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FORCING A STOCHASTIC PROCESS TO STAY IN OR TO LEAVE A GIVEN REGION[1]

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Systems defined by $dx(t) = a[x(t), t]\,dt + B[x(t), t]u(t)\,dt + N^{1/2}[x(t), t]\,dW(t)$, where $x(t)$ is the state variable, $u(t)$ is the control variable, a is a vector function, B and N are matrices and $W(t)$ is a Brownian motion process, are considered. The aim is to minimize the expected value of a cost function with quadratic control costs on the way and terminal cost function $K(T)$, where $T = \inf\{s\colon x(s) \in D \mid x(t) = x\}$, D being a given region in $\mathbb{R}^n$. The function K is taken to be 0 if $T \geq (\leq)\,\tau$, where τ is a positive constant and $+\infty$ if $T < (>)\,\tau$ when the aim is to force $x(t)$ to stay in (resp., to leave) the region C, the complement of D. A particular one-dimensional problem is solved explicitly and a risk-sensitive version of the general problem is also considered.

1. Introduction and theoretical result. Consider the dynamic system defined by

$$dx(t) = a[x(t), t]\,dt + B[x(t), t]u(t)\,dt + N^{1/2}[x(t), t]\,dW(t), \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable at time t, $u(t) \in \mathbb{R}^m$ is the control variable, a is a prescribed n-vector function, B is an $n \times m$ matrix, N is an n-square symmetric positive definite matrix and $W(t)$ is a standard n-dimensional Brownian motion. Thus, $W(t)$ is a Gaussian process with

$$\begin{aligned} E[W(t)] &= 0, \\ E[W(t)W'(t)] &= tI_n, \end{aligned} \tag{1.2}$$

I_n being the identity matrix of order n. Suppose that one wants the process $x(t)$ to stay in a given region C in $\mathbb{R}^n$ until a fixed time τ, or leave the region C before a time τ. Let

$$T(x, t) = \inf\{s\colon x(s) \in D \mid x(t) = x\}, \tag{1.3}$$

where $x \in C$ and D is the complement of C. Then, one might try to minimize the expected value of the cost function

$$J(x, t) = \int_t^T (\tfrac{1}{2}) u'(s) Q[x(s), s] u(s)\,ds + K(T), \tag{1.4}$$

where Q is an n-square symmetric positive definite matrix. If the aim is to

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keep the process $x(t)$ in the region C until a fixed time τ, then we put

$$K(T) = \begin{cases} 0 & \text{if } T \geq \tau, \\ \infty & \text{if } T < \tau, \end{cases} \tag{1.5}$$

whereas we take

$$K(T) = \begin{cases} 0 & \text{if } T \leq \tau, \\ \infty & \text{if } T > \tau, \end{cases} \tag{1.6}$$

when the aim is to make $x(t)$ leave the region C before a time τ. This problem is a special case of a problem considered by Whittle [(1982), page 289] [see also Whittle and Gait (1970)] in which the terminal cost function $K[x(T), T]$ is general. Whittle proved that if the *uncontrolled* process defined by

$$dx_1(t) = a[x_1(t), t]\,dt + N^{1/2}[x_1(t), t]\,dW(t) \tag{1.7}$$

is certain to leave the continuation region C and if there exists a positive scalar c such that the relation

$$N = cBQ^{-1}B' \tag{1.8}$$

holds, then the optimal control u^* is given by

$$u^* = -Q^{-1}B'F_x, \tag{1.9}$$

where F_x denotes the partial derivative of F with respect to x and $F = F(x, t)$ is obtained from

$$\exp[-F(x,t)/c] = E\{\exp[-K[x_1(T_1), T_1]/c] \mid x_1(t) = x\}, \tag{1.10}$$

where T_1 is the same as T, but for the uncontrolled process $x_1(t)$. The function F is in fact the minimal expected cost incurred from (x, t); that is, the cost that one can expect, starting from (x, t), if one uses the optimal policy u^*.

Now, if we define the function K as in (1.5) [resp., (1.6)], we obtain

$$\exp[-F(x,t)/c] = P[T_1 \geq \tau] \quad (\text{resp., } P[T_1 \leq \tau]). \tag{1.11}$$

Letting

$$H(x,t,\tau) = P[T_1(x,t) \geq \tau] = 1 - G(x,t,\tau), \tag{1.12}$$

we have the following proposition:

PROPOSITION 1. *If the uncontrolled process defined by* (1.7) *is certain to leave the continuation region C and if there exists a positive scalar c such that relation* (1.8) *holds, then the control u^* that minimizes the expected value of the cost function* (1.4) *with the termination cost K given by* (1.5) [*resp*., (1.6)] *is*

$$u^* = cQ^{-1}B'H_x/H \quad \left(\textit{resp.}, cQ^{-1}B'G_x/G\right). \tag{1.13}$$

Note that we deduce from (1.11) that if one uses the optimal policy u^* and if the termination cost $K(T)$ is given by (1.5) [resp., (1.6)], then the expected cost is finite whenever $P[T_1 \geq \tau] > 0$ (resp., $P[T_1 \leq \tau] > 0$); furthermore, using Theorem 1 of Lefebvre (1989), which gives the joint probability density function of T and $x(T)$ in terms of that of T_1 and $x_1(T_1)$, we can show that $P[T \geq \tau] = 1$ (resp., $P[T \leq \tau] = 1$), so that the termination cost $K(T)$ in the cost function $J(x, t)$ will actually be equal to zero.

In Section 2, a particular one-dimensional problem will be solved explicitly and, in Section 3, Proposition 1 will be generalized by considering a risk-sensitive version of the original problem. Concluding remarks are given in Section 4.

2. Explicit solution of a one-dimensional problem. In this section the one-dimensional dynamic system defined by

$$dx(t) = Bu(t)\,dt + N^{1/2}\,dW(t), \tag{2.1}$$

where B and N are positive constants, is considered. Note that the uncontrolled process

$$dx_1(t) = N^{1/2}\,dW(t) \tag{2.2}$$

is a Brownian motion process without drift and with variance parameter equal to N.

Suppose that $x(0) = x \in C = (-\infty, d)$ and let

$$T(x) = \inf\{s\colon x(s) = d \mid x(0) = x\}. \tag{2.3}$$

The problem of maximizing or minimizing the time spent by the process (2.1) in an interval has been studied by Whittle [(1982), page 290], Kuhn (1985) and Lefebvre and Whittle (1988). To achieve their objective, these authors minimized the expected value of a cost function of the form

$$J(x) = \int_0^T (\tfrac{1}{2}) Q u^2(s)\,ds + LT, \tag{2.4}$$

where Q is a positive constant and L is positive (resp., negative) if the aim is to minimize (resp., maximize) the survival time in the continuation region C. Thus, the penalty or the reward given to survival in C is proportional to the survival time in C and the aim is to maximize or minimize this survival time in C taking the quadratic control costs into account. In this note, the aim is to force the process $x(t)$ to stay in or to leave the continuation region C. Therefore, the cost function, whose expected value we want to minimize, is taken to be

$$J(x) = \int_0^T (\tfrac{1}{2}) Q u^2(s)\,ds + K(T), \tag{2.5}$$

where the terminal cost function $K(T)$ is given by (1.5) or (1.6).

Now, in this problem, we have $P[T_1(x) < \infty] = 1$ [see Cox and Miller (1965), page 221]. Furthermore, since N, B and Q are all constants, the relation

$$N = cBQ^{-1}B' = cB^2/Q \tag{2.6}$$

holds if $c = NQB^{-2}$. Hence, we may use Proposition 1 to obtain the control u^* that minimizes the expected value of the cost function (2.5). Suppose first that the aim is to force the process (2.1) to stay in the interval $(-\infty, d)$ for a time τ.

Then, we deduce from Proposition 1 that

(2.7) $$u^* = c(B/Q)H_x/H,$$

where

(2.8) $$H(x,\tau) = P[T_1(x) \geq \tau].$$

But this probability is known to be [see Ross (1989), page 456, for example]

(2.9) $$H(x,\tau) = 2\Phi\left[(d-x)/(\tau N)^{1/2}\right] - 1,$$

where Φ is the standard normal distribution function. Hence, the optimal control u^* is given by

(2.10) $$u^* = -B^{-1}\left(\frac{2N}{\pi\tau}\right)^{1/2} \frac{\exp\left[-(d-x)^2/(2\tau N)\right]}{2\Phi\left[(d-x)/(\tau N)^{1/2}\right] - 1}.$$

We see that u^* tends to 0 as x tends to $-\infty$ and $u^* \to -\infty$ as $x \to d^-$, which is what one could have expected since survival in C for a time τ is almost certain when x tends to $-\infty$ and since when x increases to d, one is willing to use as large a control as needed to avoid having to pay the infinite penalty incurred for not surviving in C for the desired length of time τ. We also find that u^* tends to zero as τ or N decreases to zero, reflecting the fact that survival in C for a time τ is almost sure in both cases, so that there is no need to use any control.

When the aim is to force the process $x(t)$ out of the interval $(-\infty, d)$, we find that

(2.11) $$u^* = B^{-1}\left(\frac{N}{2\pi\tau}\right)^{1/2} \frac{\exp\left[-(d-x)^2/(2\tau N)\right]}{1 - \Phi\left[(d-x)/(\tau N)^{1/2}\right]}.$$

This time we have, as expected, $u^* \to +\infty$ as $x \to -\infty$ and $u^* \to k$ as $x \to d^-$, where

(2.12) $$k = (2/B)[N/(2\pi\tau)]^{1/2} > 0.$$

One could wonder why u^* does not tend to zero when x increases to d; but, in this case T decreases to zero, so that the final cost will be practically equal to zero, as it should be. Furthermore, we find that

(2.13) $$\lim_{N \to 0^+} u^* = (d-x)/(B\tau).$$

Thus, in the deterministic case, the optimal control is linear in x and we deduce from (2.1) that $T = \tau$. Finally, using l'Hôpital's rule we obtain that u^* tends to ∞ as τ decreases to zero, the reason being that one must then try to leave the interval $(-\infty, d)$ almost at once.

The optimal control u^* given by formula (2.11) is, of course, very different from the control u^{**} that minimizes the expected value of the cost function (2.4) with $L > 0$; indeed, it can be shown that

(2.14) $$u^{**} = \operatorname{sgn}(B)(2L/Q)^{1/2}.$$

That is, u^{**} is a constant.

3. Risk-sensitive formulation. In this section, we suppose that one is looking for the control u^* that minimizes the risk-sensitive cost criterion

$$C(\theta) = (-1/\theta)\log\{E[\exp(-\theta J)]\}, \tag{3.1}$$

where J is defined in (1.4), θ is a parameter that measures the risk sensitivity of the optimizer and log denotes the natural logarithm [see Kuhn (1985)]. When θ tends to 0, $C(\theta)$ reduces to $E[J]$; thus, the cost criterion $C(\theta)$ generalizes the cost criterion used in Section 1. Using Kuhn's results, we can show the following proposition.

PROPOSITION 2. *If the uncontrolled process defined by* (1.7) *is certain to leave the continuation region C and if there exists a scalar c_1 such that the relation*

$$c_1 N = BQ^{-1}B' + \theta N \tag{3.2}$$

holds, then the control u^* that minimizes the cost criterion (3.1), *with the termination cost K given by* (1.5) [*resp.*, (1.6)], *is given by*

$$u^* = (1/c_1)Q^{-1}B'H_x/H \quad [\textit{resp.}, (1/c_1)Q^{-1}B'G_x/G], \tag{3.3}$$

where H and G are defined in (1.12).

In the one-dimensional problem considered in Section 2, relation (3.2) becomes

$$c_1 N = B^2/Q + \theta N, \tag{3.4}$$

which holds if we take $c_1 = B^2/(QN) + \theta$. Hence, formulae (2.10) and (2.11) could easily be modified to take the risk sensitivity of the optimizer into account.

4. Conclusion. In this note, the problem of forcing a process defined by

$$dx(t) = a[x(t),t]\,dt + B[x(t),t]u(t)\,dt + N^{1/2}[x(t),t]\,dW(t) \tag{4.1}$$

to stay in a given region in $\mathbb{R}^n$ until a fixed time τ or to leave this region before a time τ has been considered. We obtained a formula for the control u^* that minimizes the expected value of a cost criterion with quadratic control costs on the way and terminal cost equal to 0 or $+\infty$ according as the objective has been attained or not. In order to obtain the formula for u^*, the relation

$$N = cBQ^{-1}B' \tag{4.2}$$

was assumed to hold for some positive scalar c. Note that relation (4.2) always holds in the one-dimensional case when the quantities N, B and Q are all constants. Relation (4.2) will also hold, for instance, in the case of the two-dimensional dynamic system defined by

$$\begin{aligned} dx(t) &= y(t)\,dt, \\ dy(t) &= Bu(t)\,dt + N^{1/2}\,dW(t), \end{aligned} \tag{4.3}$$

if, again, B, N and the quantity Q in the cost function are constant. Furthermore, according to Whittle and Gait (1970), for quite a variety of processes relation (4.2) does not violate reality too seriously. Of course, relation (4.2), requiring the noise-power and control-power matrices to be proportional, is more likely to hold, at least approximately, when the dimension of the system is low. When the number of components of the state variable is large, it is doubtful whether one can choose a matrix Q in the cost function such that $N = cBQ^{-1}B'$, except in special cases (for instance, when the matrices N and B are diagonal).

Finally, in Section 2, a one-dimensional problem was solved explicitly using the formula obtained in Section 1 and, in Section 3, a risk-sensitive formulation of the original problem considered in this note was given.

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