The Annals of Applied Probability
1993, Vol. 3, No. 1, 154–172

UNIQUENESS OF STATIONARY ERGODIC FIXED POINT FOR A ./M/K NODE[1]

BY V. ANANTHARAM

Cornell University

We view a ./M/K node having K exponential servers of service rate μ as a map on the space of stationary ergodic arrival processes of rate λ, $\lambda < K\mu$. It is well known that the Poisson process of rate λ is a fixed point of this map. We prove there is no other fixed point.

1. Introduction. Networks of quasireversible queues with Bernoulli routing [8, 15] are widely used as models for performance analysis of computer, communication and manufacturing networks. This is because they admit *product form* stationary distributions which makes the computation of stationary performance quantities feasible. Several kinds of queueing nodes are known to be quasireversible. Perhaps the simplest among these are the FCFS exponential server nodes ./M/1, ./M/K and ./M/∞.

We may think of a queueing node in stationarity as a map converting stationary arrival processes into stationary departure processes (this has to be appropriately formulated). One of the characteristics of quasireversible nodes is that they admit Poisson processes as fixed points of the input–output map. This fact seems essential to the probabilistic understanding of the product form stationary distribution when such nodes are interconnected via Bernoulli routing to form networks [14]. Because of the close connection of the existence of Poisson fixed points with the product form nature of network stationary distributions it seems to be of some interest to learn if the input–output map of quasireversible nodes admits any other fixed points, apart from mixtures of Poisson processes, which are also trivially fixed. This problem has been floating around the community for some time.

For ./G/∞ nodes acting on stationary ergodic input processes, Vere–Jones [13] established the uniqueness of the Poisson fixed point (for a precise formulation, see [13]). A recent paper of Glynn and Whitt [5] mentions the related problem of proving that the stationary departure process from a long tandem of identical ./G/1 nodes fed by a renewal process becomes asymptotically Poisson as the length of the tandem tends to infinity. For ./M/1 nodes a natural approach to this problem would be to first prove that Poisson processes are the only stationary ergodic fixed points. Another recent contribution to the study of fixed points of the input–output map of first come first served queues in a very general setup is due to Bambos and Walrand [3].

Received May, 1991; revised February 1992.

[1]Research supported by NSF PYI award NCR 8857731, an IBM Faculty Development award, BellCore Inc. and IRI 9005849.

AMS 1991 *subject classifications*. 60K25, 60K20, 60G55.

Key words and phrases. Queueing networks, quasireversibility, stationary ergodic fixed points, Poisson processes.

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In this paper we prove that Poisson processes are the only stationary ergodic fixed points of ./M/K nodes, $K < \infty$. The proof technique is quite dependent on the exponential server assumption, so a new idea will be necessary to study more general quasireversible nodes. Unfortunately at the moment of writing we are unable to verify the related generally held belief that the stationary departure process from a long tandem of identical ./M/K nodes fed with a stationary ergodic process converges weakly to the Poisson process of the appropriate rate. For a discussion of the difficulty here, see the concluding remarks.

We will first discuss fixed points of the input–output map of a ./M/1 node. In Section 2 we review some basics of the theory of stationary point processes following Baccelli and Bremaud [2] and Resnick [12]. We also introduce the sample spaces on which we will work. In Section 3 we recall the concept of a stationary regime for the ./M/1 node fed with a stationary arrival process and prove the existence and uniqueness of stationary regimes when the arrival rate is less than the service rate. The proof is a version of the Loynes scheme; see Loynes [9], Baccelli and Bremaud [2] and Walrand [15] for discussions of this kind of construction. This construction serves to define the input–output map of a ./M/1 node of service rate μ as a map on stationary ergodic arrival processes of rate λ, $0 < \lambda < \mu$. In Section 4 we introduce a metric on the space of point processes of rate λ relative to a Poisson process of rate μ. This metric was motivated by a similar metric introduced in Anantharam [1] to study the analogous problem for quasireversible queues in discrete time; the metric in [1] is a member of a family of generalizations of Ornstein's $\bar{d}$ metric [10] introduced by Gray, Neuhoff and Shields [7]. We derive several basic properties of this metric, which are analogous to properties of the metrics in [7] derived in [7] and the textbook of Gray [6]. In Section 5 coupling is used to demonstrate that the input–output map of a ./M/1 node is contractive relative to the metric introduced in Section 4. This enables us to state and prove the main result for ./M/1 nodes in Section 6. In Section 7 we indicate the modifications necessary in the preceding proof to arrive at the uniqueness of the Poisson fixed point for ./M/K nodes, $K < \infty$. Some concluding remarks are made in Section 8.

2. Preliminaries.

In this section we recall some basic concepts from the theory of stationary point processes. The development and notation closely follows the discussion in [2]. Statements regarding the topology and measurable structure of the spaces we consider are mostly proved in [12], Chapter 3.

Let $(\mathbb{R}, \mathscr{B})$ denote the real numbers with the Borel σ field. A *counting measure* is a measure m on $(\mathbb{R}, \mathscr{B})$ such that:

(i) $m(C) \in \{0, 1, \dots, \infty\}$ for all $C \in \mathscr{B}$.
(ii) $m([a, b]) < \infty$ for all finite intervals $[a, b] \subset \mathbb{R}$.

The set of all counting measures is denoted by M. Let $C_c(\mathbb{R})$ denote the set of continuous functions on $\mathbb{R}$ with compact support. We endow M with the vague topology, which is the weakest topology under which the maps $m \to m(f)$ are continuous for all $f \in C_c(\mathbb{R})$. Let $\mathscr{M}$ be the σ-algebra generated by the functions $m \to m(C)$ as C ranges over $\mathscr{B}$. Then $\mathscr{M}$ is the Borel σ-algebra of M with the vague topology (see [12], Exercise 3.4.5). $(M, \mathscr{M})$ is called the *canonical space of point processes* on $\mathbb{R}$. It is known that the vague topology on M is metrizable as a complete separable metric space ([12], Proposition 3.17).

Given $t \in \mathbb{R}$, let δ_t denote the Dirac measure at t. Associated with each counting measure $m \in M$ there is a unique sequence of real numbers $(t_n, n \in \mathbb{Z})$ such that

$$-\infty \leq \cdots \leq t_{-1} \leq t_0 \leq 0 < t_1 \leq t_2 \leq \cdots \leq \infty$$

and

$$m(C) = \sum_{n \in \mathbb{Z}} \delta_{t_n}(C)$$

for each $C \in \mathscr{B}$, with the convention that $\delta_\infty = \delta_{-\infty} = 0$. The time t_n is called the nth point of m. Note that each t_n is a random variable on $(M, \mathscr{M})$.

For each $t \in \mathbb{R}$ we may define the *left shift* S_t: $M \to M$ by $S_t m(C) = m(C + t)$, $C \in \mathscr{B}$. S_t is an invertible measurable transformation of $(M, \mathscr{M})$. Further, the family $\{S_t\}$, $t \in \mathbb{R}$ forms a *measurable flow* on $(M, \mathscr{M})$, that is:

(i) $(t, m) \to S_t m$ is measurable with respect to $\mathscr{B} \times \mathscr{M}$ and $\mathscr{M}$.
(ii) S_t is invertible for all $t \in \mathbb{R}$.
(iii) $S_t \circ S_s = S_{t+s}$ for all t, $s \in \mathbb{R}$ and S_0 is the identity.

Let $(\Omega, \mathscr{F}, P)$ support a P preserving measurable flow $\{\theta_t\}$. A *stationary point process* is a measurable map N: $(\Omega, \mathscr{F}, P) \to (M, \mathscr{M})$ which commutes with the flows, that is,

$$N(\theta_t \omega) = S_t N(\omega) \quad \text{for all } t \in \mathbb{R}.$$

Let $\boldsymbol{\mu}$ denote the distribution of the point process N, that is,

$$P(\{\omega: N(\omega) \in A\}) = \boldsymbol{\mu}(A) \quad \text{for all } A \in \mathscr{M}.$$

A point process is often loosely identified with its distribution $\boldsymbol{\mu}$; when necessary we will therefore talk of "representations" of a point process, when we actually mean a point process. When $(\Omega, \mathscr{F}, P) = (M, \mathscr{M}, \boldsymbol{\mu})$ the point process is said to be in canonical representation. The process is called ergodic if $\boldsymbol{\mu}$ is $\{S_t\}$ ergodic. If P is $\{\theta_t\}$-ergodic, the process is said to be given in an ergodic representation. For a measure m and a function f, we write $m(f)$ for $\int f\,dm$. The rate of the point process $\boldsymbol{\mu}$ is $\boldsymbol{\mu}(m(0,1])$. Let $\mathscr{M}_S(\lambda)$ [respectively, $\mathscr{M}_S^e(\lambda)$] denote the space of stationary point process (respectively, stationary and ergodic point processes) with rate λ. We note the following:

LEMMA 1. *$\mathscr{M}_S(\lambda)$ is compact in the weak topology induced by the vague topology on M.*

PROOF. By a simple application of Markov's inequality, we can choose K_N, $N = 1, 2, \ldots$ such that for any $\boldsymbol{\mu} \in \mathscr{M}_S(\lambda)$, $\boldsymbol{\mu}(\{m(-N, N] > K_N\}) \leq \varepsilon 2^{-N}$. The set

$$A = \{m\colon m(-N, N] \leq K_N \text{ for all } N = 1, 2, \dots\}$$

is vaguely relatively compact in M, because $\sup_{m \in A} m(f) < \infty$ for all $f \in C_c(\mathbb{R})$ (see [12], Proposition 3.16). But $\boldsymbol{\mu}(\bar{A}) \geq 1 - \varepsilon$ for all $\boldsymbol{\mu} \in \mathscr{M}_S(\lambda)$, so $\mathscr{M}_S(\lambda)$ is tight and therefore relatively compact, by Prohorov's theorem (see Billingsley [4], Theorem 6.1). It is easily verified that $\mathscr{M}_S(\lambda)$ is weakly closed, completing the proof. □

We view our ./M/1 node as specified by its virtual departure process, which is a Poisson process of rate μ. Throughout this paper we will choose to represent the virtual departure process in its canonical representation. We use the notation $(\Omega_d, \mathscr{F}_d, \pi_\mu, \{\theta_t^d\})$ for the sample space of this canonical representation, with the virtual departure process specified by the $(M, \mathscr{M})$ valued random variable N_d. The nth virtual departure point is denoted t_n^d. Further, it is easily seen that π_μ is $\{\theta_t^d\}$-ergodic.

Consider feeding the node with a stationary arrival process $\mu_a \in \mathscr{M}_S(\lambda)$, $0 < \lambda < \mu$, which is independent of the virtual departure process. Let the arrival process be represented on $(\Omega_a, \mathscr{F}_a, P_a, \{\theta_t^a\})$ by the $(M, \mathscr{M})$ valued random variable N_a. In what follows we may need to use different representations for the same arrival process. The node and the arrival process can be represented together on the product space $(\Omega, \mathscr{F}, P, \{\theta_t\})$ where $\Omega = \Omega_a \times \Omega_d$ endowed with the product σ-field $\mathscr{F} = \mathscr{F}_a \times \mathscr{F}_d$ and the product distribution $P = P_a \times P_d$ and supporting the product flow $\{\theta_t\} = \{\theta_t^a \times \theta_t^d\}$, $t \in \mathbb{R}$.

Note that an ergodic μ_a can be represented by a nonergodic P_a, but a nonergodic μ_a cannot be represented by an ergodic P_a. This is an easy consequence of the definition of ergodicity. Note that if P_a is $\{\theta_t^a\}$-ergodic then P is $\{\theta_t\}$-ergodic.

We will work on the Palm space of $(\Omega, \mathscr{F}, P)$ associated to the virtual departure process. This space is $(\Omega^0, \mathscr{F}^0, P^0)$ with $\Omega^0 = \Omega \cap \{t_0^d(\omega) = 0\}$, the induced σ-field and the Palm distribution. This space admits a P^0 preserving transformation θ given by the restriction of $\theta_{t_1^d}$ to Ω^0. For the development of Palm theory, see [2].

3. Existence and uniqueness of stationary regime. It is well known that one can uniquely specify the input–output map of a ./M/1 node of service rate μ as a map on $\mathscr{M}_S^e(\lambda)$ when $0 < \lambda < \mu$. This is a consequence of the existence and uniqueness results for the Loynes problem [9] for this queue, and falls out of the more general results for G/G/1 queues (see [9], [2] and [15], Chapter 7 for discussion of these results). Since we need the details of the construction, we briefly recapitulate it in our setup.

Let the arrival and virtual departure processes be represented by $(M, \mathscr{M})$ valued random variables N_a and N_d defined on $(\Omega, \mathscr{F}, P, \{\theta_t\})$. Let $(\Omega^0, \mathscr{F}^0, P^0, \theta)$ be the Palm space of the virtual departure process. The node is said to admit a *stationary regime* if there is a nonnegative integer valued

P^0-a.s. finite random variable x_0 on $(\Omega^0, \mathscr{F}^0, P^0, \theta)$ such that

$$x_0 \circ \theta = x_0 + N_a(0, t_1^d] - 1\big(x_0 + N_a(0, t_1^d] > 0\big). \tag{3.1}$$

The reason for this terminology is that with $x_n = x_0 \circ \theta^n$, $(x_n,\ n \in \mathbb{Z})$ is a stationary version of the queue size left behind by virtual departures. Note that from a stationary regime on $(\Omega^0, \mathscr{F}^0, P^0, \theta)$ as above, one can construct a time stationary right-continuous version of the queue size process on $(\Omega, \mathscr{F}, P, \{\theta_t\})$ by the prescription

$$x_t = x_0 \circ \theta_{t_n^d} + N_a(t_n^d, t] \quad \text{if } t_n^d \le t < t_{n+1}^d.$$

THEOREM 1. *Consider a ./M/1 node of service rate μ, with an ergodic arrival process $\mu_a \in \mathscr{M}_S^e(\lambda)$ given by the $(M, \mathscr{M})$ valued random variable N_a on $(\Omega_a, \mathscr{F}_a, P_a, \{\theta_t^a\})$. The overall system given on the product representation $(\Omega, \mathscr{F}, P, \{\theta_t\})$ admits a unique stationary regime.*

PROOF. For each $m \in \mathbb{Z}$ we construct a process $(x_n^m,\ n \ge -m)$ on the Palm space of the virtual departure process by

$$x_{-m}^m = 0, \tag{3.2}$$

$$x_{n+1}^m = x_n^m + N_a(t_n^d, t_{n+1}^d] - 1\big(x_n^m + N_a(t_n^d, t_{n+1}^d] > 0\big). \tag{3.3}$$

One thinks of x_n^m as the queue size that would be left behind by the nth virtual departure if the node were started empty immediately after the $-m$th virtual departure. We claim that $x_n^{m+1} \ge x_n^m$ for all $n \ge -m$. This is easily seen by induction on n, starting with the observation that $x_{-m}^{m+1} \ge 0 = x_{-m}^m$ and then using (3.3). Thus we may define

$$x_n^\infty = \lim_{m \to \infty} x_n^m \tag{3.4}$$

for all $n \in \mathbb{Z}$. Since $x_{n+1}^m = x_n^{m+1} \circ \theta$, we have $x_{n+1}^\infty = x_n^\infty \circ \theta$. In particular, examining (3.3) for $n = 0$ in the $m = \infty$ limit and comparing with (3.1) shows that x_0^∞ is a stationary regime if it is a.s. finite.

The P^0-a.s. finiteness of x_0^∞ is argued easily from the rate condition $\lambda < \mu$. First note that the $\{\theta_t\}$-ergodicity of P implies the θ-ergodicity of P^0 [see [2], statement (8.2.1)]. Since $\{x_0^\infty = \infty\}$ is a θ-invariant event, it has P^0 probability 0 or 1. Thus to show x_0^∞ is P^0-a.s. finite it suffices to prove that $P^0(x_0^\infty = \infty) < 1$. Suppose on the contrary that $P^0(x_0^\infty = \infty) = 1$. From (3.3), using $x_n^m \le x_n^{m+1}$ and $x_{n+1}^m = x_n^{m+1} \circ \theta$, we have

$$E^0 x_n^m \le E^0 x_n^{m+1} = E^0 x_{n+1}^m = E^0 x_n^m + \lambda/\mu - E^0\big[1\big(x_n^m + N_a(t_n^d, t_{n+1}^d] > 0\big)\big].$$

Hence

$$E^0\big[1\big(x_n^m + N_a(t_n^d, t_{n+1}^d] > 0\big)\big] \le \lambda/\mu < 1. \tag{3.5}$$

But if x_n^m increases to $x_n^\infty = \infty$, the expectation on the left-hand side of (3.5) approaches 1, which is a contradiction.

The uniqueness of the stationary regime can also be argued in a standard fashion. Assume that $\tilde{x}_0$ is another stationary regime. By construction, it is easy to see that x_0^∞ is a minimal stationary regime, that is, $\tilde{x}_0 \geq x_0^\infty$ P^0-a.s. Since $\tilde{x}_0$ is a stationary regime, we have

$$\tilde{x}_0 \circ \theta = \tilde{x}_0 + N_a(0, t_1^d] - 1\big(\tilde{x}_0 + N_a(0, t_1^d] > 0\big). \tag{3.6}$$

From (3.1) for x_0^∞ and (3.6) we get

$$\begin{aligned} \tilde{x}_0 \circ \theta - x_0^\infty \circ \theta &= \tilde{x}_0 - x_0^\infty - 1\big(\tilde{x}_0 + N_a(0, t_1^d] > 0\big) \\ &\quad + 1\big(x_0^\infty + N_a(0, t_1^d] > 0\big) \\ &\leq \tilde{x}_0 - x_0^\infty. \end{aligned} \tag{3.7}$$

It follows that for each $0 \leq K < \infty$ the event $\{\tilde{x}_0 - x_0^\infty \leq K\}$ is θ-invariant. By the θ-ergodicity of P^0, there is some $0 \leq K < \infty$ for which $P^0(\tilde{x}_0 = x_0^\infty + K) = 1$. Uniqueness of stationary regime will follow if we can show that $K = 0$. Suppose on the contrary that $K > 0$. Then $\tilde{x}_0 > 0$, so (3.7) yields

$$K = K - 1 + 1\big(x_0^\infty + N_a(0, t_1^d] > 0\big). \tag{3.8}$$

But from monotone convergence, (3.5) yields

$$E^0\big[1\big(x_0^\infty + N_a(0, t_1^d] > 0\big)\big] \leq \lambda/\mu < 1,$$

which is in contradiction with (3.8). □

We now define the stationary departure process of the node on $(\Omega, \mathscr{F}, P, \{\theta_t\})$ by

$$N_e = \sum_{n \in \mathbb{Z}} \delta_{t_{n+1}^d} 1\big(x_0^\infty \circ \theta_{t_n^d} + N_a(t_n^d, t_{n+1}^d] > 0\big).$$

We next prove that this is a stationary ergodic process of rate λ, with a distribution that is independent of the representation chosen for the arrival process. This allows us to define the input–output map of the ./M/1 node, as a map on $M_S^e(\lambda)$. We denote this map by T. Our goal in the rest of the paper is to prove that the Poisson process of rate λ is the unique fixed point of this map.

THEOREM 2. *Let the situation be as in the statement of Theorem* 1, *and let us define the M valued function on $(\Omega, \mathscr{F}, P, \{\theta_t\})$ by*

$$N_e = \sum_{n \in Z} \delta_{t_{n+1}^d} 1\big(x_0^\infty \circ \theta_{t_n^d} + N_a(t_n^d, t_{n+1}^d] > 0\big).$$

Then N_e is a stationary point process. Its distribution μ_e is an element of $\mathscr{M}_S^e(\lambda)$. The same element of $\mathscr{M}_S^e(\lambda)$ results for any choice of arrival representation $(\Omega_a, \mathscr{F}_a, P_a, \{\theta_t^a\})$ for a given distribution μ_a.

PROOF. The sequence $Z_{n+1} = 1(x_0^\infty \circ \theta_{t_n^d} + N_a(t_n^d, t_{n+1}^d] > 0)$, $n \in \mathbb{Z}$, is easily verified to be a sequence of $\{0, 1\}$-valued marks on the point process N (see

[2], Section 1.3 for a discussion of marked point processes). It follows that N_e is an $(M, \mathscr{M})$ valued random variable on $(\Omega, \mathscr{F}, P, \{\theta_t\})$ commuting with the shifts, that is, it is a point process. The rate of N_e is $E[N_e(0,1]]$ by definition; we first verify that this is λ. To see this, note that

$$E[N_e(0,1]] = E[N_d(0,1]]E^0\left[1\left(x_0^\infty \circ \theta_{t_n^d} + N_a\left(t_n^d, t_{n+1}^d\right] > 0\right)\right]$$

by the definition of Palm probability [see [2], equation (2.2.4)]. Since $E[N_d(0,1]] = \mu$, we need to verify that the second term on the right is λ/μ.
From (3.3), we have

$$E^0\left[1\left(x_n^m + N_a\left(t_n^d, t_{n+1}^d\right] > 0\right)\right] = \lambda/\mu - E^0\left[x_n^{m+1} - x_n^m\right]. \tag{3.9}$$

Since x_n^m increases pointwise to the P^0-a.s. finite random variable x_n^∞ as $m \to \infty$, we have $x_n^{m+1} - x_n^m$ converging pointwise to 0 as $m \to \infty$. Further, we can write

$$\begin{aligned} x_n^{m+1} - x_n^m &= x_{n-1}^{m+1} - x_{n-1}^m - 1\left(x_{n-1}^{m+1} + N_a\left(t_{n-1}^d, t_n^d\right] > 0\right) \\ &\quad + 1\left(x_{n-1}^m + N_a\left(t_{n-1}^d, t_n^d\right] > 0\right) \\ &\le x_{n-1}^{m+1} - x_{n-1}^m. \end{aligned}$$

By working backwards in n we then get

$$x_n^{m+1} - x_n^m \le x_{-m}^{m+1} - x_{-m}^m = x_{-m}^{m+1} \le N_a\left(t_{-m-1}^d, t_{-m}^d\right].$$

Thus the variables $x_n^{m+1} - x_n^m$, $m \ge -n$, are uniformly integrable. Since they converge pointwise to zero, it follows that $E^0[x_n^{m+1} - x_n^m] \to 0$ as $m \to \infty$, so from (3.9) and monotone convergence, we get

$$E^0\left[1\left(x_n^\infty + N_a\left(t_n^d, t_{n+1}^d\right] > 0\right)\right] = \lambda/\mu$$

as desired.

μ_e is ergodic because the representation N_e is ergodic and has distribution μ_e. Thus $\mu_e \in \mathscr{M}_S^e(\lambda)$.

It remains to show that μ_e does not depend on the representation chosen for μ_a. Let $(M^{(2)}, \mathscr{M}^{(2)}, \{S_t^{(2)}\})$ be as defined at the beginning of Section 4. We observe that the map (N_a, N_d) from $(\Omega, \mathscr{F}, P, \{\theta_t\})$ to $(M^{(2)}, \mathscr{M}^{(2)}, \{S_t^{(2)}\})$ commutes with the flows and has distribution $\mu_a \times \pi_\mu$. Let the Palm space of the virtual departure process on $(M^{(2)}, \mathscr{M}^{(2)}, \mu_a \times \pi_\mu, \{S_t^{(2)}\})$ be denoted $(\Omega_c^0, \mathscr{F}_c^0, P_c^0, \theta_c)$ (c for canonical). Let $(\Omega^0, \mathscr{F}^0, P^0, \theta)$ be the Palm space of the virtual departure process on $(\Omega, \mathscr{F}, P, \{\theta_t\})$. Then $\Omega_0 = (N_a, N_d)^{-1}(\Omega_c^0)$ so that (N_a, N_d) can be viewed as a map on Palm spaces. This map commutes with the shifts, and P_c^0 is the distribution of P^0 under this map. The unique stationary regime constructed on $(\Omega_c^0, \mathscr{F}_c^0, P_c^0, \theta_c)$ by the Loynes construction when composed with (N_a, N_d) gives a stationary regime on $(\Omega^0, \mathscr{F}^0, P^0, \theta)$, which, by the uniqueness in Theorem 1, must necessarily be the unique stationary regime there. The fact that μ_e does not depend on the representation of μ_a is now obvious. □

In conclusion, we have proved that for any $\lambda < \mu$ there is a well defined map $T\colon \mathscr{M}_S^e(\lambda) \to \mathscr{M}_S^e(\lambda)$, which is the input–output map of a ./M/1 node of service rate μ.

4. A metric on arrival processes. In this section we introduce a metric on $\mathscr{M}_S(\lambda)$. Let $\mathscr{M}_S(\lambda, \lambda)$ denote the space of all probability distributions on $(M^{(2)}, \mathscr{M}^{(2)}) = (M \times M, \mathscr{M} \times \mathscr{M})$ that are $\{S_t^{(2)}\} = \{S_t \times S_t\}$ stationary and have rate λ in each marginal. Given $\mu_a, \nu_a \in \mathscr{M}_S(\lambda)$, a stationary coupling of μ_a and ν_a is specified by giving $\alpha \in \mathscr{M}_S(\lambda, \lambda)$ with marginals μ_a and ν_a, respectively. We let N_a (respectively $N_{\tilde{a}}$) be the map from $(M^{(2)}, \mathscr{M}^{(2)})$ to $(M, \mathscr{M})$ giving the first (respectively, the second) marginal. These are point processes of rate λ under α, having distributions μ_a and ν_a, respectively.

We take the canonical representation $(\Omega, \mathscr{F}, \pi_\mu, \{\theta_t^d\})$ of the Poisson process of rate μ and let $(\Omega, \mathscr{F}, P, \{\theta_t\})$ denote the product of $(M^{(2)}, \mathscr{M}^{(2)}, \alpha, \{S_t^{(2)}\})$ with $(\Omega, \mathscr{F}, \pi_\mu, \{\theta_t^d\})$. Let $(\Omega^0, \mathscr{F}^0, P^0, \theta)$ denote the Palm space of the virtual departure process, where θ is the restriction of $\theta_{t_1^d}$ to Ω^0.

For $\mu > 0$, we define

$$(4.1) \qquad \bar{\rho}_\mu(\mu_a, \nu_a) = \inf_\alpha E^0 \big| N_a(0, t_1^d] - N_{\tilde{a}}(0, t_1^d] \big|.$$

In effect one considers the expected value of the absolute difference between the number of arrivals in the first process and the second process over an exponential time of rate μ independent of the coupling and takes the infimum over all stationary couplings.

The properties of $\bar{\rho}_\mu$ relevant to our discussion are summarized in the following result.

THEOREM 3. *The $\bar{\rho}_\mu$ distance on $\mathscr{M}_S(\lambda)$ introduced above has the following properties*:

(i) *$\bar{\rho}$ is a metric.*

(ii) *The infimum in the definition* (5.1) *is a minimum, that is, there is stationary coupling that achieves the $\bar{\rho}$ distance.*

(iii) *If $\mu_a, \nu_a \in \mathscr{M}_S^e(\lambda)$, the infimum in* (5.1) *can be replaced by an infimum over stationary ergodic α. Further, this infimum is a minimum.*

PROOF. We first prove (ii) and then (i) and (iii). First note that a definition of $\bar{\rho}_\mu(\mu_a, \nu_a)$ equivalent to (4.1) is

$$(4.2) \qquad \bar{\rho}_\mu(\mu_a, \nu_a) = \inf_\alpha \int_0^\infty \mu \exp(-\mu t) \alpha\big(\big| N_a(0, t] - N_{\tilde{a}}(0, t] \big|\big)\, dt.$$

This is because the virtual departure process is a Poisson process independent of the coupled arrival processes. The alternative definition allows us to avoid dealing with Palm probabilities. Now, for any $\gamma \in \mathscr{M}_S(\lambda, \lambda)$,

$$(4.3) \qquad \int_T^\infty \mu \exp(-\mu t) \gamma\big(\big| N_a(0, t] - N_{\tilde{a}}(0, t] \big|\big)\, dt \le \int_T^\infty \mu \exp(-\mu t) 2\lambda t\, dt$$

so for any $\varepsilon > 0$, by choosing a large enough T, we can bound the left-hand side of (4.3) by ε uniformly over γ. Next, we note that for any $0 \le t \le T$ and $L > 0$, we have

$$|N_a(0,t] - N_{\tilde{a}}(0,t]|1(|N_a(0,t] - N_{\tilde{a}}(0,t]| \ge L)$$
$$\le N_a(0,T]1(N_a(0,T] \ge L) + N_{\tilde{a}}(0,T]1(N_{\tilde{a}}(0,T] \ge L).$$

We may thus write

$$\gamma(|N_a(0,t] - N_{\tilde{a}}(0,t]|1(|N_a(0,t] - N_{\tilde{a}}(0,t]| \ge L))$$
$$\le \gamma(N_a(0,T]1(N_a(0,T] \ge L) + N_{\tilde{a}}(0,T]1(N_{\tilde{a}}(0,T] \ge L)).$$

For any $\gamma \in \mathscr{M}_S(\lambda, \lambda)$ having marginals μ_a, ν_a, the expectation of the right-hand side above depends only on the marginals, and can be made arbitrarily small by choice of L. So for any $\varepsilon > 0$, we may choose L so large that

$$\text{(4.4)} \quad \int_0^T \mu \exp(-\mu t)\gamma(|N_a(0,t] - N_{\tilde{a}}(0,t]|1(|N_a(0,t] - N_{\tilde{a}}(0,t]| \ge L)) \le \varepsilon$$

uniformly over $\gamma \in \mathscr{M}_S(\lambda, \lambda)$ having marginals μ_a, ν_a. Let

$$f = \int_0^T \mu \exp(-\mu t)(|N_a(0,t] - N_{\tilde{a}}(0,t]|)1(|N_a(0,t] - N_{\tilde{a}}(0,t]| < L)\, dt.$$

Then f is a bounded continuous function on $M^{(2)}$ with the vague topology, and we have shown that

$$\text{(4.5)} \quad \gamma(f) \le \int_0^\infty \mu \exp(-\mu t)\gamma(|N_a(0,t] - N_{\tilde{a}}(0,t]|)\, dt \le \gamma(f) + 2\varepsilon,$$

where the left-hand inequality is obvious. Given $\mu_a, \nu_a \in \mathscr{M}_S(\lambda)$, let $\alpha_n \in \mathscr{M}_S(\lambda, \lambda)$ be a sequence with

$$\int_0^\infty \mu \exp(-\mu t)\alpha_n(|N_a(0,t] - N_{\tilde{a}}(0,t]|)\, dt \to \bar{\rho}_\mu(\mu_a, \nu_a).$$

An argument analogous to the proof of Lemma 1 shows that $\mathscr{M}_S(\lambda, \lambda)$ is compact. Hence $(\alpha_n)_n$ has a weakly convergent subsequence; let α be the weak limit of such a subsequence. Clearly α has first (respectively, second) marginal μ_a (respectively, ν_a). Taking the limit as $n \to \infty$ along this subsequence and using (4.5) gives

$$\begin{aligned} \int_0^\infty \mu \exp(-\mu t)\alpha(|N_a(0,t] - N_{\tilde{a}}(0,t]|)\, dt &\le \alpha(f) + 2\varepsilon \\ &= \lim_{n \to \infty} \alpha_n(f) + 2\varepsilon \\ &\le \bar{\rho}_\mu(\mu_a, \nu_a) + 2\varepsilon \end{aligned}$$

for any $\varepsilon > 0$. Letting $\varepsilon \to 0$, and by the definition of $\bar{\rho}_\mu(\mu_a, \nu_a)$ as an infimum, it must be true that

$$\int_0^\infty \mu \exp(-\mu t)\alpha(|N_a(0,t] - N_{\tilde{a}}(0,t]|)\,dt = \bar{\rho}_\mu(\mu_a, \nu_a).$$

This establishes (ii).

To establish (i), we need to establish (a) $\bar{\rho}_\mu(\mu_a, \nu_a) = \bar{\rho}_\mu(\nu_a, \mu_a)$, (b) $\bar{\rho}_\mu(\mu_a, \nu_a) = 0 \Rightarrow \mu_a = \nu_a$ and (c) the triangle inequality. Symmetry of $\bar{\rho}_\mu(\mu_a, \nu_a)$ in its two arguments, that is, (a), is obvious. To verify (b), suppose $\bar{\rho}_\mu(\mu_a, \nu_a) = 0$. Let $\alpha \in \mathscr{M}_S(\lambda, \lambda)$ be a stationary coupling achieving $\bar{\rho}_\mu(\mu_a, \nu_a)$, whose existence is ensured by (ii). We first claim that $\alpha(|N_a(0,t] - N_{\tilde{a}}(0,t]|)$ is continuous in t. This is because, for any $\delta > 0$,

$$\begin{aligned}||N_a(0,t+\delta] - N_{\tilde{a}}(0,t+\delta]| - |N_a(0,t] - N_{\tilde{a}}(0,t]||\\ \le N_a(t,t+\delta] + N_{\tilde{a}}(t,t+\delta]\end{aligned}$$

and $\alpha(N_a(t,t+\delta] + N_{\tilde{a}}(t,t+\delta]) = 2\lambda\delta$. From

$$\int_0^\infty \mu \exp(-\mu t)\alpha(|N_a(0,t] - N_{\tilde{a}}(0,t]|)\,dt = 0$$

and the continuity above, it follows that

$$\alpha(|N_a(0,t] - N_{\tilde{a}}(0,t]|) = 0 \tag{4.6}$$

for all $0 \le t < \infty$. Now, given any $0 = t_0 < t_1 < t_2 < \cdots < t_n$, we claim that

$$(\mu_a(0,t_1], \ldots, \mu_a(t_{n-1},t_n]) =_d (\nu_a(0,t_1], \ldots, \nu_a(t_{n-1},t_n]). \tag{4.7}$$

This is enough to establish the equality of μ_a and ν_a (see [12], Proposition 3.4). Establishing (4.7) is equivalent to establishing

$$(N_a(0,t_1], \ldots, N_a(t_{n-1},t_n]) =_d (N_{\tilde{a}}(0,t_1], \ldots, N_{\tilde{a}}(t_{n-1},t_n]),$$

which would follow from showing that

$$\alpha((N_a(0,t_1], \ldots, N_a(t_{n-1},t_n]) \ne (N_{\tilde{a}}(0,t_1], \ldots, N_{\tilde{a}}(t_{n-1},t_n])) = 0. \tag{4.8}$$

But the left-hand side of (4.8) is bounded above by $\sum_{i=1}^n \alpha(N_a(t_{i-1},t_i] \ne N_{\tilde{a}}(t_{i-1},t_i])$, which is zero by (4.6).

To establish the triangle inequality, let $\mu_a, \nu_a, \eta_a \in \mathscr{M}_S(\lambda)$. We want to show

$$\bar{\rho}_\mu(\mu_a, \eta_a) \le \bar{\rho}_\mu(\mu_a, \nu_a) + \bar{\rho}_\mu(\nu_a, \eta_a). \tag{4.9}$$

Let $\alpha \in \mathscr{M}_S(\lambda, \lambda)$ achieve the infimum in the definition of $\bar{\rho}_\mu(\mu_a, \nu_a)$ and let $\beta \in \mathscr{M}_S(\lambda, \lambda)$ achieve the infimum in the definition of $\bar{\rho}_\mu(\nu_a, \eta_a)$. Since $(M^{(2)}, \mathscr{M}^{(2)})$ is a complete separable metric space with its Borel σ-algebra ([12], Proposition 3.17), it is a separable standard Borel space; see Parthasarathy [11] for a definition. Therefore α admits a regular conditional distribution relative to the sub σ-algebra generated by the first marginal, and so does β (see [11], Theorem 8.1). From this it is straightforward to construct a jointly stationary coupling of μ_a and η_a with the expectation on the right of (4.1) bounded above the right hand side of (4.9) in a manner similar to the proof of

Theorem 8.3.1(b) in [6]. Since $\bar{\rho}_\mu(\mu_a, \eta_a)$ is defined through an infimum, the claimed triangle inequality follows.

Finally, we prove (iii). Let $\mu_a, \nu_a \in \mathscr{M}_S^e(\lambda)$, and let $\alpha \in \mathscr{M}_S(\lambda, \lambda)$ achieve the infimum in the definition of $\bar{\rho}_\mu(\mu_a, \nu_a)$. By appealing to the ergodic decomposition theorem for separable standard Borel spaces, we can find an $\{S_t^{(2)}\}$ invariant set $E \in \mathscr{M}^{(2)}$ with $\alpha(E) = 1$ such that for each $\omega \in E$, $\lim_{T\to\infty} \int_0^T 1(S_t^{(2)}(\omega) \in G)$ exists for all G in a countable generating field $\mathscr{G}$ for $\mathscr{M}^{(2)}$, and the probability distribution α_ω on $\mathscr{M}^{(2)}$ generated by these empirical limits is $\{S_t^{(2)}\}$ ergodic, and so that for any $f \in L^1(\alpha)$ we have $f \in L^1(\alpha_\omega)$ for all $\omega \in E$ and

$$\alpha(f) = \alpha(\alpha_\omega(f)).$$

[On the right-hand side above, we get a function $\omega \to \alpha_\omega(f)$ on E and then take its expectation relative to α.] Since μ_a and ν_a are each ergodic, each α_ω must have marginals μ_a and ν_a, respectively. The function $\omega \to \int_0^\infty \mu \exp(-\mu t)|N_a(0,t] - N_{\tilde{a}}(0,t]|\,dt$ is in $L^1(\alpha)$. Applying the ergodic decomposition shows that the infimum on the right-hand side of (4.2) over ergodic α is at least as small as that over all stationary α with the appropriate marginals and that there is some ergodic α achieving the infimum. □

5. Contractiveness of input–output map. Our main result is a consequence of the following:

THEOREM 4. *Let $\mu_a, \nu_a \in \mathscr{M}_S^e(\lambda)$, $\mu_a \neq \nu_a$. Then*

$$\bar{\rho}_\mu(T(\mu_a), T(\nu_a)) < \bar{\rho}_\mu(\mu_a, \nu_a).$$

PROOF. Since $\mu_a \neq \nu_a$, by Theorem 3(i), we know that $\bar{\rho}_\mu(\mu_a, \nu_a) > 0$. We construct the stationary ergodic coupling α achieving the minimum in the definition of $\bar{\rho}_\mu(\mu_a, \nu_a)$ and take its product with the canonical representation of the virtual departure process, letting the overall sample space be denoted $(\Omega, \mathscr{F}, P, \{\theta_t\})$. Let $(\Omega^0, \mathscr{F}^0, P^0, \theta)$ denote the Palm space of the virtual departure process. On this space we will jointly construct the stationary regimes for the two arrival processes following the Loynes scheme and a colouring idea. Before giving the formal details we sketch the idea behind the proof.

Let $N_n^a = N_a(t_{n-1}^d, t_n^d]$, $N_n^{\tilde{a}} = N_{\tilde{a}}(t_{n-1}^d, t_n^d]$ and $N_n^Y = \min(N_n^a, N_n^{\tilde{a}})$. Let $N_n^R = N_n^a - N_n^Y$ and $N_n^B = N_n^{\tilde{a}} - N_n^Y$. We think of N_n^Y "yellow" arrivals as having arrived in the interval $(t_n^d, t_{n+1}^d]$. N_n^R (respectively, N_n^B) is the number of "extra" arrivals in the first (respectively, second) arrival process in this interval, which we think of as coloured "red" (respectively, "blue"). The $\bar{\rho}_\mu$ distance between the two processes is seen to be $2\lambda/\mu - E^0 N_n^Y$.

Recall that the Loynes construction is entirely carried out on the Palm space. We ensure that immediately before a virtual departure point there cannot be both red and blue customers present in the queue. This can be done by carrying out a merging procedure immediately before releasing departures —if there are red and blue customers just prior to a virtual departure, they are merged one to one to the extent possible, becoming yellow customers. At a virtual departure just prior to which there are yellow customers we release yellow customers. We only release nonyellow customers at a virtual departure if there are no yellow customers present just prior to it.

After the Loynes construction is completed, in the limit, this colouring picture results in identifying which of the virtual departure points in the stationary regime correspond to real departures for both arrival processes and which correspond to real departures for only one or the other process. Indeed the Palm version of the former is precisely the points at which yellow customers got released. The process of points at which blue (respectively, red) and yellow customers got released is the Palm version of the departure process associated to the first (respectively, second) arrival process. (Recall the Palm space is with respect to the virtual departure process.) The Palm coupling can be transferred to give a time stationary coupling between the departure processes. Now, if we consider the expected absolute difference in number of points of the two departure processes over an exponential interval independent of these processes, it is obviously no more than the rate of "nonyellow" departures. But this rate can be expressed in terms of the Palm stationary probability that a departure is nonyellow. This can be shown to be strictly less than the Palm stationary mean number of nonyellow arrivals between virtual departures, by showing that there is a positive probability that a "merge" takes place just prior to a virtual departure time. This concludes the proof.

We now proceed to the formal details. On the Palm space $(\Omega^0, \mathscr{F}^0, P^0, \theta)$, for each $m \in \mathbb{Z}$ we construct $(z_n^m, x_n^m, \tilde{x}_n^m, x_n^{m,Y}, x_n^{m,R}, x_n^{m,B}, n \geq -m)$ by the prescription

$$
\begin{aligned}
& z_{-m}^m = x_{-m}^{m,Y} = x_{-m}^{m,R} = x_{-m}^{m,B} = x_{-m}^m = \tilde{x}_{-m}^m = 0,\\
& z_{n+1}^m = z_n^m + N_n^Y - 1\big(z_n^m + N_n^Y > 0\big),\\
& x_{n+1}^m = x_n^m + N_n^a - 1\big(x_n^m + N_n^a > 0\big),\\
& \tilde{x}_{n+1}^m = \tilde{x}_n^m + N_n^{\tilde{a}} - 1\big(\tilde{x}_n^m + N_n^{\tilde{a}} > 0\big),\\
& x_{n+1}^{m,Y} = x_n^{m,Y} + N_n^Y + \min\big(x_n^{m,R}, N_n^B\big) + \min\big(x_n^{m,B}, N_n^R\big)\\
& \qquad - 1\big(x_n^{m,Y} + N_n^Y + \min\big(x_n^{m,R}, N_n^B\big) + \min\big(x_n^{m,B}, N_n^R\big) > 0\big),\\
& x_{n+1}^{m,R} = x_{n+1}^m - x_{n+1}^{m,Y},\\
& x_{n+1}^{m,B} = \tilde{x}_{n+1}^m - x_{n+1}^{m,Y}.
\end{aligned}
\tag{5.1}
$$

We claim that z_n^m, $x_n^{m,Y}$, x_n^m and $\tilde{x}_n^m$ are nondecreasing in m for each fixed n for which they are defined, that is, $n \geq -m$. For x_n^m and $\tilde{x}_n^m$ this claim follows from Theorem 1. We easily establish $z_n^{m+1} \geq z_n^m$ by induction on n, starting with the observation that $z_{-m}^{m+1} \geq 0 = z_{-m}^m$. We establish $x_n^{m+1,Y} \geq x_n^{m,Y}$ also by induction on n, starting with the observation that $x_{-m}^{m+1,Y} \geq 0 = x_{-m}^{m,Y}$. Now suppose $x_{n-1}^{m+1,Y} \geq x_{n-1}^{m,Y}$ for some $n \geq -m$. Either $N_{n-1}^R = 0$ or

$N^B_{n-1} = 0$; assume the former. Then from (5.1) we have

$$\begin{aligned}
x_n^{m+1,Y} &= x_{n-1}^{m+1,Y} + N^Y_{n-1} + \min\left(x_{n-1}^{m+1,R}, N^B_{n-1}\right) \\
&\quad - 1\left(x_{n-1}^{m+1,Y} + N^Y_{n-1} + \min\left(x_{n-1}^{m+1,R}, N^B_{n-1}\right) > 0\right) \\
&= N^Y_{n-1} + \min\left(x_{n-1}^{m+1}, x_{n-1}^{m+1,Y} + N^B_{n-1}\right) \\
&\quad - 1\left(N^Y_{n-1} + \min\left(x_{n-1}^{m+1}, x_{n-1}^{m+1,Y} + N^B_{n-1}\right) > 0\right) \\
&\geq N^Y_{n-1} + \min\left(x^m_{n-1}, x^{m,Y}_{n-1} + N^B_{n-1}\right) \\
&\quad - 1\left(N^Y_{n-1} + \min\left(x^m_{n-1}, x^{m,Y}_{n-1} + N^B_{n-1}\right) > 0\right) \\
&= x_n^{m,Y},
\end{aligned}$$

where in the third step we used the induction hypothesis and the fact that for nonnegative integers a, b, $a \geq b$ implies $a - 1(a > 0) \geq b - 1(b > 0)$.

Thus we can define z_n^∞, $x_n^{\infty,Y}$ x_n^∞ and $\tilde{x}_n^\infty$ by taking the pointwise limit as $m \to \infty$. We can also define $x_n^{\infty,R} = x_n^\infty - x_n^{\infty,Y}$ and $x_n^{\infty,B} = \tilde{x}_n^\infty - x_n^{\infty,Y}$. Clearly x_0^∞ is the unique stationary regime corresponding to the first arrival process and $\tilde{x}_0^\infty$ is the unique stationary regime corresponding to the second arrival process.

We now define stationary departure processes N_e, $N_{\tilde{e}}$, N_y and N_z on $(\Omega, \mathscr{F}, P, \{\theta_t\})$ by

$$\begin{aligned}
N_e &= \sum_{n\in\mathbb{Z}} \delta_{t^d_{n+1}} 1\left(x_0^\infty \circ \theta_{t^d_n} + N_a\left(t^d_n, t^d_{n+1}\right] > 0\right), \\
N_{\tilde{e}} &= \sum_{n\in\mathbb{Z}} \delta_{t^d_{n+1}} 1\left(\tilde{x}_0^\infty \circ \theta_{t^d_n} + N_{\tilde{a}}\left(t^d_n, t^d_{n+1}\right] > 0\right), \\
N_y &= \sum_{n\in\mathbb{Z}} \delta_{t^d_{n+1}} 1\Big(\tilde{x}_0^{\infty,Y} \circ \theta_{t^d_n} + N^Y_n + \min\left(x_0^{\infty,B} \circ \theta_{t^d_n}, N^R_n\right) \\
&\qquad + \min\left(x_0^{\infty,R} \circ \theta_{t^d_n}, N^B_n\right) > 0\Big), \\
N_z &= \sum_{n\in\mathbb{Z}} \delta_{t^d_{n+1}} 1\left(z_0^\infty \circ \theta_{t^d_n} + N^Y_n > 0\right).
\end{aligned}$$

N_e and $N_{\tilde{e}}$ are, respectively, stationary versions of the departure processes associated to the first and second arrival processes, and each has rate λ, as proved in Theorem 2. The rate of N_z is determined by writing

$$E[N_z(0,1]] = E[N_d(0,1]]E^0\left[1\left(z_0^\infty \circ \theta_{t^d_n} + N^Y_n > 0\right)\right]$$

and then arguing exactly as in Theorem 2 to conclude that $E^0[1(z_0^\infty \circ \theta_{t^d_n} + N^Y_n > 0)]$ equals $E^0[N^Y_n]$. Thus

$$\begin{aligned}
E[N_z(0,1]] &= E^0\left[N^Y_n\right] \\
&= \mu\left(2\lambda/\mu - \bar{\rho}_\mu(\mu_a, \nu_a)\right) \qquad (5.2) \\
&= 2\lambda - \mu\bar{\rho}_\mu(\mu_a, \nu_a).
\end{aligned}$$

The rate of N_y is similarly computed to be

$$
\begin{aligned}
E\big[N_y(0,1]\big] &= \mu E^0\Big[1\Big(\tilde{x}_0^{\infty,Y}\circ\theta_{t_n^d} + N_n^Y + \min\big(x_0^{\infty,B}\circ\theta_{t_n^d}, N_n^R\big) \\
&\qquad\qquad + \min\big(x_0^{\infty,R}\circ\theta_{t_n^d}, N_n^B\big) > 0\Big)\Big] \qquad (5.3)\\
&= \mu E^0\Big[N_n^Y + \min\big(x_0^{\infty,B}\circ\theta_{t_n^d}, N_n^R\big) + \min\big(x_0^{\infty,R}\circ\theta_{t_n^d}, N_n^B\big)\Big],
\end{aligned}
$$

where the second equality comes from arguing exactly as in Theorem 2.

The pair $(N_e, N_{\tilde{e}})$ is clearly jointly stationary, and the distribution of N_e (respectively, $N_{\tilde{e}}$) is $T(\mu_a)$ [respectively, $T(\nu_a)$]. Further, since points in N_y correspond to points in both N_e and $N_{\tilde{e}}$, it is seen by definition that

$$
\begin{aligned}
\bar{\rho}_\mu(T(\mu_a), T(\nu_a)) &\le 2\lambda/\mu - \int_0^\infty \mu\exp(-\mu t)E\big[N_y(0,t]\big]\,dt \\
&= 2\lambda/\mu - 1/\mu E\big[N_y(0,1]\big].
\end{aligned}
$$

Comparing with (5.2) and (5.3), we see that to establish the claim it suffices to prove that

$$E^0\big[N_0^Y\big] < E^0\big[N_0^Y + \min\big(x_0^{\infty,B}, N_0^R\big) + \min\big(x_0^{\infty,R}, N_0^B\big)\big]$$

or equivalently that

$$0 < E^0\big[\min\big(x_0^{\infty,B}, N_0^R\big) + \min\big(x_0^{\infty,R}, N_0^B\big)\big],$$

that is, there is a positive probability of a merge prior to a virtual departure. We proceed to establish this.

On Ω^0, let A_n denote the event $\{\min(x_n^{\infty,R}, N_n^B) + \min(x_n^{\infty,B}, N_n^R) = 0\}$. Let B denote the event $\{N_0^Y \ge 3,\ N_1^R > 0,\ N_3^B > 0\}$. We claim that

$$B \subseteq (A_0 \cap A_1 \cap A_2 \cap A_3)^c.$$

To see this, it suffices to prove that $B \cap A_0 \cap A_1 \cap A_2 \subseteq A_3^c$. This is obvious. Indeed, if there are at least three arrivals common to the two arrival processes in $(0, t_1^d]$, then there must necessarily be yellow departures at t_1^d, t_2^d and t_3^d. If there is a red arrival in $(t_1^d, t_2^d]$ and a blue arrival in $(t_3^d, t_4^d]$, and there are no merges in the intervals $(t_{i-1}^d, t_i^d]$, $i = 1, 2, 3$, the red and blue arrival must necessarily merge just prior to the virtual departure at t_4^d. To complete the proof, it therefore suffices to show that $P^0(B) > 0$.

$P^0(B) > 0$ follows if we can show that there are $0 < T_1 < T_2 < T_3 < T_4 < T_5$ and $\delta > 0$ such that

$$
\begin{aligned}
&\alpha\big(N_a(0,t_1] \ge 3,\ N_{\tilde{a}}(0,t_1] \ge 3,\ N_a(t_2,t_3] > N_{\tilde{a}}(t_2,t_3], \\
&N_{\tilde{a}}(t_4,t_5] > N_a(t_4,t_5] \qquad (5.4)\\
&\qquad \text{for all } 0 < t_1 < t_2 < t_3 < t_4 < t_5 \text{ with } |t_i - T_i| < \delta,\ 1 \le i \le 5\big) > 0.
\end{aligned}
$$

This is because P-independence of the coupled arrival processes from the virtual departure processes implies their P^0-independence, and the event

$$\big\{|t_1^d - T_2| < \delta,\ |t_2^d - T_3| < \delta,\ |t_3^d - T_4| < \delta,\ |t_4^d - T_5| < \delta\big\} \qquad (5.5)$$

has positive P^0-probability. The intersection of the events in (5.4) and (5.5) therefore has positive P^0-probability. Clearly this intersection is contained in B.

It remains to show (5.4). Now we are working on the space $(M^{(2)}, \mathscr{M}^{(2)}, \alpha, \{S_t^{(2)}\})$. Since $\mu_a \neq \nu_a$, we have $\alpha(|N_a(0,T] - N_{\tilde{a}}(0,T]|) > 0$ for some $T > 0$ as can be seen from the formula (4.2) for $\bar{\rho}_\mu(\mu_a, \nu_a)$, the fact that α achieves the infimum in (4.2) and the fact that $\bar{\rho}_\mu(\mu_a, \nu_a) > 0$. Since $\alpha(N_a(0,T] - N_{\tilde{a}}(0,T]) = 0$, we must have $\alpha(N_a(0,T] > N_{\tilde{a}}(0,T]) > 0$ and also $\alpha(N_{\tilde{a}}(0,T] > N_a(0,T]) > 0$. Now, $\lim_{\delta\to 0} \alpha(N_a(T-\delta, T+\delta] > 0) = 0$ and $\lim_{\delta \to 0} \alpha(N_{\tilde{a}}(T-\delta, T+\delta] > 0) = 0$ by a simple application of the Markov inequality. It follows that for some $\delta > 0$, we simultaneously have

$$\alpha\big(N_a(0,t] > N_{\tilde{a}}(0,t] \text{ for all } |t-T| < \delta\big) > 0$$

and

$$\alpha\big(N_{\tilde{a}}(0,t] > N_a(0,t] \text{ for all } |t-T| < \delta\big) > 0.$$

Now let C_s denote the event

$$\{N_a(s, s+t] > N_{\tilde{a}}(s, s+t] \text{ for all } |t-T| < \delta\}$$

and let D_s denote the event

$$\{N_{\tilde{a}}(s, s+t] > N_a(s, s+t] \text{ for all } |t-T| < \delta\}.$$

$\bigcup_{s \geq 2T} D_s$ is mapped into itself by $\{S_t^{(2)}, t \geq 0\}$. By the ergodicity of α, and because it has nonzero probability, it has probability 1. There must therefore be some $S \geq 2T$ such that $\alpha(C_0 \cap D_S) > 0$.

Let $E_s = C_s \cap D_{S+s}$. Let T_0 be such that $\alpha(N_a(0,T_0] \geq 3,\ N_{\tilde{a}}(0,T_0] \geq 3) > 0$. Clearly we can find such T_0, and we may choose $T_0 > \delta$. The set $\bigcup_{s \geq 2T_0} E_s$ is mapped into itself by $\{S_t^{(2)}, t \geq 0\}$. By the ergodicity of α, and because it has nonzero probability, it has probability 1. There must therefore be some $S_1 \geq 2T_0$ such that

$$\alpha\big(\{N_a(0,T_0] \geq 3,\ N_{\tilde{a}}(0,T_0] \geq 3\} \cap E_{S_1}\big) > 0. \tag{5.6}$$

Let $T_1 = 2T_0$, $T_2 = S_1$, $T_3 = S_1 + T$, $T_4 = S_1 + S$ and $T_5 = T_4 + T$. Because $T_1 > \delta$, it is easily seen that (5.6) implies (5.4), completing the proof. □

6. Main result for ./M/1 nodes. We are now in a position to state and prove the uniqueness of a stationary ergodic fixed point for ./M/1 nodes.

THEOREM 5. *Consider a ./M/1 node of service rate μ. Let $\lambda < \mu$, and let $\mu_a \in \mathscr{M}_S^e(\lambda)$ be such that $T(\mu_a) = \mu_a$, that is, μ_a is a stationary ergodic fixed point of the input-output map. Then $\mu_a = \pi_\lambda$.*

PROOF. We know that $T(\pi_\lambda) = \pi_\lambda$. Suppose $\mu_a \neq \pi_\lambda$. Since $\bar{\rho}_\mu$ is a metric by Theorem 3(i), we have $\bar{\rho}_\mu(\mu_a, \pi_\lambda) > 0$. By Theorem 8, $\bar{\rho}_\mu(\mu_a, \pi_\lambda) > \bar{\rho}_\mu(T(\mu_a), T(\pi_\lambda)) = \bar{\rho}_\mu(\mu_a, \pi_\lambda)$. This is an absurdity. Hence $\mu_a = \pi_\lambda$. □

7. Multiple servers. In this section we indicate the points at which the preceding proof has to be modified to prove the uniqueness of a stationary ergodic fixed point for ./M/K nodes, $K < \infty$.

As a canonical representation of the node we choose a marked Poisson process of rate $K\mu$. The Poisson process itself is thought of as an overall virtual departure process. The sequence of marks is independent of the points, and consists of independent and uniformly distributed $\{1, \dots, K\}$ valued random variables. The interpretation is that if the mark at a point is k, $1 \le k \le K$, the corresponding overall virtual departure serves as a true departure only if the total number of customers at the node is at least k. Formally, the node is specified on a space $(\Omega_d, \mathscr{F}_d, P_d, \{\theta_t^d\})$ supporting an $(M, \mathscr{M}, \{S_t\})$ valued random variable N_d commuting with the shifts and having distribution $\pi_{K\mu}$ (i.e., a Poisson process of rate $K\mu$) and a sequence $(k_n,\ n \in \mathbf{Z})$ of marks of the process N_d. $(k_n)_n$ that are $\{1, \dots, K\}$ valued and are independent and uniformly distributed and are independent of the point process. The points of the overall virtual departure process are denoted $(t_n^d)_n$. When the arrival process is given in the representation $(\Omega_a, \mathscr{F}_a, \mu_a, \{\theta_t^a\})$, the overall system can be studied on the product sample space $(\Omega, \mathscr{F}, P, \{\theta_t\})$ as before. The constructions will be on the Palm space $(\Omega^0, \mathscr{F}^0, P^0, \theta)$ of the overall virtual departure process.

A stationary regime is a P^0-a.s. finite random variable x_0 defined on $(\Omega^0, \mathscr{F}^0, P^0, \theta)$ with

$$x_0 \circ \theta = x_0 + N_a(0, t_1^d] - 1\left(x_0 + N_a(0, t_1^d] \ge k_1\right). \tag{7.1}$$

The proof of the existence and uniqueness of a stationary regime for any $\mu_a \in \mathscr{M}_S^e(\lambda)$ for any $\lambda < K\mu$ is like Theorem 1. For each $m \in \mathbf{Z}$ we construct $(x_n^m,\ n \ge -m)$ on $(\Omega^0, \mathscr{F}^0, P^0, \theta)$ by

$$x_{-m}^m = 0, \tag{7.2}$$

$$x_{n+1}^m = x_n^m + N_a(t_n^d, t_{n+1}^d] - 1\left(x_n^m + N_a(t_n^d, t_{n+1}^d] \ge k_{n+1}\right). \tag{7.3}$$

From (7.2) and (7.3), by induction on $m \ge -n$ for fixed n, we see that x_n^m increases in m, allowing us to define

$$x_n^\infty = \lim_{m \to \infty} x_n^m \tag{7.4}$$

as in (3.4). Then x_0^∞ obeys (7.1). The P^0-a.s. finiteness of x_0^∞ when $\lambda < K\mu$ comes from taking expectation relative to P^0 in (7.3), giving

$$E^0\left[1\left(x_n^m + N_a(t_n^d, t_{n+1}^d] \ge k_{n+1}\right)\right] \le \lambda / K\mu < 1. \tag{7.5}$$

As before we argue that this implies $P^0(x_0^\infty = \infty) < 1$, and use ergodicity to conclude that this implies $P^0(x_0^\infty < \infty) = 1$. To show uniqueness, assume $\tilde{x}_0$ is another stationary regime, that is, it obeys

$$\tilde{x}_0 \circ \theta = \tilde{x}_0 + N_a(0, t_1^d] - 1\left(\tilde{x}_0 + N_a(0, t_1^d] \ge k_1\right). \tag{7.6}$$

As before one argues there must be some $0 \le K < \infty$ with $P^0(\tilde{x}_0 = x_0^\infty + K) = 1$. The problem is then to show that $K = 0$. If $K > 0$ then $P^0(\tilde{x}_0 > 1) = 1$, so subtracting (7.1) from (7.6) and multiplying by $1(k_1 = 1)$ gives

$$1(k_1 = 1) = 1\left(x_0^\infty + N_a\left(0, t_1^d\right] \geq 1, k_1 = 1\right). \tag{7.7}$$

By construction, x_0^∞ is a function of the arrival process, the overall virtual departure process up to time 0 and the marks $(k_n,\ n \leq 0)$. It follows that $\{x_0^\infty + N_a(0, t_1^d] \geq 1\}$ and $\{k_1 = 1\}$ are P^0-independent. Taking P^0-expectation in (7.7) and using $P^0(k_1 = 1) = 1/K > 0$ gives

$$P^0\left(x_0^\infty + N_a\left(0, t_1^d\right] \geq 1\right) = 1. \tag{7.8}$$

We will now argue that this is impossible. Since x_0^∞ is P^0-a.s. finite, there is some $M < \infty$ with $P^0(x_0^\infty = M) > 0$. Also, we have

$$\lim_{\delta \to 0} P^0\left(N_a(0, \delta] = 0\right) = 1.$$

Hence there is some $\delta > 0$ with

$$P^0\left(x_0^\infty = M,\ N_a(0, \delta] = 0\right) > 0.$$

Using the independence of x_0^∞ from the overall virtual departure process after time 0, we have

$$P^0\left(x_0^\infty = M,\ N_a(0, \delta] = 0,\ N_d(0, \delta] \geq M + 1\right) > 0. \tag{7.9}$$

But on the event in (7.9) we have $x_0^\infty \circ \theta_{t_M^d} + N_a(t_M^d, t_{M+1}^d] = 0$. Thus (7.9) contradicts (7.8). We must therefore have $K = 0$, that is, the stationary regime is unique.

The departure process of the ./M/K node is defined by

$$N_e = \sum_{n \in \mathbf{Z}} \delta_{t_{n+1}^d} 1\left(x_0^\infty \circ \theta_{t_n^d} + N_a\left(t_n^d, t_{n+1}^d\right] \geq k_{n+1}\right).$$

The fact that this is a stationary ergodic point process of rate λ, that is, an element of $\mathscr{M}_S^e(\lambda)$, follows mutatis mutandis from Theorem 2 as does the fact that the same distribution results whatever the representation chosen for the arrival process. This allows defining the input–output map of a ./M/K node as a map on $\mathscr{M}_S^e(\lambda)$, $\lambda < K\mu$. We denote this map by T_K.

We work with the metric $\bar{\rho}_{K\mu}$ as defined in Section 4. We prove:

THEOREM 6. *Let $\mu_a, \nu_a \in \mathscr{M}_S^e(\lambda)$, $\lambda < K\mu$, $\mu_a \leq \nu_a$. Then*

$$\bar{\rho}_{K\mu}\left(T_K(\mu_a), T_K(\nu_a)\right) < \bar{\rho}_{K\mu}(\mu_a, \nu_a).$$

This is proved by constructing processes $(z_n^m, x_n^m, \tilde{x}_n^m, x_n^{m,Y}, x_n^{m,R}, x_n^{m,B}$, $n \geq -m)$ on $(\Omega^0, \mathscr{F}^0, P^0, \theta)$, for each $m \in \mathbf{Z}$ exactly as in Theorem 4 with the obvious change in the indicator conditions. One also defines $(N_n^Y, N_n^a, N_n^{\tilde{a}}$, $n \in \mathbf{Z})$ as in Theorem 4. One shows that z_n^m, $x_n^{m,Y}$, x_n^m and $\tilde{x}_n^m$ are nondecreasing in m for each fixed n for which they are defined, thereby defining $(z_n^\infty, x_n^\infty, \tilde{x}_n^\infty, x_n^{\infty,Y}, x_n^{\infty,R}, x_n^{\infty,B}, n \in \mathbf{Z})$. One defines the point processes N_e, $N_{\tilde{e}}$, N_z and N_y as in Theorem 4, with the obvious change in indicator conditions. Now the Palm theory gives

$$E[N_z(0,1]] = E[N_d(0,1]]E^0\left[1\left(z_0^\infty \circ \theta_{t_n^d} + N_n^Y \geq k_0 \circ \theta_{t^d n+1}\right)\right].$$

One can argue exactly as in Theorem 2 to conclude that $E^0[1(z_0^\infty \circ \theta_{t_n^d} + N_n^Y \geq k_{n+1})]$ equals $E^0[N_n^Y]$, and further that

$$E^0\left[N_n^Y\right] = 2\lambda - K\mu\bar{\rho}_{K\mu}(\mu_a, \nu_a). \tag{7.10}$$

Similarly one computes

$$\begin{aligned} E\left[N_y(0,1]\right] = K\mu E^0\Big[N_n^Y &+ \min\left(x_0^{\infty,B} \circ \theta_{t_n^d}, N_n^R\right) \\ &+ \min\left(x_0^{\infty,R} \circ \theta_{t_n^d}, N_n^B\right)\Big] \end{aligned} \tag{7.11}$$

and

$$\bar{\rho}_{K\mu}(T_K(\mu_a), T_K(\nu_a)) = 2\lambda/K\mu - (1/K\mu)E\left[N_y(0,1]\right]. \tag{7.12}$$

From (7.10), (7.11) and (7.12), we see that proving Theorem 4 boils down to proving that there is a positive P^0-probability that a merge takes place just before an overall virtual departure, that is, that

$$0 < E^0\left[\min\left(x_0^{\infty,B}, N_0^R\right) + \min\left(x_0^{\infty,R}, N_0^B\right)\right].$$

The proof of this is essentially identical to the proof of Theorem 4; one just throws in the additional condition that the marks of the first three overall virtual departures after time 0 on the Palm space are all 1.

Our main result for ./M/K queues now follows. The proof is identical to that of Theorem 5. We state the result for completeness.

THEOREM 7. *Consider a ./M/K node with K servers having service rate μ. Let $\lambda < K\mu$, and let $\mu_a \in \mathscr{M}_S^e(\lambda)$ be such that $T_K(\mu_a) = \mu_a$, that is, μ_a is a stationary ergodic fixed point of the input–output map. Then $\mu_a = \pi_\lambda$.*

8. Concluding remarks. It would be interesting to prove that when a stationary ergodic arrival process of rate $\lambda < K\mu$ is put through a long tandem of ./M/K nodes of service rate μ, the stationary departure process converges weakly to a Poisson process of rate λ as the length of the tandem goes to ∞. In the preceding notation, to prove this we would have to show that for all $\mu_a \in \mathscr{M}_S^e(\lambda)$, $\lambda < K\mu$, we have $T_K^k(\mu_a) \to \pi_\lambda$ as $k \to \infty$, where T_K^k denotes T_K iterated k times. Let $W \subset \mathscr{M}_S(\lambda)$ denote the set of weak limit points of the sequence $(T_K^k(\mu_a), k \geq 1)$. W is closed, and hence compact by Lemma 1. It is easily shown that $\mu_a \to \bar{\rho}_{K\mu}(\mu_a, \pi_\lambda)$ is a lower semicontinuous function on $\mathscr{M}_S(\lambda)$, so on the compact set W, this function attains its minimum at some $\tilde{\mu}_a$. Suppose $\tilde{\mu}_a \neq \pi_\lambda$. Then, by Theorem 3(i), $\bar{\rho}_{K\mu}(\tilde{\mu}_a, \pi_\lambda) > 0$. If we could show that $\tilde{\mu}_a \in \mathscr{M}_S^e(\lambda)$, then we would arrive at a contradiction by Theorem 6. This would show that the only subsequential weak limit point of the sequence $(T_K^k(\mu_a), k \geq 1)$ is π_λ, and it is a simple step to conclude that $(T_K^k(\mu_a))_k$ converges weakly to π_λ. The missing ingredient is that it is a priori possible for the sequence of ergodic processes $(T_K^k(\mu_a))_k$ to have a nonergodic subsequential limit. Intuitively this seems impossible, but we have not yet been able to rule it out. Another way to avoid the problem is to show uniform contraction in Theorem 6, that is, to show that there is some $\alpha < 1$ with

$$\bar{\rho}_{K\mu}(T_K(\mu_a), T_K(\nu_a)) < \alpha \bar{\rho}_{K\mu}(\mu_a, \nu_a)$$

for all $\mu_a, \nu_a \in \mathscr{M}_S^e(\lambda)$.

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SCHOOL OF ELECTRICAL ENGINEERING
CORNELL UNIVERSITY
322 ENGINEERING THEORY CENTER BUILDING
ITHACA, NEW YORK 14853-5401