The Annals of Applied Probability
1991, Vol. 1, No. 1, 16–35

STEADY-STATE ANALYSIS OF RBM IN A RECTANGLE: NUMERICAL METHODS AND A QUEUEING APPLICATION

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Multidimensional reflected Brownian motions, also called regulated Brownian motions or simply RBM's, arise as approximate models of queueing networks. Thus the stationary distributions of these diffusion processes are of interest for steady-state analysis of the corresponding queueing systems. This paper considers two-dimensional semimartingale RBM's with rectangular state space, which include the RBM's that serve as approximate models of finite queues in tandem. The stationary distribution of such an RBM is uniquely characterized by a certain basic adjoint relationship, and an algorithm is proposed for numerical solution of that relationship.

We cannot offer a general proof of convergence, but the algorithm has been coded and applied to special cases where the stationary distribution can be determined by other means; the computed solutions agree closely with previously known results and convergence is reasonably fast. Our current computer code is specific to two-dimensional rectangles, but the basic logic of the algorithm applies equally well to any semimartingale RBM with bounded polyhedral state space, regardless of dimension. To demonstrate the role of the algorithm in practical performance analysis, we use it to derive numerical performance estimates for a particular example of finite queues in tandem; our numerical estimates of both the throughput loss rate and the average queue lengths are found to agree with simulated values to within about five percent.

1. Introduction. To understand the motivation for this paper, it will be useful to consider the simple queueing network pictured in Figure 1. The network consists of two single-server stations arranged in series, each with a first-in-first-out discipline; arriving customers go to station 1 first, after completing service there they go to station 2 and after completing service at station 2 they exit the system. The input process to station 1 is Poisson with average arrival rate λ, except when new arrivals are blocked. Service times at station 1 are deterministic of duration $\tau_1 = 1$ and service times at station 2 are exponentially distributed with mean $\tau_2 = 1$. There is a storage buffer in front of station k that can hold $b_k = 24$ waiting customers, $k = 1, 2$, in addition to the customer occupying the service station. When the buffer in front of station 1 is full, the Poisson input process is simply turned off, and in similar fashion server 1 stops working when the buffer in front of station 2 is full, although a

Received October 1989; revised March 1990.

AMS 1980 *subject classifications*. Primary 60J70, 60K30, 65U05; secondary 65P05, 68M20.

Key words and phrases. Reflected Brownian motion, stationary distributions, diffusion processes, numerical analysis, least squares problem, queueing networks, heavy traffic, performance analysis, finite queues in tandem.

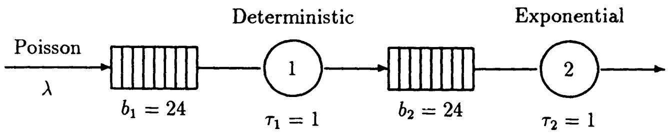

FIG. 1. *Finite queues in tandem.*

customer may still occupy station 1 when the server is idle because of such blocking. (In the literature of queueing theory, this is called communications blocking.) The steady-state performance measures on which we focus are

$$\gamma = \text{the long-run average throughput rate,}$$

$$q_k = \text{the long-run average queue length at station } k,\ k = 1, 2.$$

In these definitions, queue length means the number of customers at the station, either waiting or being served, and the average throughput rate may be equivalently viewed as (a) the average rate at which new arrivals are accepted into the system, or as (b) the average rate at which services are completed at the first station, or as (c) the average rate at which customers depart from the system.

Despite its apparent simplicity, the tandem queue described before is not amenable to exact mathematical analysis, but as an alternative to simulation one may construct and analyze what we call an approximate Brownian system model. This is a diffusion approximation of the general type suggested by heavy traffic limit theorems for queueing networks. However, no limit theorem to justify our particular approximation has been proved thus far and we will not try to provide such a formal justification in this paper. Instead, we will explain or motivate the approximate model in direct, intuitive terms and then concentrate on its analysis. As we will explain later, the two-dimensional queue length process associated with the tandem queue is represented in our approximate model by the reflected Brownian motion Z whose directions of reflection are portrayed in Figure 2. The state space of Z is a 25×25 rectangle and in the interior of this state space Z behaves as an ordinary two-dimensional Brownian motion, whose drift vector and covariance matrix

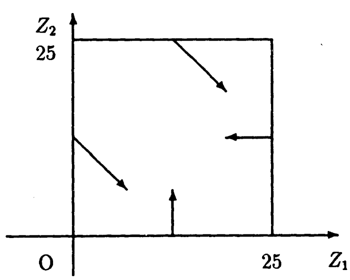

FIG. 2. *State space and directions of reflection for Z.*

TABLE 1
Performance estimates for the queueing model pictured in Figure 1

	$\lambda = 0.9$			$\lambda = 1.0$			$\lambda = 1.1$			$\lambda = 1.2$		
	γ	q_1	q_2	γ	q_1	q_2	γ	q_1	q_2	γ	q_1	q_2
SIM	0.8991	5.1291	6.2691	0.9690	13.87	11.07	0.9801	20.4801	12.3801	0.9804	22.4804	12.4804
QNET	0.8995	4.8490	6.3184	0.9688	13.75	11.25	0.9801	20.5239	12.4445	0.9807	22.2688	12.4676

depend on the data of the queueing network. At the boundary of the rectangle Z is "instantaneously reflected" (this term will be given precise meaning later) in a direction that depends on which side of the boundary is hit; for the RBM that models our tandem queueing system, the directions of reflection are precisely those pictured in Figure 2. In Section 6 of this paper we will explain why these are the appropriate directions of reflection for the tandem queue and how one computes the drift vector and covariance matrix of Z from the parameters of the queueing model.

If the stationary distribution of Z can be computed, then one can derive from it approximate values for γ, q_1 and q_2 (or whatever other steady-state performance measures may be deemed interesting). In this paper, we will describe a method for computing the stationary distributions of processes like Z and the method will be applied in an approximate analysis of the tandem queue pictured in Figure 1. The results of that analysis are summarized in Table 1, where we give performance estimates derived from the approximate Brownian model, identified in the table as QNET estimates, as well as estimates obtained via simulation. None of the QNET estimates of average queue length differs from the corresponding simulation estimate by more than 5% and the accuracy of our throughput rate estimates is equally impressive: when $\lambda = 0.9$, both simulation and QNET predict a throughput loss rate below one-tenth of 1%; when $\lambda = 1.0$, simulation and QNET predict throughput loss rates of 3.10 and 3.14%, respectively; when $\lambda = 1.1$, the limiting factor on system throughput is the average service rate of 1.0 and both simulation and QNET predict a throughput rate 1.99% below this maximum; when $\lambda = 1.2$, the maximum possible throughput rate is again 1.0 and the simulation and QNET estimates of γ are 1.96 and 1.93% below this maximum, respectively. Our analysis of tandem queues in series can be extended to allow an arbitrary renewal input process and arbitrary service time distributions (the first two moments of the interarrival and service time distributions determine the drift vector and covariance matrix of the corresponding Brownian system model); Table 1 shows in a concrete way how useful Brownian system models can be, *if* one can compute their stationary distributions. With that introduction, we now explain in more general terms the background and objectives of our study.

Multidimensional reflected Brownian motions, also called regulated Brownian motions or just RBM's, are a class of diffusion processes that arise as approximate models of queueing networks. That is, such diffusion processes constitute an alternative class of stochastic system models, sometimes called

Brownian system models, that can be used to represent congestion and delay in networks of interacting processors. Thus the stationary distributions of RBM's are of interest for steady-state analysis of the queueing systems to which they correspond. There is now a substantial literature on Brownian models of queueing networks and virtually all of the papers in that literature are devoted to one or more of the following tasks.

1. Identify the Brownian analogs for various types of conventional queueing models, explaining how the data of the approximating RBM are determined from the structure and the parameters of the conventional model. An important subtask is showing that the RBM exists and is uniquely determined by an appropriate set of axiomatic properties.
2. Prove limit theorems that justify the approximation of conventional models by their Brownian analogs under heavy traffic conditions.
3. Determine the analytical problems that must be solved in order to answer probabilistic questions associated with the RBM. These are invariably partial differential equation problems (PDE problems) with oblique derivative boundary conditions. A question of central importance, given the queueing applications that motivate the theory, is which PDE problem one must solve in order to determine the stationary distribution of an RBM.
4. Solve the PDE problems of interest, either analytically or numerically.

Most research to date has been aimed at tasks 1 through 3, and it has concentrated on network models with a single customer type; see Harrison and Williams [10] and Harrison, Williams and Chen [12] for recent surveys of work on open network models and closed network models, respectively. Two even more recent papers by Peterson [13] and by Harrison and Nguyen [8] discuss Brownian models of networks with many customer types, but there is still much to be done in that area. With regard to research category 4, for a driftless RBM in two dimensions the work of Harrison, Landau and Shepp [7] gives an analytical expression for the stationary distribution and the availability of a package for evaluation of Schwarz–Christoffel transformations makes the evaluation of associated performance measures numerically feasible (cf. [18]). For the two-dimensional case with drift, Foddy [6] found analytical expressions for the stationary distributions for certain special domains, drifts and directions of reflection, using Riemann–Hilbert techniques. In dimensions three and more, RBM's having stationary distributions of exponential form were identified in [11, 21] and these results were applied in [10, 12] to RBM's arising as approximations to open and closed queueing networks with homogeneous customer populations. However, until now there has been no general method for solving the PDE problems alluded to in 4.

If Brownian system models are to have an impact in the world of practical performance analysis, task 4 is obviously crucial. In particular, practical methods are needed for determining stationary distributions and it is very unlikely that general analytical solutions will ever be found. In this paper we describe an approach to computation of stationary distributions that seems to be widely applicable, but the method will only be developed and tested for two-dimensional RBM's with rectangular state space, and even in that limited setting our proof of convergence is incomplete. Our decision to test the computational method in this particular setting was motivated by (a) the availability of exact results for at least some RBM's with rectangular state space, which allows us to evaluate the accuracy of our method and (b) the fact that RBM's with rectangular state space include the approximate Brownian models corresponding to finite queues in tandem, which allows us to demonstrate the ultimate use of the computational method on an interesting and nontrivial class of queueing systems. As a tool for analysis of queueing systems, the computer program described in this paper is obviously limited in scope, but our ultimate goal is to implement the same basic computational approach in a general routine that can compete with software package like PANACEA [14] and QNA [20] in the analysis of large, complicated networks. Readers who would like to obtain a copy of the computer program described in this paper should contact the second-named author by electronic mail, addressing the message to fharrison@what.stanford.edu.

The paper is organized as follows. Section 2 gives a precise definition of an RBM with rectangular state space and a statement of the PDE problem that one must solve to determine its stationary distribution. In Sections 3 and 4, an algorithm is developed for solution of the PDE problem and then in Section 5 the algorithm is applied to some special cases for which exact analytical solutions are available. In Section 6 we return to discussion of the previously described tandem queue, explaining how the QNET performance estimates in Table 1 were obtained. Finally, Section 7 contains some remarks about promising directions for future research.

2. SRMB with rectangular state space.

Let S be a closed two-dimensional rectangle and $\mathcal{O}$ be the interior of the rectangle. For $i = 1, 2, 3, 4$ let F_i be the ith boundary face of S and let v_i be an inward-pointing vector on F_i with unit normal component (see Figure 3). For the purpose of this paper, we assume that (here and later the symbol $\equiv$ means equals by definition)

(1) there are positive constants a_i and b_i such that $a_i v_i + b_i v_{i+1}$ points into the interior of S from the vertex where F_i and F_{i+1} meet, $i = 1, 2, 3, 4$, where $v_5 \equiv v_1$ and $F_5 \equiv F_1$.

Also, let us define the 2×4 matrix $R \equiv (v_1, v_2, v_3, v_4)$.

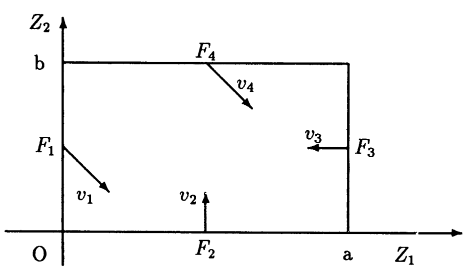

FIG. 3. *The state space S of the RBM.*

Definition 1. A semimartingale reflected Brownian motion (abbreviated as SRBM) Z associated with data (S, Γ, μ, R) is a continuous, adapted, two-dimensional process with an associated family of probability measures $\{P_x,\ x \in S\}$ defined on some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\})$ such that for each $x \in S$, we have P_x-a.s.,

$$Z(t) = X(t) + RL(t) = X(t) + \sum_{i=1}^{4} L_i(t) \cdot v_i \in S, \qquad t \geq 0, \tag{2}$$

(3) X is a two-dimensional Brownian motion with $X(0) = x$, covariance matrix Γ and drift vector μ such that $\{X(t) - \mu t, \mathscr{F}_t, t \geq 0\}$ is a martingale,

L is a continuous $\{\mathscr{F}_t\}$-adapted four-dimensional process such that

(4)
(a) $L(0) = 0$,
(b) L is nondecreasing,
(c) L_i increases only at times t such that $Z(t) \in F_i$, $i = 1, 2, 3, 4$.

Remark. The previous definition of SRBM follows from Reiman and Williams [15]. Harrison, Landau and Shepp [7] considered a more general class of RBM's which may not have a semimartingale representation as in (2). However, for the purpose of this paper, we *only* consider the class of RBM's defined in Definition 1. For the notational convenience of this paper, from now on we will *not* distinguish SRBM from RBM.

The RBM Z defined before behaves like a two-dimensional Brownian motion with drift vector μ and covariance matrix Γ in the interior $\mathscr{O}$ of its state space. When the boundary face F_i is hit, the process L_i (sometimes called the local time of Z on F_i) increases, causing an instantaneous displacement of Z in the direction given by v_i; the magnitude of the displacement is the minimal amount required to keep Z always inside S. Therefore, we call Γ, μ and R the covariance matrix, the drift vector and the reflection matrix of Z, respectively.

The following will be proved in [3], essentially by just piecing together the results in Harrison and Williams [10], Reiman and Williams [15] and Varadhan and Williams [19].

Proposition 1. *Let there be given a covariance matrix* Γ, *a drift vector* μ *and a reflection matrix* R *whose columns satisfy* (1). *Then there is an RBM associated with the data* (S, Γ, μ, R). *For each* $x \in S$, *let* Q_x *be the probability measure induced on the path space* $C([0,\infty), S) \equiv \{w\colon [0,\infty) \to S,\ w \text{ is continuous}\}$ *by any such RBM* Z *and associated measure* P_x. *Then the family* $\{Q_x,\ x \in S\}$ *is unique, it is Feller-continuous, that is,* $x \to E^{Q_x}[f(w(t))]$ *is continuous for all* $f \in C_b(S)$ *and* $t \geq 0$, *and* $w(\cdot)$ *together with* $\{Q_x,\ x \in S\}$ *is a strong Markov process. Moreover,* $\sup_{x \in S} E^{P_x}[L_i(t)] < \infty$ *for each* $t \geq 0$ *and* $i = 1, 2, 3, 4$.

For a probability measure π on S, let E^{π} denote the expectation with respect to P_{π}, where $P_{\pi}(A) \equiv \int_S P_x(A)\pi(dx)$. A probability measure π on S is called a *stationary distribution* of the RBM Z if for every bounded Borel function f on S and every $t > 0$,

$$\int_S E^{P_x}[f(Z_t)]\pi(dx) = \int_S f(x)\pi(dx).$$

Because the state space S is compact, there is a stationary distribution for Z, (cf. Dai [2]). Also, using arguments virtually identical to those in [10], one can show that

(5) Z is *ergodic* and therefore its stationary distribution π is unique.

(6) $\pi(\partial S) = 0$ and $\pi(dx) = p_0(x)\,dx$ for some density function p_0 in $\mathscr{O}$.

(7) For each i, there exists a finite Borel measure ν_i on F_i such that $E^{\pi}\{\int_0^t 1_A(Z_s)\,dL_i(s)\} = t\nu_i(A)$ for all $t \geq 0$, where A is any Borel subset of F_i. Furthermore, ν_i has a density p_i with respect to one-dimensional Lebesgue measure $d\sigma$ on F_i.

Let $C^2(S)$ be the space of twice differentiable functions whose first and second order partials are continuous on S. Because S is compact, functions in $C^2(S)$ are bounded. For $f \in C^2(S)$, applying Itô's formula to the process Z exactly as in [9] or [10], one has that

$$\begin{aligned} f(Z(t)) = f(Z(0)) &+ \sum_{i=1}^{2} \int_0^t \frac{\partial}{\partial x_i} f(Z(s))\,d\xi_i(s) + \int_0^t \mathscr{G} f(Z(s))\,ds \\ &+ \sum_{i=1}^{4} \int_0^t \mathscr{D}_i f(Z(s))\,dL_i(s), \end{aligned} \tag{8}$$

where $\xi_i(t) = X_i(t) - \mu_i t$,

$$\mathscr{G} f \equiv \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \Gamma_{ij} \frac{\partial^2 f}{\partial x_i\,\partial x_j} + \sum_{i=1}^{2} \mu_i \frac{\partial f}{\partial x_i}, \tag{9}$$

$$\mathscr{D}_i f(x) \equiv v_i \cdot \nabla f(x) \quad \text{for } x \in F_i, \qquad i = 1,2,3,4. \tag{10}$$

Again proceeding exactly as in [10], we can then take E^{π} of both sides of (8) to conclude that the stationary density p_0 and the boundary measures $\nu_i = p_i\,d\sigma$ jointly satisfy the following *basic adjoint relationship*:

$$\int_S (\mathscr{G} f \cdot p_0)\,dx + \sum_{i=1}^{4} \int_{F_i} (\mathscr{D}_i f \cdot p_i)\,d\sigma = 0 \quad \text{for all } f \in C^2(S). \tag{11}$$

The argument given in the previous paragraph shows that (11) is *necessary* for p_0 to be the stationary density of Z. The following essential complement will be proved in [3].

PROPOSITION 2. *Suppose that p_0 is a probability density function in $\mathscr{O}$ and $\nu_1, \dots, \nu_4$ are finite Borel measures on $F_1, \dots, F_4$, respectively. If they jointly* *satisfy*

$$(12) \qquad \int_S (\mathscr{G} f \cdot p_0)\, dx + \sum_{i=1}^{4} \int_{F_i} \mathscr{D}_i f \, d\nu_i = 0 \quad \textit{for all } f \in C^2(S),$$

then p_0 *is the stationary density of* Z.

In this paper we develop an algorithm for computing the stationary density p_0 and the boundary densities p_i, $i = 1, 2, 3, 4$. The logic of the algorithm applies equally well in higher dimensions, but in higher dimensions we have no analog for Proposition 2. The following conjecture is useful in proving the convergence of the algorithm that we will develop in Section 4.

CONJECTURE 1. *Suppose that* p_0 *is an integrable Borel function in* $\mathscr{O}$ *and* $\nu_1, \dots, \nu_4$ *are finite Borel measures on* $F_1, \dots, F_4$, *respectively. If they jointly satisfy the basic adjoint relationship* (12), *then* p_0 *does not change sign in* $\mathscr{O}$.

Readers might naturally assume that it is best to convert (11) into a direct PDE for p_0, but that gets very complicated because of auxiliary conditions associated with the singular parts of the boundary; we are just going to work with (11) directly. We end this section by converting (11) into a compact form that will be used in the next section. Let

$$(13) \qquad \mathscr{A} f(x) = \begin{cases} \mathscr{G} f(x), & x \in \mathscr{O}, \\ \mathscr{D}_i f(x), & x \in F_i,\ i = 1, 2, 3, 4, \end{cases}$$

$$\eta(dx) = \begin{cases} dx, & \text{in } \mathscr{O}, \\ d\sigma, & \text{on } \partial S, \end{cases}$$

and

$$(14) \qquad p(x) = \begin{cases} p_0(x), & \text{in } \mathscr{O}, \\ p_i(x), & \text{on } F_i,\ i = 1, 2, 3, 4. \end{cases}$$

With these notations, the basic adjoint relationship (11) can be rewritten as

$$(15) \qquad \int_S (\mathscr{A} f \cdot p)\, d\eta = 0 \quad \text{for all } f \in C^2(S).$$

REMARK. Strictly speaking, $\mathscr{A} f(x)$ in (13) is undefined at each vertex x. However, because the vertices have no mass under the measure η, the definition of $\mathscr{A} f$ at vertices is immaterial in this paper.

3. A least squares problem. In this section we first convert the problem of solving (15) into a *least squares problem*, and then propose an algorithm to solve the least squares problem. Our approach is similar in spirit to that of Bramble and Schatz [1], who considered a Rayleigh–Ritz–Galerkin method for solution of the Dirichlet problem using a subspace without boundary conditions. The purpose of their method was to avoid finding boundary elements when the boundary of the domain is complicated. In our problem, the domain is not complicated at all except that it is nonsmooth, but the boundary condition is implicit in (15) and is not known to us.

We start with the compact form (15) of the basic adjoint relationship (11). Let $L^2 = L^2(S, \eta)$ and denote by $\|\cdot\|$ the usual L^2 norm and by $(\cdot, \cdot)$ the usual inner product. It is evident that $\mathscr{A}f \in L^2$ for any $f \in C^2(S)$. Hence we can define

$$H = \overline{\{\mathscr{A}f\colon f \in C^2(S)\}},$$

where the closure is taken in L^2. If one assumes that the unknown density p is in L^2, then (15) says simply that $\mathscr{A}f \perp p$ for all $f \in C^2(S)$, or equivalently, $p \in H^{\perp}$. Conversely if $w \in H^{\perp}$, then w satisfies (15).

Let us assume for the moment that the unknown density function p defined by (14) is in L^2. That is, assume p_0 is square-integrable with respect to Lebesgue measure in $\mathscr{O}$ and p_i is square integrable with respect to one-dimensional Lebesgue measure on F_i, $i = 1, 2, 3, 4$. In order to construct a function $w \in H^{\perp}$, let

$$\phi_0(x) = \begin{cases} 1 & \text{for } x \in \mathscr{O}, \\ 0 & \text{for } x \in \partial S. \end{cases} \tag{16}$$

Because p_0 is a probability density, we have $(p, \phi_0) = \int_S (p \cdot \phi_0) \eta(dx) = \int_{\mathscr{O}} p_0 \, dx = 1$, so p is *not* orthogonal to ϕ_0. On the other hand, we know from (15) that $p \perp h$ for all $h \in H$ and therefore ϕ_0 is *not* in H. Let $\bar{\phi}_0$ be the projection of ϕ_0 onto H. That is,

$$\bar{\phi}_0 \equiv \arg\min_{\phi \in H} \|\phi_0 - \phi\|^2. \tag{17}$$

Because ϕ_0 is not in H, we know that

$$\tilde{\phi}_0 \equiv \phi_0 - \bar{\phi}_0 \neq 0. \tag{18}$$

Obviously, $\tilde{\phi}_0 \in H^{\perp}$. Simple algebra gives

$$\alpha \equiv \int_S \tilde{\phi}_0(x)\, dx = \int_S \big(\tilde{\phi}_0(x) \cdot \phi_0(x)\big) \eta(dx) = \big(\hat{\phi}_0, \tilde{\phi}_0\big) > 0.$$

Proposition 3. *Suppose that* $p \in L^2$. *Then* $w = (w_0, w_1, \dots, w_4) \equiv (1/\alpha)\tilde{\phi}_0$ *satisfies the basic adjoint relationship* (15) *and* $\int_S w_0 \, dx = 1$. *Therefore, assuming that Conjecture* 1 *is true, we have* $w_0 = p_0$ *almost everywhere.*

Proof. Let $w \equiv (1/\alpha)\tilde{\phi}_0$. Then, by construction, w satisfies the basic adjoint relationship (15). Conjecture 1 asserts that w_0 does not change the sign and because $\int_S w_0 \, dx = 1$, w_0 is a probability density function. Thus Proposition 2 can be applied to assert $w_0 = p_0$ a.e. □

As we will see later, the assumption that p is in L^2 is *not* satisfied in all cases of practical interest. However, when that assumption is satisfied, Proposition 3 says that in order to find the unknown stationary density p, it suffices to solve the *least squares problem* (17).

We now define some quantities that are of interest in the queueing theoretic applications of RBM. Let $q_i = \int_{\mathscr{O}}(x_i \cdot w_0(x))\,dx$, $i = 1, 2$, and $\delta_i = \int_{F_i} w_i(x)\,d\sigma$, $i = 1, 2, 3, 4$. Assuming Conjecture 1, $q_i = \int_{\mathscr{O}}(x_i \cdot p_0(x))\,dx$, which represents the long-run average value of Z_i, and $\delta_i = \int_{F_i} p_i(x)\,d\sigma$, which represents the long-run average amount of pushing per unit of time needed on boundary F_i in order to keep Z inside the rectangle S. That is, $E^{P_x}[L_i(t)] \sim \delta_i t$ as $t \to \infty$ for each $x \in S$, $k = 1, 2, 3, 4$.

4. An algorithm.

Given Proposition 3, we will now propose an algorithm for approximate computation of p based on L^2 projection. In the examples presented later, it will be seen that the algorithm works well even in cases where p is known *not* to be in L^2.

PROPOSITION 4. *Suppose that we can construct a sequence of finite dimensional subspaces $\{H_n\}$ of H such that $H_n \uparrow H$ as $n \uparrow \infty$ ($H_n \uparrow H$ means that $H_1, H_2, \dots$ are increasing and every $h \in H$ can be approximated by a sequence $\{h_n\}$ with $h_n \in H_n$ for each n). Let*

$$\psi_n \equiv \arg\min_{\phi \in H_n} \|\phi_0 - \phi\|^2.$$

Then $\|\bar{\phi}_0 - \psi_n\|^2 \to 0$ as $n \to \infty$. Furthermore, if

$$w_n \equiv \phi_0 - \psi_n, \tag{19}$$

then $w_n \to \tilde{\phi}_0$ in $L^2(S, \eta)$ as $n \to \infty$.

PROOF. We can find an orthonormal basis $\{\phi_i\}_{i \ge 1}$ in H, such that $\{\phi_1, \dots, \phi_n\}$ is an orthonormal basis for H_n. Then

$$\bar{\phi}_0 = \sum_{i=1}^{\infty} (\phi_0, \phi_i)\phi_i \quad \text{and} \quad \psi_n = \sum_{i=1}^{n} (\phi_0, \phi_i)\phi_i.$$

Hence

$$\|\bar{\phi}_0 - \psi_n\|^2 = \sum_{i=n+1}^{\infty} (\phi_0, \phi_i)^2 \to 0.$$

Let $w_n \equiv \phi_0 - \psi_n$. Then

$$\|w_n - \tilde{\phi}_0\|^2 = \|(\phi_0 - \psi_n) - (\phi_0 - \bar{\phi}_0)\|^2 = \|\psi_n - \bar{\phi}_0\|^2 \to 0. \qquad \square$$

Now the problem is to find the projections ψ_n. The way to find the projection is standard. Suppose that $\{\phi_1, \dots, \phi_n\}$ is a basis for H_n (it need not be an orthonormal basis). Then

$$\psi_n = \sum_{i=1}^{n} a_i \phi_i$$

for some $a_1, a_2, \dots, a_n$. Let T denote the transpose operator. Then $(a_1, a_2, \dots, a_n)^T$ is the unique solution x of the *normal* equations

$$Ax = b, \tag{20}$$

where

$$A = \begin{pmatrix} (\phi_1, \phi_1) & \cdots & (\phi_1, \phi_n) \\ \vdots & \ddots & \vdots \\ (\phi_n, \phi_1) & \cdots & (\phi_n, \phi_n) \end{pmatrix}, \tag{21}$$

and

$$b = \begin{pmatrix} (\phi_0, \phi_1) \\ \vdots \\ (\phi_0, \phi_n) \end{pmatrix}. \tag{22}$$

Because A is positive definite, the normal equations do have a unique solution. Finally, we have

$$w_n = \phi_0 - \sum_{i=1}^{n} a_i \phi_i.$$

As pointed out in Serbin [16, 17], the normal matrix A in the normal equations (20) is generally ill-conditioned. There are many alternatives for solving the normal equations. However, we have chosen to use Gram–Schmidt orthogonalization to find the projections ψ_n directly.

There are many ways to choose the approximating subspaces H_n, each of which yields a different version of the algorithm. We choose H_n as

$$H_n = \text{span of } \{\mathscr{A} f_{k,i} \colon k = 1, 2, \dots, n;\ i = 0, 1, \dots, k\},$$

where $f_{k,i} = x_1^i x_2^{k-i}$. The dimension of H_n is

$$\frac{(n+1)(n+2)}{2} - 1.$$

PROPOSITION 5. *If H_n is defined as before, then $H_n \uparrow H$. Let w_n be defined as in* (19); *then $\alpha_n \equiv (w_n, w_n) \neq 0$. Therefore we can define $p^n \equiv (1/\alpha_n) w_n$. Furthermore, if $p \in L^2$, then $p^n \to (1/\alpha)\tilde{\phi}_0$ in L^2 as $n \to \infty$.*

PROOF. The proof of $H_n \uparrow H$ is an immediate consequence of Proposition 7.1 and Remark 6.2 in the appendices of Ethier and Kurtz [5]. Because $\phi_0 \notin H_n$ for each n, we know that $w_n \neq 0$ and $\alpha_n \equiv (w_n, w_n) \neq 0$. Hence we can define $p^n \equiv (1/\alpha_n) w_n$. If we assume $p \in L^2$, then $\alpha \neq 0$. Because $w_n \to \tilde{\phi}_0$ and $\alpha_n \to \alpha$, it is immediate that $p^n \to (1/\alpha)\tilde{\phi}_0$. □

Several considerations lie behind our choice of approximating subspaces. First, more complicated subspaces (e.g., those used in the finite element method) may lead to problems in higher dimensions. Second, when low order polynomials are substituted into (11), one obtains *exact* relations among some quantities associated with the stationary density p. These relations resemble *energy preserving* relations in the finite element method. We believe that this property will enhance the accuracy of our computational method. Finally, as the following section will show, our choice seems to give reasonably good results.

PROPOSITION 6. *Suppose that* $p \in L^2$. *Let* p^n *be defined as in Proposition* 5. *Let* $q_1^{(n)} = \int_{\mathscr{O}}(x_1 \cdot p^n(x))\,dx$ *and* $q_2^{(n)} = \int_{\mathscr{O}}(x_2 \cdot p^n(x))\,dx$. *Then* $q_1^{(n)} \to q_1$ *and* $q_2^{(n)} \to q_2$ *as* $n \to \infty$.

PROOF.

$$|q_1^{(n)} - q_1| \le \int_{\mathscr{O}} x_1 |p^n(x) - w_0(x)|\,dx = \int_{\mathscr{O}} x_1 \left| p^n(x) - \frac{1}{\alpha}\tilde{\phi}_0(x) \right| dx$$

$$\le \left(\int_S x_1^2\,dx \right)^{1/2} \left(\int_S \left| p^n(x) - \frac{1}{\alpha}\tilde{\phi}_0(x) \right|^2 dx \right)^{1/2}$$

$$\le \left(\frac{a^3 b}{3} \right)^{1/2} \left\| p^n - \frac{1}{\alpha}\tilde{\phi}_0 \right\|^2 \to 0$$

as $n \to \infty$, where the constants a and b are indicated in Figure 3. Similarly we have $q_2^{(n)} \to q_2$ as $n \to \infty$. □

REMARK. If $p \notin L^2$, then $\tilde{\phi}_0$ in (18) must be zero in L^2. However, as stated in Proposition 5, each p^n is well defined. We conjecture that p^n still converges to p in the weak sense and the weak convergence would imply $q_i^{(n)} \to q_i$ as $n \to \infty$, $i = 1, 2$.

5. Comparison with SC solutions. In this section we consider a special case of the RBM described in Section 2, comparing results obtained with our algorithm against a known analytic solution. The special case considered has $\mu = 0$ and $\Gamma = 2I$ (I is the 2×2 identity matrix), so our differential operator $\mathscr{G}$ is the ordinary Laplacian. Before going further, we introduce some additional notation. Let n_i be the unit normal vector on F_i and θ_i be the angle between the vector v_i and the normal n_i, with θ_i being positive when v_i lies right of n_i as one traces the boundary counter clockwise and nonpositive otherwise, $i = 1, 2, 3, 4$. Let $\beta_i = 2(\theta_{i+1} - \theta_i)/\pi$, $i = 1, 2, 3, 4$ with $\theta_5 \equiv \theta_1$. It can be shown that (1) is equivalent to $\beta_i > -1$ for all i. From the results in [7] it follows that p_0 is always square integrable in $\mathscr{O}$ w.r.t. Lebesgue measure, whereas p_i is square integrable on F_i w.r.t. one-dimensional Lebesgue measure if and only if

$$\beta_i > -\tfrac{1}{2}, \qquad i = 1, 2, 3, 4. \tag{23}$$

Hence we conclude that $p \in L^2(S, \eta)$ if and only if (23) is true.

In addition to the restrictions mentioned earlier, we assume that $\theta_1 = \pi/4$, $\theta_2 = 0$, $\theta_3 = 0$ and $\theta_4 = -\pi/4$. The corresponding reflection matrix is

$$R = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \end{pmatrix}. \tag{24}$$

We fix the height of the rectangle at $b = 1$ and let the length a of the rectangle change freely. The RBM in this case corresponds to the heavy traffic limit of two balanced finite queues in tandem. It is easy to calculate that $\beta_1 = -\frac{1}{2}$, $\beta_2 = 0$, $\beta_3 = -\frac{1}{2}$ and $\beta_4 = 1$. Therefore (23) is not satisfied and hence $p \notin L^2(S, \eta)$. Readers will see that our algorithm gives very accurate approximations even in this case. This is consistent with our conjecture explained at the end of Section 4.

For various values of the length parameter a, Table 2 compares two different estimates of $q_1, q_2, \delta_1, \delta_2, \delta_3, \delta_4$. The QNET estimate is that obtained with our algorithm (see Sections 3 and 4), using $n = 6$. The SC estimate is that obtained by Trefethen and Williams [18] using a software package called SCPACK: For the special case under discussion (the restriction to two dimensions and the assumption of zero drift are both essential), Harrison, Landau and Shepp [7] used complex variable methods to compute the stationary density function p_0 in terms of a certain Schwarz–Christoffel transformation and then SCPACK allows numerical evaluation of these formulas. The SC estimates on our Table 2 are taken from Table 2 on page 244 of [18] and the rows labelled DIFF give the absolute differences between those SC estimates and our QNET estimates. It should be mentioned that our algorithm also applies to problems with nonzero drift and its basic logic extends readily to higher dimensions; neither of those statements is true of the methods used in [7] and [18]. Incidentally, the QNET estimates in Table 2 were obtained using $n = 6$ and double precision on a VAX machine; about 32 seconds of CPU time were required to generate all the figures in the table.

6. Analysis of finite queues in tandem. This section is devoted to an analysis of the particular tandem queueing system pictured in Figure 1. A similar treatment of open queueing networks (without buffer storage limitations) has been given by Harrison and Williams [10] and from that earlier work it should be clear how one may extend the current analysis to allow general interarrival and service distributions. Let $A = \{A(t),\ t \geq 0\}$ be a Poisson process with intensity parameter λ, let $S_2 = \{S_2(t),\ t \geq 0\}$ be another independent Poisson process with intensity parameter 1 and let S_1 be the deterministic counting process defined by $S_1(t) = n$ for $n \leq t < n + 1$ and $n = 0, 1, \ldots$. (The letter A is mnemonic for arrival, whereas S_k is a service process associated with station k.)

TABLE 2
Comparisons with SCPACK *when* $n = 6$

	a	q_1	q_2	δ_1	δ_2	δ_3	δ_4
QNET	0.5	0.258229	0.380822	1.848991	2.413695	2.413695	0.564704
SC		0.258585	0.380018	1.871418	2.412890	2.412890	0.541472
DIFF		−0.000356	0.000804	−0.022427	0.000805	0.000805	0.023232
QNET	1.0	0.551325	0.448675	0.805813	1.611625	1.611625	0.805813
SC		0.551506	0.448494	0.805295	1.610589	1.610589	0.805295
DIFF		−0.000181	0.000181	0.000518	0.000036	0.000036	0.000518
QNET	1.5	0.878800	0.471640	0.466710	1.340876	1.340876	0.874166
SC		0.879534	0.471624	0.446669	1.340225	1.340225	0.893557
DIFF		−0.000734	0.000016	0.020041	0.000651	0.000651	−0.019391
QNET	2.0	1.238442	0.483103	0.292077	1.206981	1.206981	0.914904
SC		1.239964	0.482830	0.270736	1.206445	1.206445	0.935709
DIFF		−0.001522	0.000273	0.021341	0.000536	0.000536	0.020805
QNET	2.5	1.625775	0.489845	0.1888642	1.131142	1.131142	0.942499
SC		1.628342	0.489146	0.171214	1.130587	1.130587	0.959373
DIFF		−0.002567	0.000699	0.017428	0.000555	0.000555	−0.016874
QNET	3.0	2.036371	0.494084	0.122836	1.085136	1.085136	0.962300
SC		2.040075	0.492970	0.110891	1.084582	1.084582	0.973691
DIFF		−0.003704	0.001114	0.011945	0.000554	0.000554	0.003308
QNET	3.5	2.466108	0.496881	0.079113	1.056112	1.056112	0.976999
SC		2.471022	0.495381	0.072873	1.055585	1.055585	0.982712
DIFF		−0.004914	0.001500	0.006240	0.000527	0.000527	−0.005713
QNET	4.0	2.911243	0.498826	0.048974	1.037364	1.037364	0.988391
SC		2.917572	0.496936	0.048334	1.036868	1.036868	0.988534
DIFF		−0.006329	0.001890	0.000640	0.000496	0.000496	−0.000143

Let $Q_k(t)$ be the total number of customers occupying station k at time t, either waiting or being served ($k = 1, 2$) and let

$$B_0(t) = \text{measure}\{s \in [0,t]: Q_1(s) < 25\},$$

$$B_1(t) = \text{measure}\{s \in [0,t]: Q_1(s) > 0 \text{ and } Q_2(s) < 25\},$$

$$B_2(t) = \text{measure}\{s \in [0,t]: Q_2(s) > 0\}.$$

Thus $B_0(t)$ represents the amount of time in the interval $[0, t]$ that the arrival process to station 1 is turned on, $B_1(t)$ is the amount of time in that interval during which server 1 is busy and $B_2(t)$ is the amount of time in the interval during which server 2 is busy. Assuming for convenience that there are no customers in the system at time zero, we then have

$$Q_1(t) = A(B_0(t)) - S_1(B_1(t)), \quad t \geq 0, \tag{25}$$

$$Q_2(t) = S_1(B_1(t)) - S_2(B_2(t)), \quad t \geq 0. \tag{26}$$

Next define centered processes $\hat{A}$, $\hat{S}_1$ and $\hat{S}_2$ via $\hat{A}(t) = A(t) - \lambda t$, $\hat{S}_1(t) = S_1(t) - t$ and $\hat{S}_2(t) = S_2(t) - t$. Then the representations (25) and (26) can be rewritten as

$$Q_1(t) = \hat{A}(B_0(t)) - \hat{S}_1(B_1(t)) + \lambda B_0(t) - B_1(t), \tag{27}$$

$$Q_2(t) = \hat{S}_1(B_1(t)) - \hat{S}_2(B_2(t)) + B_1(t) - B_2(t). \tag{28}$$

To establish the connection between our two-dimensional queue length process $Q(t)$ and the RBM's studied earlier, it will be convenient to define nondecreasing processes $I_k(t) = t - B_k(t)$ for $k = 0, 1, 2$ and $t \geq 0$. One interprets $I_0(t)$ as the total amount of time during the interval $[0, t]$ that input to station 1 is turned off and $I_1(\cdot)$ and $I_2(\cdot)$ are the cumulative idleness processes for server 1 and server 2, respectively. Thus we have that

$$I_0(\cdot) \text{ increases only when } Q_1(\cdot) = 25, \tag{29}$$

$$I_1(\cdot) \text{ increases only when } Q_1(\cdot) = 0 \text{ or } Q_2(\cdot) = 25, \tag{30}$$

$$I_2(\cdot) \text{ increases only when } Q_2(\cdot) = 0. \tag{31}$$

Setting $\mu_1 = \lambda - 1$, $\mu_2 = 0$, $\xi_1(t) = \hat{A}(B_0(t)) - \hat{S}_1(B_1(t))$ and $\xi_2(t) = \hat{S}_1(B_1(t)) - \hat{S}_2(B_2(t))$, we observe that (27) and (28) can be rewritten as

$$Q_1(t) = [\xi_1(t) + \mu_1 t] + I_1(t) - \lambda I_0(t), \tag{32}$$

$$Q_2(t) = [\xi_2(t) + \mu_2 t] - I_1(t) + I_2(t). \tag{33}$$

Let us now set $L_3(t) = \lambda I_0(t)$ and $L_2(t) = I_2(t)$ and split $I_1(\cdot)$ into two parts via

$$L_1(t) = \int_0^t 1_{\{Q_2(s) < 25\}}\, dI_1(s) \tag{34}$$

and

$$L_4(t) = \int_0^t 1_{\{Q_2(s) = 25\}}\, dI_1(s). \tag{35}$$

Let S be the 25×25 rectangle pictured in Figure 1 and let $F_1, \dots, F_4$ be the four line segments that make up the boundary of S as in Figure 2. From (30) we see that $I_1 = L_1 + L_4$ and (29)–(31) and (34)–(35) together imply that

$$L_i \text{ increases only at times } t \text{ when } Q(t) \in F_i, \qquad i = 1, 2, 3, 4. \tag{36}$$

Defining a two-vector μ, a two-dimensional process $\xi(t)$ and a four-dimensional process $L(t)$ in the obvious way, we can then write (32)–(33) in matrix-vector form as

$$Q(t) = [\xi(t) + \mu t] + RL(t), \qquad t \geq 0, \tag{37}$$

where

$$R = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \end{pmatrix}. \tag{38}$$

Comparing (36) and (37) with (2)–(4) we see that if ξ were a Brownian motion, then Q would be an RBM of the type defined in Section 2. Thus, to form an approximate Brownian system model, it remains only to determine the natural Brownian approximation for ξ. The processes B_0, B_1 and B_2 are continuous and nondecreasing, and using regenerative process theory, one can show that there exists a constant $\gamma > 0$ (the system throughput rate) such that

$$(39) \qquad \lambda B_0(t) \sim \gamma t, \qquad B_1(t) \sim \gamma t \quad \text{and} \quad B_2(t) \sim \gamma t \quad \text{almost surely}$$

as $t \to \infty$. From this and familiar properties of the Poisson process, it follows that the processes $\{n^{-1/2}\xi(nt),\ t \geq 0\}$, indexed by $n = 1, 2, \dots$, converge weakly as $n \to \infty$ to a Brownian motion with zero drift and covariance matrix

$$(40) \qquad \Gamma = \gamma I,$$

where I is the 2×2 identity matrix. Thus the natural Brownian approximation for ξ has drift zero and covariance matrix γI and we are led to

(41) approximate Q by a reflected Brownian motion Z whose drift vector is $\mu = (\lambda - 1, 0)^T$, whose covariance matrix is $\Gamma = \gamma I$, whose state space S is a 25×25 rectangle and whose reflection matrix R is given by (38).

An obvious problem with the proposed approximation (41) is that the throughput rate γ is unknown; in fact, a central purpose of our analysis is to determine γ. The obvious remedy for this problem is to adopt the iterative procedure described in the following paragraph.

Suppose for the moment that the covariance matrix Γ is given and let $Z(t) \equiv X(t) + RL(t)$ be the RBM that approximates $Q(t)$ in accordance with (41). Recall from that last paragraph of Section 2 that δ_k is the constant such that

$$(42) \qquad E[L_k(t)] \sim \delta_k t \quad \text{as } t \to \infty, \qquad k = 1, 2, 3, 4.$$

If one puts the linear test functions $f(x) = x_1$ and $f(x) = x_2$ into the basic adjoint relationship (11), specializing this relationship to the particular drift vector and reflection matrix proposed in (6), one obtains

$$(43) \qquad (\lambda - 1) + (\delta_1 + \delta_4) - \delta_3 = 0,$$

$$(44) \qquad -(\delta_1 + \delta_4) + \delta_2 = 0.$$

Now how does one estimate the throughput rate γ for our tandem queue from the steady-state performance characteristics of the Brownian system model? In the definition of RBM in Section 2, the boundary processes L_k are primitive, but to arrive at the ultimate representation (37) for our queue length process $Q(t)$ we defined $L_1(t), \dots, L_4(t)$ in terms of $I_1(t), \dots, I_3(t)$, which were defined in turn via $I_k(t) = t - B_k(t)$, $k = 1, 2, 3$. Thus, given the boundary processes L_k of the Brownian system model, one naturally defines an associated triple of processes $B_1(t), B_2(t), B_3(t)$ by inverting those original relationships, eventually arriving at

$$B_0(t) = t - \frac{1}{\lambda} L_3(t), \qquad B_2(t) = t - L_2(t), \tag{45}$$
$$B_1(t) = t - [L_1(t) + L_4(t)].$$

Given (39), (42) and (45), any of the following three equations provides an equally attractive estimate of the throughput rate γ in terms of the constants δ_k associated with the reflected Brownian motion Z:

$$\gamma = \lambda - \delta_3, \tag{46}$$

$$\gamma = 1 - \delta_2, \tag{47}$$

$$\gamma = 1 - (\delta_1 + \delta_4). \tag{48}$$

From (43) and (44) we see that (46)–(48) are equivalent relationships, which is as it should be. Then the following iterative procedure naturally suggests itself: Start with a trial value of γ (say, $\gamma = 1$), set $\Gamma = \gamma I$ to complete the data set of the RBM, from that data compute the steady-state performance characteristic δ_2 of the Brownian system model, use (47) to determine a new value of γ, and repeat the process until convergence is obtained. Table 3 shows the results that we obtained with this procedure using data for the tandem queue with $\lambda = 0.9$ and using the algorithm described in Section 4 (with $n = 7$) to compute the value of δ_2 for each trial value of γ. The QNET estimates for q_1, q_2 and γ that we reported earlier in Table 1 for $\lambda = 0.9$ are those in the final column of Table 3. The other QNET estimates reported in Table 1 were obtained in identical fashion, except those corresponding to $\lambda = 1.0$. The tandem queue with $\lambda = 1.0$ gives rise to an approximate Brownian system model with zero drift, and in that special case one can use a scaling argument to determine the QNET estimate of γ in a single pass, without iterative computation. The scaling argument is of interest in its own right, so we will explain it in detail. To begin, consider a standardized RBM $Z^*(t) \equiv X^*(t) + RL^*(t)$ with covariance matrix $\Gamma^* = 2I$ and drift vector $\mu^* = 0$, whose state space S^* is a 1×1 square and whose reflection matrix R is given by (38). The steady state performance characteristics of Z^* were computed and dis-

TABLE 3
Iterative calculation of γ for the case $\lambda = 0.9$

Iteration number	**1**	**2**	**3**
Trial value of γ	1.0	0.898706	0.899547
Computed value of q_1	5.3243	4.8450	4.8490
Computed value of q_2	6.7470	6.3146	6.3184
Computed value of δ_2	0.10294	0.100453	0.100459
Computed value of γ	0.898706	0.899547	0.899541

played earlier in Table 2, where we found that

(49) $$\delta_2^* = 1.61, \quad q_1^* = 0.55 \quad \text{and} \quad q_2^* = 0.45.$$

Given constants $a > 0$ and $\beta > 0$, one can define three new processes X, L and Z via

(50) $$X(t) = aX^*(\beta t), \qquad L(t) = aL^*(\beta t) \quad \text{and} \quad Z(t) = aZ^*(\beta t).$$

Obviously $Z(t) = X(t) + RL(t)$, and one can easily verify that Z is an RBM with covariance matrix $\Gamma = 2a^2\beta I$ and drift vector $\mu = 0$, whose state space S is an $a \times a$ square and whose reflection matrix R is the same as that for Z^*. Recall that the constants δ_i^* (associated with Z^*) and δ_i (associated with Z) are defined so that

(51) $$E[L_i^*(t)] \sim \delta_i^* t \quad \text{and} \quad E[L_i(t)] \sim \delta_i t$$

as $t \to \infty$, $i = 1, 2, 3, 4$. Thus, from (50) and (51),

(52) $$\delta_i = a\beta\delta_i^*, \qquad i = 1, 2, 3, 4,$$

whereas the constants q_i^* (associated with Z^*) and q_i (associated with Z) are related by

(53) $$q_i = aq_i^*, \qquad i = 1, 2.$$

When $\lambda = 1.0$, the reflected Brownian motion Z that approximates our two-dimensional queue length process Q has covariance matrix $\Gamma = \gamma I$ and drift vector $\mu = 0$; its state space S is a 25×25 square and its reflection matrix R is given by (38). Thus, to represent this process Z in terms of the standardized process Z^* via (50), we must have $a = 25$ and $2a^2\beta I = \gamma I$; so that required scaling is

(54) $$a = 25 \quad \text{and} \quad \beta = \gamma/2(25)^2.$$

As explained earlier, we want to find a value for γ such that the corresponding value of δ_2 satisfies

(55) $$1 - \delta_2 = \gamma.$$

Thus, combining (49) with (52) through (55), one concludes that

(56) $$\gamma = 1 - \delta_2 = 1 - a\beta\delta_2^* = 1 - 25\left[\gamma/2(25)^2\right]\delta_2^*$$

and hence

(57) $$\gamma = \left[(1 + \delta_2^*/2(25))\right]^{-1} = (1 + 1.61/50)^{-1} = 0.9688.$$

Of course, (49) and (53) together give

(58) $$q_1^* = 25(0.55) = 13.75 \quad \text{and} \quad q_2^* = 25(0.45) = 11.25$$

and (56) and (57) are the QNET estimates reported in Table 1 for $\lambda = 1.0$.

7. Concluding remarks. Let us return to the setting of Section 3, where the problem of computing the stationary density p was cast as a least squares problem. The treatment given there can be generalized in the following way, which may be important for both practical and theoretical purposes. (In this section the letter q will be reused with a new meaning, but that should cause no confusion.) Let q_0 be a strictly positive function on the interior $\mathscr{O}$ of the rectangle S and let $q_1, \dots, q_4$ be strictly positive functions on the boundary surfaces $F_1, \dots, F_4$, respectively. Defining

$$q(x) = \begin{cases} q_0(x) & \text{in } \mathscr{O}, \\ q_i(x) & \text{on } F_i,\ i = 1,2,3,4, \end{cases} \tag{59}$$

we call q a reference density and we define a corresponding reference measure ν via

$$\nu(dx) = \begin{cases} q_0(x)\,dx & \text{in } \mathscr{O}, \\ q_i(x)\,d\sigma & \text{on } F_i,\ i = 1,2,3,4. \end{cases} \tag{60}$$

If we work in the Hilbert space $L^2(S, \nu)$ rather than the space $L^2(S, \eta)$ used in Section 3, then the focus is on the unknown function r defined by

$$r(x) = \begin{cases} p_0(x)/q_0(x) & \text{in } \mathscr{O}, \\ p_i(x)/q_i(x) & \text{on } F_i,\ i = 1,2,3,4. \end{cases} \tag{61}$$

That is, with the inner product defined by $(f, g) = \int_S (f \cdot g)\,d\nu$, our basic adjoint relationship (11) says that $\mathscr{A}f \perp r$ for all $f \in C_2(S)$ and hence one may proceed exactly as in Sections 3 and 4 to devise an algorithm for approximate computation of r by projection in $L^2(S, \nu)$. Of course, the final estimate of r is converted to an estimate of p via $p = rq$, where q is the reference density chosen.

A different computational procedure is obtained depending on how one chooses the reference density q and the functions $f_1, f_2, \dots$ that are used to build up the approximating subspaces $H_1, H_2, \dots$ via $H_n = \operatorname{span}\{\mathscr{A}f_1, \dots, \mathscr{A}f_n\}$; recall that in Section 4 we took $f_1, f_2, \dots$ to be polynomial functions, but other choices are obviously possible. One wants to choose q and $f_1, f_2, \dots$ in such a way that the inner products $(\mathscr{A}f_m, \mathscr{A}f_n)$ can be determined analytically and in such a way as to accelerate convergence of the algorithm. From a theoretical standpoint, the freedom to choose q is important because one may have $r \in L^2(S, \nu)$ even though $p \notin L^2(S, \eta)$ and thus a judicious choice of reference density may enable a rigorous proof of convergence in $L^2(S, \nu)$. From a practical standpoint, one may be able to choose q in such a way that convergence is accelerated, taking q to be a best guess of the unknown density p based on either theory or prior computations. In a future paper [4] we will discuss computation of stationary distributions on *unbounded* regions, where a proper choice of reference density is essential to efficient computation.

Acknowledgments. The authors would like to thank Ruth Williams for her invaluable assistance and Christoph Loch for providing the simulation estimates in Table 1.

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