetilde{S}^{(n)}_t - \widetilde{S}^{(n)}_\delta\big) &\le \varepsilon_0 + 8(1+\varepsilon_0)\cdot(t-\delta)^2/2 - \big(2(t-\delta-\varepsilon_0)\big)^2 \\ &\le \varepsilon_0(1+4t^2+4t);\end{aligned}$$

hence by Chebyshev's inequality,

$$\limsup_n P^0_\omega\big(|\widetilde{S}^{(n)}_t - \widetilde{S}^{(n)}_\delta - E^0_\omega(\widetilde{S}^{(n)}_t - \widetilde{S}^{(n)}_\delta)| \ge \varepsilon/4\big) \le 16(1+4t^2+4t)\cdot \varepsilon_0/\varepsilon^2.$$

Taking ε_0 so small that $\varepsilon_0 < \varepsilon/16$ and $16(1+4t^2+4t)\cdot\varepsilon_0/\varepsilon^2 \le \varepsilon/4$, we obtain (3.27). $\square$

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