

ON THE ADJUSTMENT COEFFICIENT, DRAWDOWNS AND LUNDBERG-TYPE BOUNDS FOR RANDOM WALK

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Consider a random walk whose (light-tailed) increments have positive mean. Lower and upper bounds are provided for the expected maximal value of the random walk until it experiences a given drawdown d . These bounds, related to the Calmar ratio in finance, are of the form $(\exp\{\alpha d\} - 1)/\alpha$ and $(K \exp\{\alpha d\} - 1)/\alpha$ for some $K > 1$, in terms of the adjustment coefficient α ($E[\exp\{-\alpha X\}] = 1$) of the insurance risk literature. Its inverse $\frac{1}{\alpha}$ has been recently derived by Aumann and Serrano as an index of riskiness of the random variable X .

This article also complements the Lundberg exponential stochastic upper bound and the Crámer–Lundberg approximation for the expected minimum of the random walk, with an exponential stochastic lower bound. The tail probability bounds are of the form $C \exp\{-\alpha x\}$ and $\exp\{-\alpha x\}$, respectively, for some $\frac{1}{K} < C < 1$.

Our treatment of the problem involves Skorokhod embeddings of random walks in martingales, especially via the Azéma–Yor and Dubins stopping times, adapted from standard Brownian motion to exponential martingales.

1. Introduction.

Drawdowns of Brownian motion with positive drift. Let $\{W(t) \mid t \geq 0, W(0) = 0\}$ be standard Brownian motion (SBM) and let $\{B(t) \mid B(t) = \mu t + \sigma W(t), t \geq 0\}$ be Brownian motion (BM) with drift $\mu > 0$ and diffusion parameter $\sigma \in (0, \infty)$. For $d > 0$, define the stopping time

$$(1.1) \quad \tau_d^{\text{BM}} = \min \left\{ t \mid \max_{0 \leq s \leq t} B(s) \geq B(t) + d \right\}$$

to be the first time to achieve a *drawdown* of size d . That is, τ_d^{BM} is the first time that BM has gone down by d from its record high value so far. As motivated by Taylor [21], an investor that owns a share whose value at time t is $V_t = V_0 \exp(B(t))$ may consider selling it at time τ_d^{BM} (for some $d > 0$) because it has lost for the first time some fixed fraction $1 - \exp(-d)$ of its previously held highest value $V_0 \exp(M_d)$ [where $M_d = M_d^{\text{BM}} = \max_{0 \leq s \leq \tau_d^{\text{BM}}} B(s) = B(\tau_d^{\text{BM}}) + d$], a possible indication of change of drift.

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As pointed out in Meilijson [18], drawdowns are *gaps* for Dubins and Schwarz [12], *extents* for Goldhirsch and Noskovicz [13] and *downfalls* for Douady, Shiryaev and Yor [9]. Taylor [21] (see also [18]) presents a closed-form formula for the joint moment generating function of τ_d^{BM} and $B(\tau_d^{\text{BM}})$, from which it follows that M_d^{BM} is exponentially distributed, with expectation

$$(1.2) \quad E[M_d^{\text{BM}}] = \frac{\sigma^2}{2\mu} \left(\exp\left\{ \frac{2\mu}{\sigma^2} d \right\} - 1 \right).$$

A direct proof of the mean (1.2) and (exponential) distribution of M_d^{BM} is presented in Section 4.

Maximum of Brownian motion with negative drift. The maximum $\max(BM) = \inf_{t>0}\{B(t)\}$ is well known to have the exponential distribution

$$(1.3) \quad P(\max(BM) > x) = 1 \wedge \exp\left\{ -\frac{2|\mu|}{\sigma^2} x \right\}.$$

This article contributes to the generalization of (1.2) and (1.3) from BM to random walks (RW). There is a rather vast literature on the maximum of RW with negative drift. Kingman [16] showed that $P(\max(RW) > x) \approx 1 \wedge \exp\{-\frac{2|\mu|}{\sigma^2}x\}$ for small μ , Siegmund [19] studied first-order corrections to this approximation via renewal-type overflow distributions and Chang and Peres [8] developed asymptotic expansions of $P(\max(RW) > x)$ for the Gaussian case. Blanchet and Glynn [6] improved on these approximations. In the insurance risk literature, exponential bounds and approximations of $P(\max(RW) > x)$ are referred to as Lundberg's inequality or Crámer–Lundberg approximations (see Asmussen's comprehensive treatise [1]).

This paper is methodologically different from the above; instead of relying on change of measure and renewal theory, our setup involves exponential martingales and Skorokhod embeddings, in a way reminiscent of Wald's [22] method for deriving the OC characteristic of the Sequential Probability Ratio Test. As part of the change, we will give up on trying to save the inaccurate role of $\frac{2|\mu|}{\sigma^2}$ as the exponential rate in the questions under study, in favor of the so-called *adjustment coefficient* of the insurance risk literature, provided by the α solving $E[\exp\{-\alpha X\}] = 1$. However, the rate $\frac{2|\mu|}{\sigma^2}$ will stay around: the RW will be coupled with a BM for which $\frac{2|\mu|}{\sigma^2}$ is α .

More explicitly, using Skorokhod ([20] and also [4, 7, 10, 17]) embeddings, mean-zero RW can be viewed as optional sampling of SBM. This idea will be mimicked here to embed the exponential martingale $\exp\{-\alpha S_n\}$ into the martingale $\exp\{-\alpha B(t)\}$. This method could be useful in obtaining other approximate extensions of pricing under log-normal models to more general distributions.

Aumann and Serrano [3] asked a scalar index of *riskiness* $Q(X)$ of the random variable (r.v.) X to satisfy a *homogeneity* axiom $Q(tX) = tQ(X)$ and a *duality*

axiom that models the increased preference of a more risk-averse individual for constant wealth w over random wealth $w + X$. The unique solution (up to a multiplicative constant) is the inverse $\frac{1}{\alpha}$ of the adjustment coefficient. The role played by α in our subject matter is clearly consistent with riskiness—a large α corresponds to low risk, as it (i) protects against heavy initial losses before eventual divergence of the RW to ∞ , and (ii) makes the RW reach high yield before experiencing sizable drawdowns.

The *Calmar ratio* (see Atiya and Magdon-Ismail [2] and the implementation of their work in the Matlab financial toolbox) of a financial asset with positive drift is a measure of the likely drawdown in the logarithm of its price in a given interval of time, such as a year. Since height (and time) are exponential in the drawdown, the Calmar ratio is heavily influenced by the length of this time interval. Besides, typical drawdown in a given time span is harder to analyze than our subject matter, typical height (or time) to achieve a given drawdown. We propose the use of the adjustment coefficient or its inverse as a Calmar-type measure of the risk of a financial asset, and provide simple approximate formulas to quantify its effects. The more commonly used *Sharpe index*, or ratio of net drift (drift minus market interest rate) to volatility (standard deviation), lets volatility penalize the asset even when it favors gains. In contrast, drawdown-based indices measure risk in a more reasonable asymmetric sense.

2. Results. From now on, we only consider BM and RW with positive drift and thus unify the presentation of the two problems, by switching from the commonly studied maximum of BM and RW with negative drift to the equivalent treatment of the minimum of BM and RW with positive drift.

PROPERTY PMLLT. The distribution of the r.v. X satisfies Property PMLLT (positive mean, light left tail) if $E[X] \in (0, \infty)$, $P(X < 0) > 0$ and there is $\alpha > 0$ such that $E[\exp\{-\alpha X\}] = 1$.

Since (if finite) the moment generating function $\Psi(t) = E[\exp\{tX\}]$ is strictly convex with $\Psi'(0) = E[X] > 0$ and $\Psi(t) \rightarrow \infty$ as $|t| \rightarrow \infty$, such α exists and is unique as long as the moment generating function is finite wherever relevant. This assumption is satisfied, for example, for Gaussian r.v.'s and for r.v.'s bounded from below. If $X \sim N(\mu, \sigma^2)$, then α is indeed $\frac{2\mu}{\sigma^2}$ [see (1.2)].

Besides α , we need other characteristics of the distribution F of X :

$$\begin{aligned}
 d^+ &= \frac{1}{\alpha} \sup_{0 < x < es_F} -\log(E[e^{-\alpha(X-x)} | X \geq x]), \\
 d^- &= \frac{1}{\alpha} \sup_{ei_F < x < 0} \log(E[e^{\alpha(x-X)} | X < x]), \\
 d_0 &= d^+ + d^-,
 \end{aligned}
 \tag{2.1}$$

where $es_F = \sup\{y|F(y) < 1\}$ and $ei_F = \inf\{y|F(y) > 0\}$ are the *essential supremum* and *infimum* of F . By Jensen’s inequality, d^+ is bounded from above by the simpler and more natural $\sup_x E[X - x|X \geq x]$ and d^- is accordingly bounded from below. These constants are defined in terms of excesses of the r.v. X itself, unlike the Siegmund or Crámer–Lundberg approximations, built in terms of the renewal overflow distribution of the random walk with X -increments.

Let $X_i, i = 1, 2, \dots$ (generically, X) be i.i.d. PMLLT F -distributed random variables and let $S_0 = 0; S_n = \sum_{i=1}^n X_i$ be the corresponding random walk (RW). The definition of α makes $\exp\{-\alpha S_n\}$ a martingale with mean 1.

Drawdowns, maximal heights M_d^{RW} achieved prior to drawdowns and the corresponding stopping times τ_d^{RW} can be defined for random walk in much the same way they are defined for Brownian motion. Restating Lundberg’s inequality as the RHS of (2.3), the purpose of this paper is to prove the other three inequalities in (2.2) and (2.3).

THEOREM 1. *Let X satisfy Property PMLLT and let d^+, d^-, d_0 be as defined in (2.1). Then*

$$(2.2) \quad \frac{e^{\alpha d} - 1}{\alpha} \leq E[M_d^{RW}] \leq \frac{e^{\alpha(d+d_0)} - 1}{\alpha}$$

and, for $x > 0$,

$$(2.3) \quad e^{-\alpha(x+d^-)} \leq P(-\min(RW) > x) \leq e^{-\alpha x}.$$

Furthermore, the upper bound in (2.2) is a stochastic inequality: M_d^{RW} is stochastically smaller than the exponentially distributed random variable $M_{d+d_0}^{BM}$.

Theorem 1 will be proved in Section 6 after developing some background material in Section 5.

We thus have lower and upper bounds for $E[M_d^{RW}]$ whose ratio stays bounded as d increases, provided d_0 is finite. These bounds clearly show that drawdowns are logarithmic in the highest value achieved so far, and precisely identify the exponential rate α at which the latter grows as a function of the former. We do not provide a stochastic lower bound for M_d^{RW} . In contrast, (2.3) provides stochastic upper and lower bounds on the minimum of RW.

The upper bound in (2.2) can be improved by letting d_0 depend on d and be defined as d_0 in (2.1) but restricting the maximization to $x \in (0, \min(es_F, d))$ and $x \in (\max(ei_F, -d), 0)$. This improved upper bound is finite for every d .

3. A few examples.

Example 1: The Gaussian case. Let ϕ and Φ stand respectively for the standard normal density and cumulative distribution function.

LEMMA 1. *Let $X \sim N(\mu, \sigma^2)$. Then*

$$(3.1) \quad \alpha = \frac{2\mu}{\sigma^2}; \quad e^{\alpha d^+} = e^{\alpha d^-} = \frac{\Phi(\mu/\sigma)}{1 - \Phi(\mu/\sigma)}.$$

COROLLARY 1. *Let $X \sim N(\mu, \sigma^2)$. Then*

$$(3.2) \quad \frac{e^{2\mu/\sigma^2 d} - 1}{2\mu/\sigma^2} \leq E[M_d] \leq \frac{(\Phi(\mu/\sigma)/(1 - \Phi(\mu/\sigma)))^2 e^{2\mu/\sigma^2 d} - 1}{2\mu/\sigma^2},$$

$$(3.3) \quad \frac{1 - \Phi(\mu/\sigma)}{\Phi(\mu/\sigma)} e^{-2\mu/\sigma^2 x} \leq P(-\min(RW) > x) \leq e^{-2\mu/\sigma^2 x}.$$

REMARK 1. If we view the normal random walk as sampling Brownian motion with drift μ and diffusion coefficient σ at regular intervals, α is independent of the grid length δ but the three d 's are not, predictably vanishing with the grid length: by (3.1), $d^+ = d^- = \frac{1}{\alpha} \log \frac{\Phi(\mu/\sigma\sqrt{\delta})}{1 - \Phi(\mu/\sigma\sqrt{\delta})} \rightarrow 0$ as $\delta \downarrow 0$.

PROOF OF LEMMA 1. The LHS of (3.1) is well known and easy to obtain from the formula $\exp\{\mu + \sigma^2 t^2/2\}$ of the moment generating function of the normal distribution. As for the RHS, it requires evaluating via

$$(3.4) \quad \begin{aligned} E[e^{-\beta Z} | Z > z] &= e^{1/2\beta^2} \frac{1}{\sqrt{2\pi}} \frac{\int_z^\infty \exp\{-1/2(t + \beta)^2\} dt}{1 - \Phi(z)} \\ &= e^{1/2\beta^2} \frac{1 - \Phi(z + \beta)}{1 - \Phi(z)} \end{aligned}$$

the expressions

$$(3.5) \quad \begin{aligned} E[e^{-\alpha X} | X > x] &= E[e^{-\alpha(\mu + \sigma Z)} | \mu + \sigma Z > x] \\ &= e^{-2\mu^2/\sigma^2} E\left[e^{-2\mu/\sigma Z} \Big| Z > \frac{x - \mu}{\sigma} \right] \frac{1 - \Phi((x + \mu)/\sigma)}{1 - \Phi((x - \mu)/\sigma)}, \end{aligned}$$

$$(3.6) \quad E[e^{-\alpha X} | X < x] = \frac{\Phi((x + \mu)/\sigma)}{\Phi((x - \mu)/\sigma)},$$

from which the RHS of (3.1) follows, at least in the sense of plugging $x = 0$. To see that $x = 0$ is indeed the correct choice for each side, observe that for the normal distribution the residual distributions $\mathcal{L}(X - x | X > x)$ are ordered by monotone likelihood ratio, thus also by stochastic inequality $[\frac{\phi(x_1+t)}{1-\Phi(x_1)} / \frac{\phi(x_2+t)}{1-\Phi(x_2)} = K(x_1, x_2) \exp\{(x_2 - x_1)t\}$ is a monotone function of t]. Hence, the expectations

of monotone functions (e.g., exponential) are ordered accordingly. This argument applies equally to the two tails. \square

Relevant material on monotone likelihood ratio and stochastic ordering can be found in Lehmann and Romano ([14], Section 3.4).

Example 2: The double exponential case. Let X have density $p\theta \exp\{-\theta x\}$ for $x > 0$ and $(1-p)\mu \exp\{\mu x\}$ for $x < 0$, with $p > p_0 = \frac{\theta}{\mu+\theta}$ to achieve $E[X] > 0$. Then

$$(3.7) \quad E[X] = \frac{p}{\theta} - \frac{1-p}{\mu}; \quad \alpha = p\mu - (1-p)\theta$$

with the corresponding bound ingredients

$$(3.8) \quad e^{\alpha d^+} = \frac{(\mu + \theta)p}{\theta}; \quad e^{\alpha d^-} = \frac{\mu}{(\mu + \theta)(1-p)}; \quad e^{\alpha d_0} = \frac{p\mu}{(1-p)\theta}.$$

There is no clear-cut inequality between α and the Gaussian-like version $\frac{2E[X]}{\sqrt{\text{Var}[X]}}$. The rate α exceeds $\frac{2E[X]}{\sqrt{\text{Var}[X]}}$ iff p is in the interval with endpoints p_0 and $[2\frac{\mu}{\theta} - 1]p_0$. Hence, if $\mu \leq \theta$ (heavier left tail), the rate α is below $\frac{2E[X]}{\sqrt{\text{Var}[X]}}$ for all feasible p . If $\mu > 2\theta$, the opposite inequality holds for all feasible p . In the complementary, intermediate case, the latter holds only for p between p_0 and $[2\frac{\mu}{\theta} - 1]p_0$.

Example 3: The shifted exponential case. Let X have exponential distribution with mean $\frac{1}{\theta}$ shifted down by $\Delta < \frac{1}{\theta}$ so as to allow negative values and still preserve positive mean. It is easier to express the inverse function to α :

$$(3.9) \quad \frac{1}{\alpha} \log\left(1 + \frac{\alpha}{\theta}\right) = \Delta$$

from which

$$(3.10) \quad d^+ = \Delta; \quad e^{\alpha d^+} = 1 + \frac{\alpha}{\theta}; \quad e^{\alpha d^-} = \frac{1 - e^{-(\theta+\alpha)\Delta}}{1 - e^{-\theta\Delta}};$$

$$e^{\alpha d_0} = 1 + \frac{\alpha/\theta}{1 - e^{-\theta\Delta}}.$$

The Gaussian-motivated rate $\frac{2E[X]}{\sqrt{\text{Var}[X]}}$ is $2\theta(1 - \theta\Delta)$, always smaller than α . That is, a random walk with shifted exponential increments gets to higher heights before a given drawdown than a normal one with the same mean and variance.

Example 4: A dichotomous case. Let $P(X = -1) = 1 - p$ and $P(X = 1) = p > \frac{1}{2}$. Then $\alpha = \log \frac{p}{1-p}$ and, obviously, $d^+ = d^- = 1$. As is well known from the Gambler's ruin problem, the probability of reaching +1 before (integer) $-d$

is $P^+ = \frac{1 - \exp\{-\alpha d\}}{1 - \exp\{-\alpha(d+1)\}}$. M_d^{RW} is nothing but the number of independent such attempts until a first “failure.” Hence, it is (-1) plus a geometric r.v., and its mean is

$$(3.11) \quad E(M_d^{\text{RW}}) = -1 + \frac{1}{1 - P^+} = \frac{p}{2p - 1}(e^{\alpha d} - 1).$$

For noninteger d , the ceiling of d should be substituted in (3.11). Even without doing so, the LHS of (2.2) is verified, because $\alpha = \log \frac{p}{1-p} > \frac{2p-1}{p}$. To ascertain the RHS, take $E(M_{d+1}^{\text{RW}})$ as worst-case ceiling and check that $\frac{p}{2p-1}(e^{\alpha(d+1)} - 1)$ is below the bound $(\exp\{\alpha(d+2)\} - 1)/\alpha$.

Just as in the shifted exponential case, the rate of growth α exceeds the rate $\frac{2E[X]}{\text{Var}[X]} = \frac{1}{2}(\frac{1}{1-p} - \frac{1}{p})$ that would have been obtained in the Gaussian case. However, we have the following example:

Example 5: A skew dichotomous case. Let $P(X = -1) = \frac{b}{1+b}$ and $P(X = b(1 + \varepsilon)) = \frac{1}{1+b}$. The mean is $E[X] = \frac{\varepsilon b}{1+b}$ so let us take $\varepsilon = 0.2$ to achieve positive mean and $b = 0.1$ to tilt the distribution toward bigger losses. Then $\alpha = 0.318$ but $\frac{2E[X]}{\text{Var}[X]} = 0.351$. This shows that even for dichotomous variables the inequality between the two can go both ways, like in Example 2 under lighter left tail.

In all the previous examples, the distribution F has nondecreasing failure rate and the “excess lifetime” over x looks shorter as x increases. That is why the d ’s are attained at $x = 0$ [see (2.1)]. This is not always the case: it is easy to produce a four-point distribution with one negative atom in which d^+ will be the distance between the two rightmost atoms.

Example 6: A power-law right tail. If F is light left tailed but behaves like power law at the right tail, then α is finite but d^+ is infinite because its maximand behaves like $\log x$. To wit,

$$(3.12) \quad E[e^{-\alpha(X-x)} | X > x] = \frac{\gamma}{x} \int_0^\infty \frac{e^{-\alpha t}}{(1 + t/x)^{\gamma+1}} dt \approx \frac{\gamma}{\alpha x}$$

so $-\frac{1}{\alpha} \log(E[\exp\{-\alpha(X-x)\} | X > x]) = \log(x) + o(1)$. Although much smaller than $E[X-x | X > x] = O(x)$ [see the sentence following (2.1)], it still goes to ∞ . However, the improved definition of d^+ sets it as $\log(d)$ up to a vanishing term.

This example illustrates that yield-to-drawdown, while at least as high as the Brownian lower bound, may in principle be superexponential.

4. Miscellaneous.

The record high value M_d^{BM} is exponentially distributed. This is so because as long as first hitting times of positive heights occur before achieving a drawdown of d , these times are renewal times: knowing that $M_d^{\text{BM}} > x$ is the same as knowing that B has not achieved a drawdown of d by the time it first reaches height x . But then it starts anew the quest for a drawdown.

A direct argument for (1.2). Since the mean-1 martingale $\exp\{-\alpha B\}$ stopped at τ_d^{BM} is uniformly bounded, it is also uniformly integrable. Hence,

$$(4.1) \quad 1 = E[e^{-\alpha B(\tau_d^{\text{BM}})}] = E[e^{-\alpha M_d^{\text{BM}}}]e^{\alpha d}.$$

Since M_d^{BM} is exponentially distributed, $E[\exp\{-\alpha M_d^{\text{BM}}\}] = \frac{1}{1+\alpha E[M_d^{\text{BM}}]}$.

5. Skorokhod embeddings in martingales. The problem as posed and solved by Skorokhod in [20] is the following: given a distribution F of a r.v. Y with mean zero and finite variance, find a stopping time τ in SBM W , with finite mean, for which $W(\tau)$ is distributed F . The Chacon–Walsh [7] family of solutions is easiest to describe: Express Y as the limit of a martingale $Y_n = E[Y|\mathcal{F}_n]$ with dichotomous transitions (i.e., the conditional distribution of Y_{n+1} given \mathcal{F}_n is a.s. two-valued), and then progressively embed this martingale in W by a sequence of first exit times from open intervals.

Dubins [10] was the first to build such a scheme, letting \mathcal{F}_1 decide whether $Y \geq E[Y]$ or $Y < E[Y]$ by a first exit time of W starting at $E[Y]$ from the open interval $(E[Y|Y < E[Y]], E[Y|Y \geq E[Y]])$. It then proceeds recursively. For example, if the first step ended at $E[Y|Y \geq E[Y]]$, then the second step ends when W , restarting at $E[Y|Y \geq E[Y]]$, first exits the open interval $(E[Y|E[Y] \leq Y < E[Y|Y \geq E[Y]]], E[Y|Y \geq E[Y|Y \geq E[Y]]])$.

One of the analytically most elegant solutions to Skorokhod’s problem is the Azéma–Yor stopping time T_{AY} (see Azéma and Yor [4] and Meilijson [17]), defined in terms of $H_F(x) = E[Y|Y \geq x] = \int_x^\infty y dF(y)/(1 - F(x-))$, the upper barycenter function of F , as

$$(5.1) \quad T_{\text{AY}} = \min \left\{ t \mid \max_{0 \leq s \leq t} W(s) \geq H_F(W(t)) \right\}.$$

Among all uniformly integrable càdlàg martingales with a given final or limiting distribution, SBM stopped at the Azéma–Yor stopping time to embed this distribution is extremal, in the sense that it stochastically maximizes the maximum of the martingale (see Dubins and Gilat [11] and Azéma and Yor [4]). That is, if T_{AY} embeds F , then $M_{T_{\text{AY}}}$ is stochastically bigger than the maximum of any such martingale.

The connection of the Azéma–Yor stopping time to the Chacon–Walsh family becomes apparent (see Meilijson [17]) if the r.v. Y has finite support $\{x_1 < \dots < x_k\}$. In this case, let \mathcal{F}_n be the σ -field generated by $\min(Y, x_{n+1})$, that is, let the atoms of Y be incorporated one at a time, in their natural order: the first stage decides whether $Y = x_1$ (by stopping there) or otherwise (by temporarily stopping at $E[Y|Y > x_1]$), etc. This is precisely the Azéma–Yor stopping rule: stop as soon as a value of Y is reached after having visited the conditional expectation of Y from this value and up.

Clearly, there is a mirror-image notion T_{AY-} to Azéma and Yor’s stopping time that stochastically minimizes the minimum of the martingale. Simply put, apply T_{AY} to embed the distribution of $-X$ in $-W$.

The stopping time T_{DAY} to be applied in the next section is a hybrid of the Dubins and Azéma and Yor stopping times. It starts as the Dubins stopping time by a first-exit time of SBM W from the interval $(E[Y|Y < E[Y]], E[Y|Y \geq E[Y]])$. If exit occurred at the top, it proceeds by embedding the law $\mathcal{L}(Y|Y \geq E[Y])$ by T_{AY} in the remainder SBM starting at $E[Y|Y \geq E[Y]]$. If, on the other hand, exit occurred at the bottom, it proceeds by embedding the law $\mathcal{L}(Y|Y < E[Y])$ by T_{AY-} in the remainder SBM starting at $E[Y|Y < E[Y]]$.

Once a distribution F is embeddable in SBM W , so is the random walk with increments distributed F . Plainly, embed X_1 at time τ_1 , then use the same rule to embed X_2 at time τ_2 in the SBM $W'(t) = W(\tau_1 + t) - W(\tau_1)$, etc. Skorokhod’s original idea was to infer the Central Limit Theorem for $\frac{S_n}{\sqrt{n}}$ from the Law of Large Numbers for $\frac{\sum_{i=1}^n \tau_i}{n}$. This idea was extended by Holewijn and Meilijson [15] from random walks to martingales with stationary ergodic increments, to obtain a simple proof of the Billingsley and Ibragimov [5] CLT.

6. Proof of Theorem 1. Assume the X_i to be i.i.d. PMLLT random variables. Just as the random walk S_n can be embedded in SBM, the exponential martingale $\exp\{-\alpha S_n\}$ can be embedded in the continuous-time continuous martingale $\exp\{-\alpha B(t)\}$, where the BM B has drift μ and diffusion coefficient σ such that $\frac{2\mu}{\sigma^2} = \alpha$. At the time the RW reaches drawdown at least d , BM has also gone down by at least d , but may have gone higher in the meantime. Thus, τ_d^{BM} is a.s. smaller than τ_d^{RW} . Now we may compute, under the obvious property $\Delta \geq d$ a.s. of $\Delta = M_d^{RW} - B(\tau_d^{RW})$,

$$(6.1) \quad \begin{aligned} \mu E[\tau_d^{BM}] &= E[B(\tau_d^{BM})] = E[M_d^{BM}] - d \\ &\leq \mu E[\tau_d^{RW}] = E[B(\tau_d^{RW})] = E[M_d^{RW}] - E[\Delta] \end{aligned}$$

so $E[M_d^{RW}] \geq E[M_d^{BM}]$. We have proved the LHS inequality in (2.2) by the method of coupling.

The RHS inequality in (2.2) is proved by using the Dubins–Azéma and Yor stopping time τ_{DAY} for the above embedding. If this embedding in $\exp\{-\alpha W\}$ ends up with W below (resp. above) some $x > 0$ (nonpositive), the underlying process could not have reached the exponential barycenter height above (below) the closest support point to the right (left) of x , because then the stopped value would have been from this support rightward (leftward). For the first increment of RW following the maximal (minimal) value the relevant x is 0, but for values embedded later the starting x is lower (higher). It should now be clear that the BM path cannot reach as far up as $M_d^{RW} + d^+$ nor as far down as $B(\tau_d^{RW}) - d^-$ before

RW achieves drawdown d . Hence, BM cannot reach drawdown $d + d_0$ before RW reaches drawdown d , or $\tau_{d+d_0}^{\text{BM}} \geq \tau_d^{\text{RW}}$ a.s. The RHS inequality in (2.2) follows.

This RHS inequality holds stochastically, since the cumulative maximum of BM exceeds the cumulative maximum of the embedded RW timewise. Thus, it holds a fortiori if the former is measured later than the latter. This argument fails for the LHS because then the latter is measured before the former.

The proof of (2.3) follows similar coupling lines: the two claimed stochastic inequalities are pointwise (a.s.) satisfied by the embedded random walk versus BM. Whatever the Skorokhod embedding stopping time be, if RW ever goes below $-x$, then a fortiori the BM it samples has gone below $-x$, thus the RHS. To prove the LHS, assume embedding to be by the Azéma–Yor stopping time. If BM embeds a RW visit at height y , it will stop and embed the next RW visit at the latest upon reaching as far down as $y - \frac{1}{\alpha} \log(E[e^{\alpha(y-X)} | X < y]) \geq y - d^-$. Hence, if BM ever falls below $-x - d^-$, it must have embedded a RW visit at height below $-x$.

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