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EFFECTIVE RESISTANCE OF RANDOM TREES

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We investigate the effective resistance R_n and conductance C_n between the root and leaves of a binary tree of height n. In this electrical network, the resistance of each edge e at distance d from the root is defined by $r_e = 2^d X_e$ where the X_e are i.i.d. positive random variables bounded away from zero and infinity. It is shown that $\mathbf{E}R_n = n\mathbf{E}X_e - (\mathbf{Var}(X_e)/\mathbf{E}X_e)\ln n + O(1)$ and $\mathbf{Var}(R_n) = O(1)$. Moreover, we establish sub-Gaussian tail bounds for R_n. We also discuss some possible extensions to supercritical Galton–Watson trees.

1. Introduction. Electric networks have been known to be closely related to random walks and their investigation often offers an elegant and effective way of studying properties of random walks. See Doyle and Snell [9] and Lyons and Peres [15] for very nice introductions to the subject. For the better understanding of certain random walks in random environments, it is natural to study random electric networks, that is, electric networks in which edges are equipped with independent random resistances. This model was studied by Benjamini and Rossignol [5] who considered the case of the cubic lattice $\mathbb{Z}^d$ where the resistance of each edge is an independent copy of a Bernoulli random variable. Using an inequality of Falik and Samorodnitsky [11], they proved that point-to-point effective resistance has submean variance in $\mathbb{Z}^2$, whereas the variance is of the order of the mean when $d \geq 3$. In this paper, we study the corresponding problem for binary trees.

An electric network is a locally finite connected graph $G = (V, E)$ with vertex set V and edge set E such that each edge $e \in E$ is equipped with a number $r_e \geq 0$ called *resistance*. (In this paper we only consider finite graphs.) Alternatively, an edge is associated with a conductance $c_e = 1/r_e$. The *effective resistance* between two disjoint sets of vertices $A, B \subset V$ is defined as follows: assign "voltage" (or potential) $U(u) = 1$ to each vertex $u \in A$ and $U(v) = 0$ for all $v \in B$. If G is finite then the function U can be extended, in a unique way, to all vertices in V according to two basic laws given by *Ohm's law* and *Kirchhoff's node law*. In order to describe these laws, we need the notion of *current*. Given two vertices

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$u, v \in V$ joined by and edge $e \in E$, the current flowing from u to v is a real number $i(u, v)$. Ohm's law states that for each edge of the graph, $i(u, v)r_e = U(u) - U(v)$. Kirchhoff's node law postulates that for any vertex $u \notin A \cup B$, $\sum_{v:v\sim u} i(u, v) = 0$. (For the proof that these two laws uniquely determine the function $U : V \to [0, 1]$, see [9] or [15].) Now the *effective conductance* between the vertex sets A and B is defined as the total current flowing into the network, that is,

$$C(A \leftrightarrow B) = \sum_{u\in A} \sum_{v:v\sim u} i(u, v).$$

The effective resistance between A and B is $R(A \leftrightarrow B) = 1/C(A \leftrightarrow B)$.

Several useful tricks of network reduction are known that help simplify resistance calculations. Since in this paper we focus on trees, it suffices to recall two of the simplest rules. One of them states that two resistors in series are equivalent to a single resistor whose resistance is the sum of the original resistances. The other rule states that two conductors in parallel are equivalent to a single conductor whose conductance is the sum of the original conductances. Apart from these two simple rules, a formula called *Thomson's principle* will be useful for our purposes. Thomson's principle gives an explicit expression for the effective resistance. It states that

$$R(A \leftrightarrow B) = \inf_{\Theta\in F} \sum_{e\in E} r_e \Theta(e)^2, \tag{1}$$

where the infimum is taken over the set F of all *unit flows*. A unit flow is a function Θ over the set of oriented edges $\{(u, v) : u \sim v\}$ which is antisymmetric [i.e., $\Theta(u, v) = -\Theta(v, u)$], satisfies $\sum_{v:v\sim u} \Theta(u, v) = 0$ for any vertex $u \notin A \cup B$, and has

$$\sum_{u\in A} \sum_{v\notin A:v\sim u} \Theta(u, v) = \sum_{v\in B} \sum_{u\notin B:u\sim v} \Theta(u, v) = 1.$$

It can be shown that the unique unit flow Θ^* which attains the above infimum is proportional to the current $i(u, v)$ (see, e.g., Doyle and Snell [9], page 50).

In this paper we consider the case of a complete infinite binary tree T with root r. (All results carry over trivially to infinite b-ary trees for integers $b > 2$.) The *depth* $d(v)$ of a node v in T is the number of edges on the path from the root to v. We say that an edge e has depth d if there are d edges on the path starting with edge e and ending at the root. The resistance of an edge e at depth d is defined by $2^{d-1}X_e$ where the X_e are independent copies of some strictly positive random variable with finite mean. This exponential weighting corresponds to the "critical" (with respect to transience/recurrence) case of the biased random walk in random environment obtained by traversing an edge e, starting from either endpoint, with probability proportional to its conductance (the inverse of its resistance). This type of exponential scaling of resistances was considered, for example, by Lyons [13]. He showed that in an infinite rooted tree with branching number b, if the (deterministic) resistance of an edge equals λ^d then the effective resistance between the root and "infinity" is infinite if $\lambda > b$ and finite if $\lambda < b$. Thus, our choice of scaling corresponds to the critical case. Similar biased random walks have been studied in depth by Pemantle [17], Lyons [13] and Lyons and Pemantle [14], who beautifully characterize the type of such random walks in many situations. (However, our model does not quite fit within their framework, as the transition probabilities fail to satisfy a certain independence requirement.) Unfortunately, we do not immediately see how to translate our results into results about biased random walks or random walks in random environments. Also, it is likely that such results would only be new if we could also extend our results to the more general setting of Galton–Watson trees. For more background on the connection between effective resistance of networks and random walks, see Doyle and Snell [9], Lyons and Peres [15], Peres [18] or Soardi [19].

For a random network such as that described in the previous paragraph, interesting and nontrivial behavior emerges. Let R_n be the effective resistance between the root r and the set of vertices at depth n, and let μ and σ^2 be the mean and variance of X_e, respectively. The primary results of our paper are that as long as X_e is bounded away from both zero and infinity,

$$\mathbf{E}R_n = \mu n - \frac{\sigma^2}{\mu}\ln n + O(1) \quad \text{and} \quad \mathbf{E}[|R_n - \mathbf{E}R_n|^q] = O(1) \qquad \text{for all } q \geq 1.$$

(These results are contained in Theorems 5 and 7.) We also derive correspondingly precise results about the conductance $C_n = 1/R_n$. Interestingly, in order to estimate the expected resistance, our main tool is a sharp upper bound for the variance of the conductance (and thereby for the variance of the resistance). Intuitively, concentration of the conductance implies that the behavior of the electric network is not very different from the one with deterministic resistances $2^d\mu$. Thus, Section 2 is devoted to the variance of the conductance C_n. In particular, we show that $\mathbf{Var}[C_n] = O(n^{-4})$. In Section 3 we derive the bounds for the expected resistance and conductance mentioned above. In Section 4 we establish sub-Gaussian tail bounds for R_n. The proof is based on Thomson's formula (1) and relies on an exponential concentration inequality due to Boucheron et al. [7].

Finally, in Section 5 we briefly discuss the ways in which one might attempt to extend our results to supercritical Galton–Watson processes (in this case the appropriate scaling for the resistances is $[\mathbf{E}Z_1]^d$ for edges at depth d, where Z_1 is the number of offspring of the root). In the Galton–Watson setting, it makes sense to first condition on the tree, then study the conditional behavior of the effective resistance and of Θ^*. In Section 5 we shall also observe that if the random variable X_e is constant then the "scaled analogue" of Question 4.1 from Lyons, Pemantle and Peres [16] is easily answered; motivated by this, we suggest a more general question.

From this point on, we assume X_e is any random variable taking values in some interval $[a, b]$ with $0 < a < b$. Most our arguments rely on a recursive decomposition of the tree. This decomposition is made easier by rooting the tree at an edge

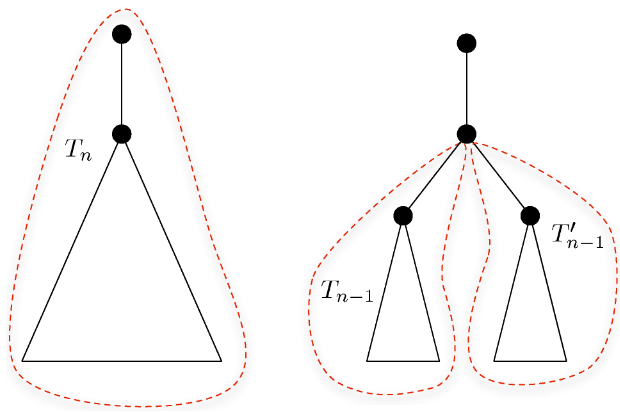

FIG. 1. *Rooting the binary tree rooted at an edge instead of a vertex simplifies the recursive decomposition.*

instead of at a vertex; this trick was also used in [16] to facilitate conductance computations. More precisely, for $n \geq 1$ we define the tree T_n as follows: the root has one single child whose subtree is a complete binary tree with $n-1$ levels (so 2^{n-1} leaves). Then, T_n decomposes exactly into a single edge connected (in series) to two independent copies of T_{n-1} (in parallel) as shown in Figure 1. We let R_n be the effective resistance of T_n taken between the root and the leaves. Let $C_n = 1/R_n$ be the corresponding effective conductance, so in particular R_1 is distributed as X and C_1 is distributed as $1/X$. The difference between R_n and the effective resistance of the complete binary tree of height $n-1$ is at most b, so bounds on the moments of the former immediately imply corresponding bounds for the latter.

We close this introduction by noting that the results of Benjamini and Rossignol [5] are proved by adapting an argument first used by Benjamini, Kalai and Schramm [6] to prove submean variance bounds for first-passage percolation on $\mathbb{Z}^2$. Addario-Berry and Reed [1] have studied first-passage percolation on supercritical Galton–Watson processes; though their approach is entirely different from ours, their result is strikingly similar: under suitable assumptions on the edge lengths (which in their case are i.i.d.), the height of the first-passage percolation cluster of (weighted) diameter n has expected value $\alpha n - \beta \ln n + O(1)$, for computable constants α and β, and has bounded variance. We are not sure whether this similarity is more than a coincidence.

2. The variance of the conductance.

The purpose of this section is to derive an upper bound for the variance of the conductance C_n. We start by noticing that R_n and C_n admit the following scalings.

LEMMA 1. *When $a \leq X \leq b$, we have $an \leq R_n \leq bn$ and $1/b \leq nC_n \leq 1/a$, for all $n \geq 1$.*

The lemma follows from Rayleigh's monotonicity law (see [9], page 53) by bounding the resistance of T_n between that of two deterministic networks in which the random variables either always take their minimum value a or their maximum value b. We first derive a bound on $\mathbf{Var}[C_n]$. Using Chebyshev's inequality, this bound yields a quadratically decaying tail bound for R_n.

THEOREM 2. *There exists a constant K depending only on a and b such that $\mathbf{Var}[C_1] \le K$, $\mathbf{Var}[R_1] \le K$ and for all integers $n \ge 2$,*

$$\mathbf{Var}[C_n] \le K \cdot \left(\sum_{i=1}^{n-1} \frac{2^{1-i}}{(n-i)^4} + \frac{1}{2^{n-1}} \right) \le \frac{2^{10} K}{n^4} \quad \textit{and} \quad \mathbf{Var}[R_n] \le K.$$

Our main tool in proving Theorem 2 is the Efron–Stein inequality, which provides an upper bound on the variance of functions of independent random variables.

THEOREM 3 (Efron and Stein [10], Steele [20]). *Let Y_i, $i \ge 1$, be independent random variables, and let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a measurable function of n variables. Then,*

$$\mathbf{Var}[f(Y_1, \dots, Y_n)] \\ \le \frac{1}{2} \cdot \sum_{i=1}^{n} \mathbf{E}\big[\big(f(Y_1, \dots, Y_i, \dots, Y_n) - f(Y_1, \dots, Y_i', \dots, Y_n)\big)^2\big],$$

where Y_i', $i \ge 0$, are independent copies of Y_i, $i \ge 0$.

PROOF OF THEOREM 2. It clearly suffices to treat the case $n \ge 2$. We decompose T_n into three independent conductors C_1, $C_{n,1}$ and $C_{n,2}$ as shown in Figure 2. Then, C_n is a function of these three independent random variables:

$$C_n = \frac{C_1 \cdot (C_{n,1} + C_{n,2})}{C_1 + C_{n,1} + C_{n,2}}. \tag{2}$$

By the Efron–Stein inequality and the symmetry of $C_{n,1}$ and $C_{n,2}$, we have

$$\begin{aligned} \mathbf{Var}[C_n] &\le \mathbf{E}\left[\left(\frac{C_1 \cdot (C_{n,1} + C_{n,2})}{C_1 + C_{n,1} + C_{n,2}} - \frac{C_1 \cdot (C_{n,1}' + C_{n,2})}{C_1 + C_{n,1}' + C_{n,2}}\right)^2\right] \\ &\quad + \frac{1}{2} \cdot \mathbf{E}\left[\left(\frac{C_1 \cdot (C_{n,1} + C_{n,2})}{C_1 + C_{n,1} + C_{n,2}} - \frac{C_1' \cdot (C_{n,1} + C_{n,2})}{C_1' + C_{n,1} + C_{n,2}}\right)^2\right], \end{aligned} \tag{3}$$

where $C_{n,1}'$ is an independent copy of $C_{n,1}$. Letting $\alpha = C_{n,1} + C_{n,2}$, the second term of (3) reduces to

$$\mathbf{E}\left[\left(\frac{C_1 \alpha}{C_1 + \alpha} - \frac{C_1' \alpha}{C_1' + \alpha}\right)^2\right].$$

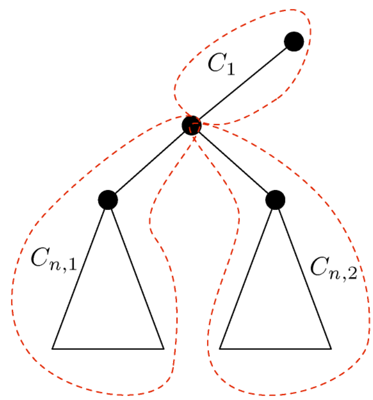

FIG. 2. *The decomposition of T_n into three conductors C_1, $C_{n,1}$ and $C_{n,2}$ at the origin of our recurrence relations.*

Observe that, since $1/b \le C_1, C_1' \le 1/a$, we have

$$\left|\frac{C_1\alpha}{C_1+\alpha} - \frac{C_1'\alpha}{C_1'+\alpha}\right| = \left|\frac{\alpha^2(C_1 - C_1')}{(C_1+\alpha)(C_1'+\alpha)}\right| \le b^2\alpha^2\left(\frac{1}{a} - \frac{1}{b}\right).$$

Hence

$$\frac{1}{2}\mathbf{E}\left[\left(\frac{C_1\alpha}{C_1+\alpha} - \frac{C_1'\alpha}{C_1'+\alpha}\right)^2\right] \le b^4\left(\frac{1}{b} - \frac{1}{a}\right)^2 \frac{1}{2}\mathbf{E}[(C_{n,1}+C_{n,2})^4].$$

Since both $C_{n,1}$ and $C_{n,2}$ are distributed as $C_{n-1}/2$, by Lemma 1, we have $C_{n,1} + C_{n,2} \le 1/(a(n-1))$, and this yields

$$\frac{1}{2}\mathbf{E}\left[\left(\frac{C_1\alpha}{C_1+\alpha} - \frac{C_1'\alpha}{C_1'+\alpha}\right)^2\right] \le \frac{1}{2}\left(\frac{b}{a}\right)^4\left(\frac{1}{b} - \frac{1}{a}\right)^2 \frac{1}{(n-1)^4} \stackrel{\text{def}}{=} \frac{K_0}{(n-1)^4}. \tag{4}$$

We now use the first term on the right-hand side of (3) to devise a recurrence. We have

$$\left|\frac{C_1(C_{n,1}+C_{n,2})}{C_1+C_{n,1}+C_{n,2}} - \frac{C_1(C_{n,1}'+C_{n,2})}{C_1+C_{n,1}'+C_{n,2}}\right| = \frac{C_1^2|C_{n,1}-C_{n,1}'|}{(C_1+C_{n,1}+C_{n,2})(C_1+C_{n,1}'+C_{n,2})} \le |C_{n,1}-C_{n,1}'|.$$

Accordingly,

$$\mathbf{E}\left[\left(\frac{C_1(C_{n,1}+C_{n,2})}{C_1+C_{n,1}+C_{n,2}}-\frac{C_1(C'_{n,1}+C_{n,2})}{C_1+C'_{n,1}+C_{n,2}}\right)^2\right] \leq \mathbf{E}[(C_{n,1}-C'_{n,1})^2] = 2\cdot \mathbf{Var}[C_{n,1}] = \frac{1}{2}\cdot \mathbf{Var}[C_{n-1}].$$

Therefore, letting $K_1 = \max\{K_0, \mathbf{Var}[C_1]\}$ and recalling (4) we have the following recurrence relation:

$$\mathbf{Var}[C_n] \leq \frac{K_1}{(n-1)^4} + \frac{1}{2}\cdot \mathbf{Var}[C_{n-1}].$$

Expanding the recurrence yields readily

$$\mathbf{Var}[C_n] \leq K_1 \cdot \sum_{i=1}^{n-1} \frac{2^{1-i}}{(n-i)^4} + \frac{1}{2^{n-1}} \mathbf{Var}[C_1] \leq K \cdot \sum_{i=1}^{n-1} \frac{2^{1-i}}{(n-i)^4} + \frac{K}{2^{n-1}}.$$

Since

$$\sum_{i=1}^{n-1} \frac{2^{1-i}}{(n-i)^4} + \frac{1}{2^{n-1}} \leq \sum_{i=1}^{\lfloor n/2\rfloor} \frac{2^{1-i}}{(n-i)^4} + \sum_{i=\lfloor n/2\rfloor+1}^{n-1} \frac{2^{1-i}}{(n-i)^4} + \frac{1}{2^{n-1}} \leq \frac{2^5}{n^4}\sum_{i\geq 1} 2^{-i} + 2^{1-\lfloor n/2\rfloor}\sum_{i\geq 1} 2^{-i} \leq \frac{2^{10}}{n^4},$$

the claimed bound on the variance holds as long as $K \geq K_1$. Finally, for all n, by Lemma 1 we have

$$\mathbf{Var}[R_n] \leq \mathbf{E}\left[\left(R_n - \frac{1}{\mathbf{E}[C_n]}\right)^2\right] = \mathbf{E}\left[\left(\frac{\mathbf{E}[C_n]-C_n}{C_n\mathbf{E}[C_n]}\right)^2\right] \leq b^4 n^4\, \mathbf{Var}[C_n], \tag{5}$$

so letting $K = \max\{b^4, 1\}\cdot K_1$, the proof is complete. □

We remark that this theorem is tight up to a constant factor unless X is deterministic (in which case $\mathbf{Var}[C_n] = 0$). Indeed, by considering equation (2), since $C_{n,1}$ and $C_{n,2}$ are both of order n^{-1}, we see that fluctuations of constant size in the value of C_1 change C_n by order n^{-2}. Such fluctuations occur with positive probability, so we must have that $\mathbf{Var}[C_n] \geq \varepsilon n^{-4}$ for some $\varepsilon > 0$ depending on X.

By Chebyshev's inequality, (5) also implies tail bounds on the probability that the resistance R_n deviates from $1/\mathbf{E}C_n$.

COROLLARY 4. *There exists a constant K, such that for all $t > 0$ and $n \geq 1$, we have*

$$\left\{\left|R_n - \frac{1}{\mathbf{E}C_n}\right| > t\right\} \leq \frac{K}{t^2}.$$

It follows that $\mathbf{E}|R_n - 1/\mathbf{E}C_n| \leq 1 + K$.

PROOF. The first claim is immediate using (5) and Chebyshev's inequality; to see the second claim, observe that

$$\mathbf{E}|R_n - 1/\mathbf{E}C_n| \leq 1 + \int_1^\infty \{|R_n - 1/\mathbf{E}C_n| \geq x\}\,dx \leq 1 + K\int_1^\infty \frac{dx}{x^2} = 1 + K.$$

□

3. The expected resistance and conductance. In this section we give precise locations for the expected values $\mathbf{E}C_n$ and $\mathbf{E}R_n$, respectively. Let $\sigma^2 = \mathbf{Var}[X]$ and let $\mu = \mathbf{E}X$.

THEOREM 5. *There exist constants M_1 and M_2 depending only on a and b such that for all integers $n \geq 2$,*

$$\left|\mathbf{E}R_n - \mu n + \frac{\sigma^2}{\mu}\ln n\right| \leq M_1 \quad \textit{and} \quad \left|\mathbf{E}C_n - \frac{1}{\mu n} - \frac{\sigma^2 \ln n}{\mu^3 n^2}\right| \leq \frac{M_2}{n^2}.$$

We remark that since $\mathbf{Var}[R_n]$ is certainly bounded from below by a positive constant (unless X is deterministic, in which case we know R_n precisely), we have determined the value of $\mathbf{E}R_n$ up to the order of its standard deviation. Furthermore, since $\mathbf{Var}[C_n]$ is of order n^{-4}, we have likewise determined $\mathbf{E}C_n$ up to the order of its standard deviation.

The techniques we use to handle the recurrence relation have been used by de Bruijn [8] to analyze slowly converging sequences and by Flajolet and Odlyzko [12] in the context of heights of simple trees.

PROOF OF THEOREM 5. We focus on $\mathbf{E}C_n$. By Corollary 4, bounds on $\mathbf{E}C_n$ immediately yield bounds on $\mathbf{E}R_n$. As in the proof of Theorem 2, we decompose T_{n+1} into three independent conductors C_1, $C_{n+1,1}$ and $C_{n+1,2}$ (see Figure 2). Let C_n and C'_n be independent copies of the conductance between the root and level n. Since

$$C_{n+1} = \frac{C_1 \cdot (C_{n+1,1} + C_{n+1,2})}{C_1 + C_{n+1,1} + C_{n+1,2}},$$

$C_{n+1,1}$ and $C_{n+1,2}$ are both distributed as $C_n/2$, and C_1 is distributed as $1/X$, we have, in distribution,

$$C_{n+1} = \frac{C_n + C'_n}{2} \cdot \frac{1}{1 + X((C_n + C'_n)/2)}, \tag{6}$$

where X is independent of all the other random variables appearing in (6). The second factor in (6) can be rewritten as

$$
\begin{aligned}
\frac{1}{1+X((C_n+C_n')/2)} &= 1 - X\left(\frac{C_n+C_n'}{2}\right) + X^2\cdot\left(\frac{C_n+C_n'}{2}\right)^2 \\
&\quad - X^3\cdot\left(\frac{C_n+C_n'}{2}\right)^3\cdot\frac{1}{1+X((C_n+C_n')/2)}.
\end{aligned}
\tag{7}
$$

Using the equality (7) to replace the term $1/(1+X(C_n+C_n')/2)$ in (6) and taking expectations, we obtain

$$
\begin{aligned}
\mathbf{E}C_{n+1} = \mathbf{E}C_n &- \frac{\mathbf{E}X}{2}\cdot(\mathbf{E}[C_n^2]+[\mathbf{E}C_n]^2) \\
&+ \frac{\mathbf{E}[X^2]}{4}\cdot(\mathbf{E}[C_n^3]+3\mathbf{E}[C_n^2]\mathbf{E}C_n) \\
&- \mathbf{E}\left[\frac{X^3(C_n+C_n')^4}{16(1+X((C_n+C_n')/2))}\right],
\end{aligned}
\tag{8}
$$

where we have used the equalities $\mathbf{E}[(C_n+C_n')^2] = 2(\mathbf{E}[C_n^2]+[\mathbf{E}C_n]^2)$ and $\mathbf{E}[(C_n+C_n')^3] = 2(\mathbf{E}[C_n^3]+3\mathbf{E}[C_n^2]\mathbf{E}C_n)$. By Lemma 1, we have deterministically

$$
\frac{a^3}{b^4n^4}\cdot\frac{1}{1+b/(an)} \le \frac{X^3(C_n+C_n')^4}{16(1+X((C_n+C_n')/2))} \le \frac{b^3}{a^4n^4},
$$

so (8) yields

$$
\begin{aligned}
\mathbf{E}[C_{n+1}] = \mathbf{E}[C_n] &- \frac{\mathbf{E}X}{2}\cdot(\mathbf{E}[C_n^2]+[\mathbf{E}C_n]^2) \\
&+ \frac{\mathbf{E}[X^2]}{4}\cdot(\mathbf{E}[C_n^3]+3\mathbf{E}[C_n^2]\mathbf{E}C_n) + O(n^{-4}),
\end{aligned}
\tag{9}
$$

where the order notation $O(\cdot)$ depends only on a and b. We observe that, by Theorem 2,

$$
\mathbf{E}[C_n^2]+[\mathbf{E}C_n]^2 = \mathbf{Var}[C_n]+2[\mathbf{E}C_n]^2 = 2[\mathbf{E}C_n]^2 + O(n^{-4}).
\tag{10}
$$

Furthermore, since $\mathbf{E}[(C_n-\mathbf{E}C_n)^3] = O(n^{-1})\cdot\mathbf{Var}[C_n] = O(n^{-5})$ by Theorem 2, we have

$$
\begin{aligned}
\mathbf{E}[C_n^3] &= \mathbf{E}[(C_n-\mathbf{E}C_n)^3] + 3\mathbf{E}[C_n^2]\mathbf{E}C_n - 3[\mathbf{E}C_n]^3 + [\mathbf{E}C_n]^3 \\
&= O(n^{-5}) + 3(\mathbf{Var}[C_n]+[\mathbf{E}C_n]^2)\mathbf{E}C_n - 2[\mathbf{E}C_n]^3 \\
&= 3\,\mathbf{Var}[C_n]\mathbf{E}C_n + [\mathbf{E}C_n]^3 + O(n^{-5}) \\
&= \mathbf{E}[C_n]^3 + O(n^{-5}),
\end{aligned}
$$

so

$$\mathbf{E}[C_n^3] + 3\mathbf{E}[C_n^2]\mathbf{E}C_n = 4\mathbf{E}[C_n]^3 + 6\,\mathbf{Var}[C_n]\mathbf{E}C_n + O(n^{-5})$$
$$= 4\mathbf{E}[C_n]^3 + O(n^{-5}).$$

Combining (9), (10) and (11), we obtain

$$\mathbf{E}C_{n+1} = \mathbf{E}C_n - \mathbf{E}X[\mathbf{E}C_n]^2 + \mathbf{E}X^2[\mathbf{E}C_n]^3 + O(n^{-4}).$$

Dividing through by $\mathbf{E}C_{n+1}\mathbf{E}C_n$ and letting $x_n = 1/\mathbf{E}C_n$ gives

(11) $$x_n = x_{n+1} - \mathbf{E}X \cdot \frac{x_{n+1}}{x_n} + \mathbf{E}[X^2] \cdot \frac{x_{n+1}}{x_n^2} + O(n^{-2}).$$

We let $\delta_n = x_{n+1}/x_n - 1$ and let $\varepsilon_n = x_{n+1}/x_n^2$, and remark that δ_n and ε_n are both $O(n^{-1})$. Summing (11) gives

(12) $$x_{n+1} = n\mathbf{E}X + \mathbf{E}X \cdot \sum_{i=1}^{n} \delta_i - \mathbf{E}[X^2] \sum_{i=1}^{n} \varepsilon_i + O(1).$$

Since both δ_i and ε_i are $O(i^{-1})$, (12) immediately yields the bound

(13) $$x_{n+1} = n\mathbf{E}X + O(\ln n) = (n+1)\mathbf{E}X + O(\ln n),$$

a bound we will bootstrap to prove the theorem. From (11) we have

$$\frac{x_{n+1}}{x_n} = 1 + \frac{\mathbf{E}X + \mathbf{E}X \cdot \delta_n - \mathbf{E}[X^2] \cdot \varepsilon_n}{x_n} + O(n^{-3}),$$

so since x_{n+1}/x_n also equals $1 + \delta_n$, solving for δ_n we obtain

(14) $$\delta_n = \frac{\mathbf{E}X - \mathbf{E}[X^2]\varepsilon_n}{x_n - \mathbf{E}X} + O(n^{-3}) = \frac{\mathbf{E}X}{x_n - \mathbf{E}X} + O(n^{-2})$$

as long as n is large enough to ensure that $x_n - \mathbf{E}X$ does not happen to be zero (say $n \ge n_0$ for some fixed n_0 depending only on a and b). Similarly,

(15) $$\varepsilon_n = \frac{1}{x_n} \cdot \frac{x_{n+1}}{x_n} = \frac{1}{x_n} + \frac{\delta_n}{x_n} = \frac{1}{x_n} + O(n^{-2})$$

for all $n \ge 1$. Combining (12), (14) and (15) gives the identity

$$x_{n+1} - n\mathbf{E}X = [\mathbf{E}X]^2 \sum_{i=n_0}^{n} \frac{1}{x_i - \mathbf{E}X} - \mathbf{E}[X^2] \sum_{i=n_0}^{n} \frac{1}{x_i} + O(1)$$

(16) $$= [\mathbf{E}X]^3 \sum_{i=n_0}^{n} \frac{1}{x_i(x_i - \mathbf{E}X)} - \mathbf{Var}[X] \sum_{i=n_0}^{n} \frac{1}{x_i} + O(1)$$

$$= -\mathbf{Var}[X] \sum_{i=n_0}^{n} \frac{1}{x_i} + O(1).$$

Since $x_i = i\mathbf{E}X + O(\ln i)$ by (13), we have

$$\sum_{i=n_0}^{n} \frac{1}{x_i} = \sum_{i=n_0}^{n} \frac{1}{i\mathbf{E}X + O(\ln i)} = \sum_{i=n_0}^{n} \left(\frac{1}{i\mathbf{E}X} + \frac{O(\ln i)}{(i\mathbf{E}X)^2}\right) = \frac{\ln n}{\mathbf{E}X} + O(1),$$

so (16) yields

$$x_{n+1} - n\mathbf{E}X = \frac{\mathbf{Var}[X]}{\mathbf{E}X} \ln n + O(1).$$

The first assertion of the theorem follows immediately, and second assertion of the theorem follows since $x_{n+1} = \mathbf{E}R_{n+1} + O(1)$ by Corollary 4. $\square$

4. Sub-Gaussian tails bounds for the resistance. In this section we show that the resistance R_n does not only have a bounded variance but all its moments are also bounded and satisfies a sub-Gaussian tail inequality. In order to show this, we use a strengthening of the Efron–Stein inequality developed by Boucheron et al. [7], together with Thomson's formula. This flow-based formulation of the resistance was used by Benjamini and Rossignol [5] to show submean variance bounds for the random resistance in $\mathbb{Z}^2$. Given a graph G, let $E(G)$ be the set of edges of G. Recall that if F denotes the set of unit flows from the root r to depth n in T_n, then

$$R_n = \inf_{\Theta\in F}\left\{\sum_{e\in E(T_n)} r_e \Theta(e)^2\right\}. \tag{17}$$

Furthermore, there is a unique unit flow Θ^* which attains the above infimum. As observed by Benjamini and Rossignol [5], it is a straightforward consequence of the Efron–Stein inequality that

$$\mathbf{Var}[R_n] \le \frac{(b-a)^2}{2} \sum_{e\in E(T_n)} \mathbf{E}[\Theta^*(e)^4]. \tag{18}$$

We now describe the result we use from [7] and how it can be combined with Thomson's formula to obtain a sub-Gaussian tail bound for R_n.

Suppose we are given independent random variables $\mathbf{U} = (U_1, \ldots, U_m)$ and a real-valued function $Z = f(U_1, \ldots, U_m)$. For integers $i = 1, \ldots, m$, let U_i' be an independent copy of u, and let $Z_i' = f(U_1, \ldots, U_{i-1}, U_i', U_{i+1}, \ldots, U_m)$. Let

$$V^+ = \sum_{i=1}^{m} (Z - Z_i')_+^2 \quad \text{and let} \quad V^- = \sum_{i=1}^{m} (Z - Z_i')_-^2,$$

where $(\cdot)_+$ and $(\cdot)_-$ denote the positive and negative parts, respectively. Boucheron et al. [7] prove that if there exists a constant C such that $V_+ \le C$ almost surely, then

$$\{Z > \mathbf{E}Z + t\} \le e^{-t^2/4C},$$

and if $V_- \le C$ almost surely, then

$$\{Z < \mathbf{E}Z - t\} \le e^{-t^2/4C}.$$

We shall apply these bounds with $\mathbf{U} = (X_e)_{e\in E(T_n)}$ and $Z = R_n$. From now on, X'_e denotes an independent copy of X_e, and $R_n^{(e)'}$ denotes the resistance of T_n when X_e is replaced by X'_e while all other resistances are kept unchanged. If Θ^* denotes the unit flow attaining the infimum in the expression of R_n by Thomson's formula, and $\Theta^{*,e}$ is the minimizing unit flow when X_e is replaced by X'_e, then by Thomson's formula and the deterministic bound $|X_e - X'_e| \le 2^{d(e)}(b-a)$,

$$\begin{aligned}(R_n - R_n^{(e)})_+ &\le (X_e - X'_e)_+ \cdot \Theta^{*,e}(e)^2,\\ &\le (b-a)2^{d(e)}\Theta^{*,e}(e)^2\end{aligned}$$

and similarly,

$$(R_n - R_n^{(e)})_- \le (b-a)2^{d(e)}\Theta^*(e)^2.$$

Thus,

$$V_+ \le (b-a)^2 \sum_{e\in E(T_n)} 2^{2d(e)}\Theta^{*,e}(e)^4 \quad \text{and} \quad V_- \le (b-a)^2 \sum_{e\in E(T_n)} 2^{2d(e)}\Theta^*(e)^4.$$

The key argument is the following deterministic bound.

LEMMA 6. *The optimal unit flow of any edge $e \in E(T_n)$ satisfies, deterministically,*

$$\Theta^*(e) \le \frac{bn}{a(n-d(e)+1)2^{d(e)-1}}.$$

PROOF. Let v denote the endpoint of $e \in E(T_n)$ closer to the root r of T_n. If $U(v)$ is the voltage at vertex v when the unit current Θ^* flows from the root r to the leaves and the leaves have voltage 0, then by Ohm's law,

$$R_{n,e}\Theta^*(e) = U(v) \le U(r) = R_n \le bn,$$

where $R_{n,e}$ is the effective resistance of the subtree rooted at v. On the other hand, by Rayleigh's monotonicity law (see [9], page 53) and since all X_e's are at least a, we have

$$R_{n,e} \ge a(n-d(e)+1)2^{d(e)-1}.$$

Comparing the two bounds, we obtain the bound of the lemma. □

Since the upper bound of Lemma 6 does not depend on the random values of the resistances X_e, the same inequality holds for $\Theta^{*,e}(e)$ as well. Thus, we have

$$V_+ \le \frac{2^4 b^4 (b-a)^2}{a^4} \sum_{e \in E(T_n)} \left(\frac{n}{n - d(e) + 1} \right)^4 2^{-2d(e)}$$
$$= \frac{2^4 b^4 (b-a)^2}{a^4} \sum_{i=1}^{n} \left(\frac{n}{n+1-i} \right)^4 2^{-i}$$

and the same upper bound applies to V_- as well. Since the sum on the right-hand side is bounded, there exists a constant $C = C(a, b)$ such that both $V_+ \le C$ and $V_- \le C$. As a consequence, we have the following sub-Gaussian concentration inequality.

THEOREM 7. *There exists a constant C depending on a and b only such that for every $t > 0$,*

$$\{|R_n - \mathbf{E}R_n| > t\} \le 2e^{-t^2/4C}.$$

5. Concluding remarks. We conclude by discussing some possible extensions to branching random networks. When discussing the possible analogues of our results for branching processes, the following formulation of branching processes is useful. We start from a single "root edge" uv and let v be the root of a supercritical branching process with branching distribution B that satisfies $\{B = 0\} = 0$. We use $\mathcal{T}$ to refer to this edge-rooted branching process. We say that a node $w \ne u$ has depth i if there are i edges on the path from v to w (v has depth 0). For $i \ge 0$, we let Z_i be the number of nodes of $\mathcal{T}$ at depth i—so in particular $Z_0 = 1$.

A branching random network is simply an edge-rooted branching process $\mathcal{T}$ as above. To each edge e at depth d, we assign a random resistance $r_e = [\mathbf{E}B]^d X_e$, where the X_e are independent, identically distributed positive random variables taking values in $[a, b]$ as in the binary case. As before, we let C_n (resp., R_n) be the effective conductance (resp., effective resistance) from the root to depth n. As in the binary case, the above scaling most naturally corresponds to the critical case of a random walk in a random environment on branching processes with push-back. In particular, if B is deterministically 2 then we recover the model of the previous sections.

It is easily seen that C_n is not concentrated. Let B_1 be the number of children of the root—then as in the case of binary branching, we may first decompose T_n into independent conductors C_1 and $C_{n,1}, \dots, C_{n,B_1}$, so that

$$C_n = \frac{C_1(C_{n,1} + \cdots + C_{n,B_1})}{C_1 + C_{n,1} + \cdots + C_{n,B_1}}.$$

If C_n is concentrated, then $C_{n,1} + \cdots + C_{n,B_1}$ is well approximated by $B_1 \cdot \mathbf{E}C_{n-1}/\mathbf{E}B_1$, so C_n is close to

$$\frac{C_1 \cdot B_1 \cdot \mathbf{E}C_{n-1}/\mathbf{E}B_1}{C_1 + B_1 \cdot \mathbf{E}C_{n-1}/\mathbf{E}B_1} = \frac{\mathbf{E}C_{n-1}}{\mathbf{E}B_1} \cdot \left(\frac{1}{B_1} + \frac{\mathbf{E}C_{n-1}}{C_1 \cdot \mathbf{E}B_1}\right)^{-1}.$$

But the latter expression is *not* concentrated—a constant change in B_1 changes this expression by a constant factor. This should not be surprising: if the root has many offspring, the conductance is likely to be much (a constant factor) higher than if the root has a single child. It seems likely that at least the first-order behavior of C_n and R_n is governed by $W = \lim_{n\to\infty} Z_n/[\mathbf{E}B]^n$. If the resistance random variable X is constant (say $X = 1$) then this is easily seen: the series-parallel laws give $R_n = \sum_{i=0}^{n} [\mathbf{E}B]^i/Z_i$, and $\mathbf{E}B^i/Z_i$ tends to $1/W$ $\mathcal{T}$-a.s., so R_n/n tends to $1/W$ $\mathcal{T}$-a.s. If we additionally assume that B has finite variance then this convergence is also in expectation [2], Theorem I.6.2, and it immediately follows that $C_n/\mathbf{E}C_n$ tends to $W/\mathbf{E}W$ a.s. and in expectation. In particular, this implies that $\lim_{n\to\infty} C_n/\mathbf{E}C_n$ has absolutely continuous distribution (as long as B is not constant; see [2], Theorem I.10.4), which is the "scaled analogue" of Question 4.1 from Lyons, Pemantle and Peres [16] mentioned in the Introduction.

We would expect that even when X is not constant, for *any* λ with $1 \le \lambda \le \mathbf{E}B$, in the network where the resistances of edges at depth i are scaled by λ^i, $\lim_{n\to\infty} C_n/\mathbf{E}C_n$ has an absolutely continuous distribution. In the special case that $\lambda = \mathbf{E}B$, we would venture that R_n/n tends to $\mathbf{E}X/W$, $\mathcal{T}$-a.s. and in expectation. However, we were unable to extend the arguments used to prove Theorems 2 and 5 to the branching process. In particular, once we condition on $\mathcal{T}$, our techniques for manipulating (3) and (6) in order to devise recurrences fail, most notably because in this setting the identical distribution of subtrees at equal depth is lost. It seems plausible that as in the case of binary branching $R_n - W \cdot (n\mathbf{E}X)$ is $O(\ln n)$, again $\mathcal{T}$-a.s. and in expectation, but the coefficient of $\ln n$ also seems likely to depend on $\mathcal{T}$ and the precise nature of this dependence is unclear to us.

Finally, observe that Barndorff-Nielsen [3] and Barndorff-Nielsen and Koudou [4] have noticed an interesting link between inverse Gaussian (or reciprocal inverse Gaussian) random variables and effective resistances of random networks: if resistances of the edges are distributed like i.i.d. inverse Gaussian random variables, then the effective resistance of the entire tree is distributed like a reciprocal inverse Gaussian.

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