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ON LADDER HEIGHT DISTRIBUTIONS OF GENERAL RISK PROCESSES

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We consider a continuous-time risk process $\{Y_a(t);\ t \ge 0\}$ defined for a stationary marked point process $\{(T_n, X_n)\}$, where $Y_a(0) = a$ and $Y_a(t)$ increases linearly with a rate c and has a downward jump at time T_n with jump size X_n for $n \in \{1, 2, \dots\}$. For $a = 0$, we prove that, under a balance condition, the descending ladder height distribution of $\{Y_0(t)\}$ has the same form as the case where $\{(T_n, X_n)\}$ is a compound Poisson process. This generalizes the recent result of Frenz and Schmidt, in which the independence of $\{T_n\}$ and $\{X_n\}$ is assumed. In our proof, a differential equation is derived concerning the deficit Z_a at the ruin time of the risk process $\{Y_a(t)\}$ for an arbitrary $a \ge 0$. It is shown that this differential equation is also useful for proving a continuity property of ladder height distributions.

1. Introduction. In collective risk theory [see, e.g., Gerber (1979) or Grandell (1991)], the following stochastic model is considered. For each real a, a continuous-time risk process $\{Y_a(t);\ t \ge 0\}$ is defined by

$$Y_a(t) = a + ct - \sum_{k=1}^{\Phi(t)} X_k, \tag{1.1}$$

where $\{\Phi(t);\ t \ge 0\}$ is a counting process whose jump sizes equal 1, $\{X_n\}$ is a sequence of nonnegative random variables and c is a constant with $0 < c < +\infty$. The ruin time τ_a of the risk process $\{Y_a(t)\}$ is defined by

$$\tau_a = \inf\{t \ge 0 \colon Y_a(t) < 0\}, \tag{1.2}$$

where $\tau_a = \infty$ if $Y_a(t) \ge 0$ for every $t \ge 0$. Furthermore, the deficit Z_a at time τ_a is given by

$$Z_a = \begin{cases} -Y_a(\tau_a), & \text{if } \tau_a < \infty, \\ \infty, & \text{if } \tau_a = \infty. \end{cases}$$

This Z_a can be used for describing the severity of ruin [see, e.g., Gerber, Goovaerts and Kaas (1987)]. In the present paper, our main concern is with the distribution of the deficit Z_0, which is the first descending ladder height of the risk process $\{Y_0(t);\ t \ge 0\}$.

The following fact is well known [see, e.g., Asmussen (1987), Theorem 2.3(a), page 206, or Feller (1971), Chapter XII]. Under the conditions that:

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1. $\{\Phi(t);\ t \geq 0\}$ is a homogeneous Poisson process with intensity λ,
2. $\{X_n\}$ is a sequence of independent and identically distributed random variables with distribution function F,
3. $\{\Phi(t);\ t \geq 0\}$ and $\{X_n\}$ are independent of each other,

the distribution of Z_0 is given by

$$P(Z_0 < x) = \frac{\lambda}{c}\int_0^x (1 - F(u))\, du; \qquad x \geq 0, \tag{1.3}$$

provided that

$$\lambda E(X_0) < c. \tag{1.4}$$

Note that from (1.3) it follows in particular that

$$P(\tau_0 < \infty) = \frac{\lambda}{c} E(X_0), \tag{1.5}$$

which is called the ruin probability of the risk process $\{Y_0(t);\ t \geq 0\}$.

Recently it was shown that the independence assumptions in conditions 1 and 2 can be weakened to get formulas (1.3) and (1.5). Björk and Grandell (1985) [see also Grandell (1991)] proved that (1.5) holds for an arbitrary counting process $\{\Phi(t);\ t \geq 0\}$ with homogeneous increments (but not necessarily independent and Poisson distributed) if conditions 2 and 3 hold. In this case, condition (1.4) has been slightly modified to

$$\overline{\Phi} E(X_0) < c \quad \text{a.s.}, \tag{1.6}$$

where $\overline{\Phi} = \lim_{t \to \infty}(1/t)\Phi(t)$. Asmussen and Schmidt (1992) gave two alternative proofs for showing (1.3) and, consequently, (1.5) for an arbitrary stationary sequence $\{X_n\}$ provided that conditions 1 and 3 hold and that

$$\lambda \overline{X} \leq c \quad \text{a.s.}, \tag{1.7}$$

where $\overline{X} = \lim_{n \to \infty}(1/n)\sum_{k=1}^{n} X_k$. It should be noted that in (1.7) and, consequently, in (1.4), equality is allowed. Furthermore, Frenz and Schmidt (1992) extended one of the proofs in Asmussen and Schmidt (1992) to the case where $\{\Phi(t);\ t \geq 0\}$ has homogeneous increments and $\{X_n\}$ is a stationary sequence provided, however, that condition 3 holds and that

$$\overline{\Phi}\,\overline{X} \leq c \quad \text{a.s.} \tag{1.8}$$

Frenz and Schmidt (1992) also showed that a slightly modified version of (1.5) can be obtained without any independence assumptions. This follows from the correspondence between the risk process and the work load process of a single-server queue. Concerning this correspondence, further results will be given in Lemma 3.3.

Unfortunately, for $x < \infty$, the probabilities $P(Z_0 < x)$ cannot be obtained in such a direct way from queueing theory. Nevertheless, the question arises whether (1.3) can also be proved without any independence assumptions.

The main result of the present paper answers this question positively. Conditions 1, 2 and 3 are not necessary for the validity of (1.3), provided that the jump epochs T_n of $\{\Phi(t)\}$ and the jump sizes X_n form a stationary marked point process $\{(T_n, X_n)\}$ (Theorem 2.1). The risk processes mentioned before, which all satisfy condition 3, are included in our model as special cases. Our approach, however, is different from those in the previous papers. We first consider the deficit Z_a for a given initial value $Y_a(0) = a$, and derive a differential equation concerning Z_a (Theorem 2.2). In this derivation, we use some results of the theory of stationary point processes; in particular, the notion of the Palm distribution and an inversion formula that expresses the time-stationary distribution by its Palm distribution. Then (1.3) is obtained by integrating the differential equation, where, now, F denotes the Palm mark distribution function of the stationary marked point process $\{(T_n, X_n)\}$. Those results are given in Section 2 and proved in Section 3. We note that, under conditions 1–3 and (1.4), the related formulas have been discussed in Dufresne and Gerber (1988a, b) and Gerber, Goovaerts and Kaas (1987) (see Remarks 2.2 and 3.1). Furthermore, the differential equation is useful for approximating the probabilities $P(Z_0 < x)$ when (1.8) fails but $(\lambda/c)E_0(X_0)$ nearly equals 1, where λ is the intensity of $\{\Phi(t)\}$ and E_0 denotes the expectation with respect to the Palm distribution. This will be discussed in Section 4.

2. Main results. For a marked point process, we pay particular attention to the epochs T_n where $\{\Phi(t);\ t \geq 0\}$ has jumps. The X_n are the marks of these jump epochs. In the following, $(\Omega, \mathscr{F}, P)$ denotes the basic probability space.

Let $\{(T_n, X_n);\ n = 0, \pm 1, \dots\}$ be a sequence consisting of ordered real-valued random variables T_n and of positive random variables X_n, which are defined on Ω. We assume that $\{(T_n, X_n);\ n = 0, \pm 1, \dots\}$ is a stationary marked point process; that is,

$$\#\{n: |T_n| < d\} < \infty \quad \text{a.s. for every } d < \infty$$

and, for any real t, the distribution of the shifted sequence $\{(T_n + t, X_n);\ n = 0, \pm 1, \dots\}$ is the same as that of $\{(T_n, X_n)\}$. The marked point process $\{(T_n, X_n)\}$ can be given by the random counting measure Ψ with

$$\Psi(B \times L) = \#(n: (T_n, X_n) \in B \times L), \qquad B \in \mathscr{B}(R),\ L \in \mathscr{B}(R^+),$$

where $\mathscr{B}(R)$ and $\mathscr{B}(R^+)$ are the Borel σ-fields on R and R^+, respectively. Let $\mathbb{M}$ be the set of all locally finite counting measures on $(R \times R^+, \mathscr{B}(R) \times \mathscr{B}(R^+))$ and $\mathscr{M}$ be the σ-field generated by all sets of the form

$$\{\psi: \psi \in \mathbb{M},\ \psi([a,b] \times [u,v]) = j\},$$
$$j = 0, 1, \dots, \quad -\infty < a \leq b < +\infty, \qquad 0 < u \leq v < +\infty.$$

Let $\{\theta_t;\ -\infty < t < +\infty\}$ be a group of shift operators on Ω satisfying $\Psi \circ \theta_t (B \times L) = \Psi((B + t) \times L)$ for any t, where $\Psi \circ \theta_t(\omega) = \Psi(\theta_t(\omega))$ for $\omega \in \Omega$ and $B + t = \{s: s \in R,\ s - t \in B\}$. Without loss of generality, for example, by putting $\Omega = \mathbb{M}$ and $\mathscr{F} = \mathscr{M}$ if necessary, we can assume the existence of $\{\theta_t\}$ and that it is measurable on $(\Omega, \mathscr{F})$. Then the stationarity assumption of a marked point process means that

$$P(\Psi \circ \theta_t \in A) = P(\Psi \in A) \quad \text{for every } A \in \mathscr{M}.$$

We assume that the intensity $\lambda = E(\Psi([0,1] \times R^+))$ of Ψ is positive and finite. More details concerning stationary marked point processes can be found, for example, in Franken, König, Arndt and Schmidt (1982), Daley and Vere-Jones (1988) and König and Schmidt (1992).

We define the stochastic process $\{\Phi(t);\ t \geq 0\}$ appearing in (1.1) by

$$\Phi(t) = \#\{n: T_n \in (0,t]\}\ [\equiv \Psi((0,t] \times R^+)]. \tag{2.1}$$

The stationarity of the nonmarked point process $\{T_n\}$ is equivalent to the homogeneous increments of $\{\Phi(t);\ t \geq 0\}$. Furthermore, it is easy to see that $\{X_n\}$ is a stationary sequence if $\{T_n\}$ and $\{X_n\}$ are independent of each other; that is, if condition 3 is fulfilled. However, we emphasize that in the following we will not use this independence assumption, relying instead on Palm distributions. Define the probability measure P_0 on $(\mathbb{M}, \mathscr{M})$ by

$$P_0(A) = \lambda^{-1} E\big(\#\{n: T_n \in (0,1],\ \Psi \circ \theta_{T_n} \in A\}\big) \quad \text{for every } A \in \mathscr{M}.$$

P_0 is called the Palm distribution of Ψ. Note that, for every $A \in \mathscr{M}$, the probability $P_0(A)$ is the ratio of two intensities. Namely, $P_0(A)$ is the quotient of the mean number of the those points T_n per time unit, for which the random counting measure Ψ seen from T_n has the property A, divided by the mean number of all points T_n per time unit. Furthermore, $P_0(A)$ can be interpreted as the conditional probability of the event $\{\Psi \in A\}$ given the condition that Ψ has an atom at the origin. However, one should be careful of this interpretation because the conditioning event $\{\Psi(\{0\} \times R^+) > 0\}$ has probability zero [see, e.g., Franken, König, Arndt and Schmidt (1982)]. It is well known that, on the probability space $(\mathbb{M}, \mathscr{M}, P_0)$, the X_n always form a stationary sequence and that under condition 3 we have $P_0(X_n \leq x) = P(X_n \leq x)$. Thus, in the general case, we put $F(x) = P_0(X_n \leq x)$, which is called the Palm mark distribution function of the stationary marked point process $\{(T_n, X_n)\}$. This leads us to the following result.

THEOREM 2.1. *Let the risk process $\{Y_0(t)\}$ be defined by the stationary marked point process $\{(T_n, X_n)\}$ via* (1.1) *and* (2.1). *Then, the distribution of Z_0 is given by* (1.3) *provided that* (1.8) *holds, where λ denotes the intensity and F denotes the Palm mark distribution function of $\{(T_n, X_n)\}$.*

For proving Theorem 2.1, we use the following interesting result, which is valid without any balance condition such as (1.8).

THEOREM 2.2. *For the risk process $\{Y_a(t)\}$, the probabilities $P(Z_a < x)$ are Lipschitz continuous with respect to a and possess right-hand derivatives $(d^+/da)P(Z_a < x)$ for which*

$$\frac{d^+}{da} P(Z_a < x) = \frac{\lambda}{c}\Big(P_0(Z_a < x) - P_0(Z_{a - X_0} < x,\ X_0 \leq a) - \big(\overline{F}(a) - \overline{F}(a+x)\big)\Big) \tag{2.2}$$

holds for any $a,\ x \geq 0$, where $\overline{F}(x) = 1 - F(x)$.

REMARK 2.1. Let us consider the case of a homogeneous Poisson process $\{T_n\}$ assuming conditions 2 and 3. For simplicity, we assume that $c = 1$. We have $P(Z_a < x) = P_0(Z_a < x)$ by the property of independently marked Poisson processes that $P(\{(T_n, X_n);\ n \geq 1\} \in (\cdot)) = P_0(\{(T_n, X_n);\ n \geq 1\} \in (\cdot))$. Therefore, Theorem 2.2 yields

$$\frac{d^+}{da} P(Z_a < x) = \lambda\Big(P(Z_a < x) - P\big(Z_{a - X_0} < x,\ X_0 \leq a\big) - \big(\overline{F}(a) - \overline{F}(a + x)\big)\Big). \tag{2.3}$$

An integrated version of (2.3) has been derived in Gerber, Goovaerts and Kaas (1987), formula (5) [see also formula (11) of Dufresne and Gerber (1988a)]. Furthermore, a differential equation similar to (2.3) is given in Dufresne and Gerber (1988b), formula (5), for the surplus $Z_a^- = Y_a(\tau_a - 0)$ immediately before the ruin time τ_a. Note that, for $x = \infty$, (2.3) is derived in Feller [(1971), Section VI.5]. Related results are also found in Section XI.7 of Feller (1971). For some classes of distribution functions F of the jump size X_n, it is possible to solve the differential equation (2.3) explicitly; see, for example, Asmussen and Schmidt (1992) for F being phase type, Dufresne and Gerber (1988b) for combinations of exponential distributions and Gerber, Goovaerts and Kaas (1987) for combinations of gamma distributions and for constant jump size.

REMARK 2.2. It is sufficient to prove Theorems 2.1 and 2.2 for the case $c = 1$. That is, if $c \neq 1$, we consider a normalized risk process $\{Y^*_{a/c}(t)\}$ defined by

$$Y^*_{a/c}(t) = \frac{1}{c} Y_a(t) = t - \sum_{k=1}^{\Phi(t)} \frac{1}{c} X_k + \frac{a}{c}.$$

Let $\tau^*_{a/c}$ and $Z^*_{a/c}$ be the ruin time and deficit, respectively, and let X^*_n be the nth jump height of $\{Y^*_{a/c}(t)\}$. Then, it is easy to see that $\tau_a = \tau^*_{a/c}$, $Z_a = cZ^*_{a/c}$ and $X_n = cX^*_n$. Hence, all results of Theorems 2.1 and 2.2 for $c = 1$ can be transferred to the case $c \neq 1$.

3. Proofs.

By Remark 2.2 we assume without loss of generality that $c = 1$ throughout this section.

PROOF OF THEOREM 2.2. For this purpose, two lemmas are prepared, which need the following notation. For $t \geq 0$ and real a, let

$$\tau_a(t) = \inf\{u \geq t \colon Y_0(u) - Y_0(t) + a < 0\},$$

$$Z_a(t) = \begin{cases} -\big(Y_0(\tau_a(t)) - Y_0(t) + a\big), & \text{if } \tau_a(t) < \infty, \\ \infty, & \text{if } \tau_a(t) = \infty. \end{cases}$$

Note that $\tau_a(0) = \tau_a$, $Z_a(0) = Z_a$ and

$$Z_a(u) = Z_{a + T_1 - u - X_1}(T_1) \quad \text{for } 0 \leq u < T_1, \tag{3.1}$$

where, if $a + T_1 - u - X_1 < 0$, then $\tau_{a + T_1 - u - X_1}(T_1) = T_1$ by our definition.

LEMMA 3.1. *For any a, $x \ge 0$, we have*

$$P(Z_a < x) = \lambda \int_0^{+\infty} P_0\big(T_1 > u, Z_{a+T_1-u-X_1}(T_1) < x\big)\, du. \tag{3.2}$$

PROOF. Because

$$\begin{aligned} Y_a(u) \circ \theta_t &= u - \sum_{0 < T_k \circ \theta_t \le u} X_k \circ \theta_t + a \\ &= u - \sum_{t < T_k \le t+u} X_k + a \\ &= Y_0(t+u) - Y_0(t) + a, \end{aligned}$$

we have

$$\tau_a \circ \theta_t = \inf\{u \ge 0 \colon Y_a(u) \circ \theta_t < 0\} = \tau_a(t) - t$$

and

$$Y_a(\tau_a) \circ \theta_t = Y_0(\tau_a(t)) - Y_0(t) + a,$$

which imply that $Z_a \circ \theta_t = Z_a(t)$ for $t \ge 0$. Hence, from the inversion formula for point processes [see, e.g., (1.2.19) of Franken, König, Arndt and Schmidt (1982)], we get

$$P(Z_a < x) = \lambda \int_0^{+\infty} P_0(T_1 > u, Z_a(u) < x)\, du.$$

Thus, by (3.1), this gives (3.2). □

LEMMA 3.2. *The probabilities $P(Z_a < x)$ satisfy the Lipschitz condition with respect to a on R^+; that is, there exists a constant $\delta > 0$ satisfying*

$$|P(Z_{a+h} < x) - P(Z_a < x)| \le \delta h, \qquad (\forall\, a, h \ge 0). \tag{3.3}$$

PROOF. We calculate $P(Z_{a+h} < x) - P(Z_a < x)$ by using Lemma 3.1:

$$\begin{aligned} &\lambda\bigg(\int_0^{\infty} P_0\big(T_1 > u, Z_{a+h+T_1-u-X_1}(T_1) < x\big)\, du \\ &\qquad - \int_0^{\infty} P_0\big(T_1 > u, Z_{a+T_1-u-X_1}(T_1) < x\big)\, du\bigg) \\ &= \lambda\bigg(\int_{-h}^{\infty} P_0\big(T_1 > u+h, Z_{a+T_1-u-X_1}(T_1) < x\big)\, du \\ &\qquad - \int_0^{\infty} P_0\big(T_1 > u, Z_{a+T_1-u-X_1}(T_1) < x\big)\, du\bigg) \\ &= \lambda\bigg(\int_{-h}^{0} P_0\big(T_1 > u+h, Z_{a+T_1-u-X_1}(T_1) < x\big)\, du \\ &\qquad - \int_0^{\infty} P_0\big(u \le T_1 < u+h, Z_{a+T_1-u-X_1}(T_1) < x\big)\, du\bigg), \end{aligned} \tag{3.4}$$

where the first equation is obtained by changing the variable u to $u' = u - h$. Note that the first integral of the last right-hand side of (3.4) is obviously less than h. The second integral is bounded by

$$\int_0^\infty P_0(u \le T_1 < u + h)\, du \le E_0\left(\int_{(T_1 - h)^+}^{T_1} du\right) \le h,$$

where E_0 denotes the expectation with respect to P_0 to $x^+ = \max(0, x)$. Thus we get (3.3) with $\delta = 2\lambda$. □

Now, we can finish the proof of Theorem 2.2. Because for each realization of $\{(T_n, X_n)\}$ the corresponding realization of $\tau_{a+u}(T_1)$ converges to that of $\tau_a(T_1)$ as u tends to zero from above, $Z_{a+T_1+u-X_1}(T_1)$ and $Z_{a+u-X_1}(T_1)$ converge to $Z_{a+T_1-X_1}(T_1)$ and $Z_{a-X_1}(T_1)$, respectively, as $u \downarrow 0$. Hence, we have, by using the bounded convergence theorem,

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{1}{h} \int_{-h}^0 & P_0\big(T_1 > u + h, Z_{a+T_1-u-X_1}(T_1) < x\big)\, du \\
&= E_0\left(\lim_{h \downarrow 0} \frac{1}{h} \int_{-h}^0 I_{\{T_1 > u+h,\, Z_{a+T_1-u-X_1}(T_1) < x\}}\, du\right) \\
&= P_0\big(Z_{a+T_1-X_1}(T_1) < x\big),
\end{aligned}$$

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{1}{h} \int_0^\infty & P_0\big(u \le T_1 < u + h, Z_{a+T_1-u-X_1}(T_1) < x\big)\, du \\
&= E_0\left(\lim_{h \downarrow 0} \frac{1}{h} \int_{T_1-h}^{T_1} I_{\{Z_{a+T_1-u-X_1}(T_1) < x\}}\, du\right) \\
&= P_0\big(Z_{a-X_1}(T_1) < x\big),
\end{aligned}$$

where I denotes the indicator function. Hence, by dividing both sides of (3.4) by h and letting h tend to zero, from (3.2) and (3.4), we get

$$\frac{d^+}{da} P(Z_a < x) = \lambda\Big(P_0\big(Z_{a+T_1-X_1}(T_1) < x\big) - P_0\big(Z_{a-X_1}(T_1) < x\big)\Big).$$

On the other hand, from (3.1), we get

$$P_0\big(Z_{a+T_1-X_1}(T_1) < x\big) = P_0\big(Z_a(0) < x\big) = P_0(Z_a < x).$$

Furthermore, using the identity $Z_a \circ \theta_t = Z_a(t)$ for $t \ge 0$, from the invariance of the Palm distribution P_0 with respect to the shift θ_{T_1} [see, e.g., Franken, König, Arndt and Schmidt (1982)], we get

$$\begin{aligned}
P_0\big(Z_{a-X_1}(T_1) < x\big) &= P_0\big(Z_{a-X_1}(T_1) < x, X_1 \le a\big) \\
&\quad + P_0\big(Z_{a-X_1}(T_1) < x, X_1 > a\big) \\
&= P_0\big(Z_{a-X_0}(0) < x, X_0 \le a\big) + P_0(a < X_0 \le a + x)
\end{aligned}$$

and, consequently,

$$\frac{d^+}{da}P(Z_a < x) = \lambda\Big(P_0(Z_a < x) - P_0(Z_{a-X_0} < x, X_0 \le a) - P_0(a < X_0 \le a + x)\Big).$$

Thus, Theorem 2.2 is obtained. □

PROOF OF THEOREM 2.1. First we remark that it is enough to prove the theorem under the condition

$$\overline{\Phi}\overline{X} < 1 \quad \text{a.s.} \tag{3.5}$$

To show this we can proceed similarly as in the proof of Lemma 1 of Frenz and Schmidt (1992). For $\varepsilon > 1$, let $\Psi^{(\varepsilon)}$ be the marked point process $\{(\varepsilon T_n, X_n)\}$. Clearly, $\Psi^{(\varepsilon)}$ is again shift-invariant and has intensity λ/ε. Furthermore, for the limit $\overline{\Phi}^{(\varepsilon)} = \lim_{t\to\infty}(1/t)\Psi^{(\varepsilon)}((0, t] \times R^+)$, we have $\overline{\Phi}^{(\varepsilon)} = (1/\varepsilon)\overline{\Phi}$ and, consequently,

$$\overline{\Phi}^{(\varepsilon)}\overline{X} < 1 \quad \text{a.s.},$$

provided that (1.8) holds for $c = 1$. Assume now that Theorem 2.1 is true under condition $\overline{\Phi}\overline{X} < c$ a.s., which is equivalent to (3.5) by Remark 2.2. Let $\{Y_0^{(\varepsilon)}(t)\}$ be the risk process generated by $\Psi^{(\varepsilon)}$. Because

$$Y_0^{(\varepsilon)}(t) = \varepsilon\Big|\frac{t}{\varepsilon} - \sum_{0 < T_n \le t/\varepsilon} X_n,$$

we get, by applying Theorem 2.1 for $\{Y_0(t)\}$ with $c = \varepsilon$,

$$P\big(Z_0^{(\varepsilon)} < x\big) = \frac{\lambda}{\varepsilon}\int_0^x (1 - F(u))\, du, \tag{3.6}$$

where $Z_0^{(\varepsilon)}$ is the first decreasing ladder height; that is, the deficit at the ruin time of the risk process $\{Y_0^{(\varepsilon)}(t)\}$. Let $\tau_0^{(\varepsilon)} = \inf\{t \ge 0: Y_0^{(\varepsilon)}(t) < 0\}$ and $\Omega^{(\varepsilon)} = \{\omega: \tau_0^{(\varepsilon)}(\omega) = \varepsilon\tau_0(\omega)\}$. Because $Z_0^{(\varepsilon)} = (1 - \varepsilon)\tau_0 + Z_0$ on $\Omega^{(\varepsilon)}$ and $\lim_{\varepsilon\downarrow 1} P(\Omega^{(\varepsilon)}) = 1$, we have $\lim_{\varepsilon\downarrow 1} Z_0^{(\varepsilon)} = Z_0$ a.s. From this and (3.6), we see that condition (1.8) can be replaced by (3.5) in Theorem 2.1.

Note that (3.5) implies

$$\lambda E_0(X_0) < 1. \tag{3.7}$$

This is proved by using the individual ergodic theorem. Let $\mathscr{I}$ be the invariant sub-σ-field of $\mathscr{F}$ with respect to $\{\theta_t\}$. Then we have

$$E(\overline{\Phi}\overline{X}) = E\left(E\left(\sum_{k=1}^{\Phi(1)} X_k | \mathscr{I}\right)\right) = E\left(\sum_{k=1}^{\Phi(1)} X_k\right) = \lambda E_0(X_0).$$

Hence (3.5) implies (3.7). See Lemma 2.1 of Miyazawa (1979) for the detailed calculations. It should be noted that (3.7) is really weaker than (3.5) and that we do need (3.5) in the following proof.

Let us consider a single server queue with the input $\{(-T_{-n}, X_{-n})\}$, where T_{-n} and X_{-n} denote the arrival and service times of the nth customer, respectively. Note that $\{(-T_{-n}, X_{-n})\}$ is the time-reversed process of $\{(T_n, X_n)\}$, and hence is stationary, too. Furthermore, the left-hand side of condition (3.5) is invariant under time reversion. Thus, there exists a work load process $\{V(t)\}$ that is finite and stationary with respect to P under the condition (3.5). We here assume that $V(t)$ is left-continuous for all t. Define $W_n = V(-T_{-n})$. Then, $\{W_n\}$ is a sequence of waiting times of customers and $W_0 = V(0)$ is finite with respect to P_0. For details of these facts, refer, for example, to Franken, König, Arndt and Schmidt (1982). Note that (3.7) is not sufficient for the finiteness of $V(t)$ and W_n. We will use the following fact in our proof.

LEMMA 3.3. *Under condition* (3.5), *we have, for any* $a \geq 0$,

$$\{V(0) > a\} = \{Z_a < +\infty\} \tag{3.8}$$

and, consequently,

$$\lim_{a \to \infty} P(Z_a < +\infty) = \lim_{a \to \infty} P_0(Z_a < +\infty) = 0. \tag{3.9}$$

PROOF. By the well-known construction for $\{V(t)\}$ due to Loynes (1962), we have

$$\begin{aligned} V(0) &= \sup\left\{ \sum_{\{k:\, -u < -T_{-k} < 0\}} X_{-k} - u \colon u \geq 0 \right\} \\ &= \sup\left\{ \sum_{\{k:\, 0 < T_k < u\}} X_k - u \colon u \geq 0 \right\} = -\inf\{Y_0(u) \colon u \geq 0\}, \end{aligned}$$

where the first equality follows from the left continuity of the work load $V(t)$ and the fact that $V(0)$ is determined by the time-reversed input $\{(-T_{-n}, X_{-n})\}$. Hence, $\{Z_a < +\infty\} = \{\inf\{Y_0(u) + a \colon u \geq 0\} < 0\}$ implies (3.8). Because $P(V(0) < +\infty) = P_0(V(0) < +\infty) = 1$, this proves (3.9). □

By Lemma 3.2, $P(Z_a < x)$ has bounded variation with respect to a on each finite interval $[0, h]$, which implies that, as a function of a, the probabilities $P(Z_a < x)$ define a signed measure. Because the $P(Z_a < x)$ have right-hand derivatives,

$$\int_0^h \frac{d^+}{da} P(Z_a < x)\, da = P(Z_h < x) - P(Z_0 < x) \qquad \text{for } 0 < h < \infty. \tag{3.10}$$

By Lemma 3.3, the right-hand side of (3.10) converges to $-P(Z_0 < x)$ as h tends to infinity. We now integrate both sides of (2.2) of Theorem 2.2 over $[0, h]$. Then, the sum of the first two terms of the right-hand side of (2.2) is bounded by

$$\begin{aligned}
&\lambda\left|\int_0^h P_0(Z_a < x)\,da - \int_0^h P_0(Z_{a-X_0} < x, X_0 \le a)\,da\right| \\
&\quad = \lambda\left|E_0\left(\int_0^h I_{\{Z_a < x\}}\,da - \int_0^{(h-X_0)^+} I_{\{Z_a < x\}}\,da\right)\right| \\
&\quad = \lambda E_0\left(\int_{(h-X_0)^+}^h I_{\{Z_a < x\}}\,da\right) \\
&\quad \le \lambda E_0\left(X_0 I_{\{Z_{(h-X_0)^+} < +\infty\}}\right) \\
&\quad = \lambda E_0\left(X_0 I_{\{V(0)+X_0 > h\}} I_{\{X_0 \le h\}}\right) + \lambda E_0\left(X_0 I_{\{X_0 > h\}}\right),
\end{aligned}$$

where Lemma 3.3 has been used to get the last inequality. Using again Lemma 3.3 we see that the first term on the right-hand side of this inequality converges to zero as h tends to infinity, whereas the second term does so because $E_0(X_0) < \infty$ by (3.7). On the other hand, for the last term of the right-hand side of (2.2) we get

$$\begin{aligned}
&-\lambda\int_0^h (\overline{F}(a) - \overline{F}(a+x))\,da \\
&\qquad \to -\lambda\int_0^x \overline{F}(a)\,da, \quad h \to +\infty.
\end{aligned} \tag{3.11}$$

This completes the proof of Theorem 2.1. □

4. Related results. In Theorem 2.1, if (1.8) fails, the formula (1.5) remains true in a slightly modified form. It is not difficult to see that, in general,

$$P(\tau_0 < \infty) = P(Z_0 < \infty) = E\left(\min\left\{1, \frac{\overline{\Phi}}{c}\overline{X}\right\}\right)$$

holds. If the stationary marked point process $\{(T_n, X_n)\}$ is ergodic, then this formula simplifies to

$$P(\tau_0 < \infty) = \min\left\{1, \frac{\lambda}{c}E_0(X_0)\right\}.$$

However, it seems to be much more complicated to determine the distribution of Z_0 if the balance condition (1.8) does not hold. It turns out that, in general, in contrast to the case $\overline{\Phi}\overline{X} \le c$, the distribution of Z_0 essentially depends on the whole distribution of the point process $\{T_n\}$, and not only on its intensity λ.

In many cases, however, the distribution Z_0 should converge to that given by (1.3) if the intensity λ converges to $c[E_0(X_0)]^{-1}$ from above. Our conjecture is that this continuity property holds for a large class of stationary marked point processes $\{(T_n, X_n)\}$ and that, consequently, (1.3) can be used to approximate the probabilities $P(Z_0 < x)$ when $\overline{\Phi}\overline{X} \le c$ does not hold with probability 1, but $(\lambda/c)E_0(X_0)$ differs only a little from 1. The following results show a possible way how to verify this.

THEOREM 4.1. *Assume that $E_0(X_0) < \infty$. If, for every $x \ge 0$, the limit $p(x) = \lim_{a\to\infty} P(Z_a < x)$ exists and if*

$$p(x) = \lim_{a\to\infty} P_0(Z_a < x | X_0) \quad a.s., \tag{4.1}$$

then we have

$$P(Z_0 < x) = \frac{\lambda}{c}\int_0^x \overline{F}(u)\,du + \left(1 - \frac{\lambda}{c}E_0(X_0)\right)p(x) \tag{4.2}$$

and, consequently,

$$\sup_{0\le x<\infty}\left|P(Z_0 < x) - \frac{\lambda}{c}\int_0^x \overline{F}(u)\,du\right| \le \left|1 - \frac{\lambda}{c}E_0(X_0)\right|. \tag{4.3}$$

PROOF. Proceeding similarly as in the proof of Theorem 2.1, from (2.2) we get

$$\begin{aligned} P(Z_h < x) - P(Z_0 < x) &= \frac{\lambda}{c}E_0\left(\int_{(h-X_0)^+}^h I_{\{Z_a < x\}}\,da\right) \\ &\quad - \frac{\lambda}{c}\int_0^h \left(\overline{F}(a) - \overline{F}(a+x)\right) da \end{aligned}$$

for $0 < h < \infty$. Because of (3.11) and our assumptions, for proving (4.2) it suffices to consider the term

$$\begin{aligned} &E_0\left(\int_{(h-X_0)^+}^h I_{\{Z_a < x\}}\,da\right) \\ &\quad = E_0\left(\int_0^{\min(X_0,h)} I_{\{Z_{h-u} < x\}}\,du\right) \\ &\quad = E_0\left(\int_0^{\min(X_0,h)} P_0(Z_{h-u} < x | X_0)\,du\right) \\ &\quad = E_0\left(I_{\{X_0 < u_0\}}\int_0^{X_0} P_0(Z_{h-u} < x|X_0)\,du\right) \\ &\qquad + E_0\left(I_{\{X_0 \ge u_0\}}\int_0^{\min(X_0,h)} P_0(Z_{h-u} < x|X_0)\,du\right) \end{aligned}$$

for every $u_0 \le h$. Thus, because $E_0(X_0) < \infty$, from (4.1) we get

$$\lim_{h\to\infty} E_0\left(\int_{(h-X_0)^+}^h I_{\{Z_a < x\}}\,da\right) = E_0(X_0)p(x).$$

This completes the proof. □

REMARK 4.1. Note that, within the family of stationary marked point processes $\{(T_n, X_n)\}$ satisfying (4.1), the continuity property (4.3) of the distribution Z_0 is uniform in the sense that the bound $|1 - (\lambda/c)E_0(X_0)|$ in (4.3) does not depend on the form of the distribution of $\{(T_n, X_n)\}$, but only on its intensity λ.

Finally, we discuss condition (4.1). For arbitrary $0 \le a < \infty$, the event $\{Z_a < x\}$ can be expressed by the records of the dependent random walk $\{Y_n; n \ge 1\}$, where

$$Y_n = Y(T_n) = T_n - \sum_{k=1}^{n} X_k.$$

For showing this we define the ladder epochs $\tau_0^{(n)}$ by $\tau_0^{(1)} = \tau_0$, $\tau_0^{(n+1)} = \tau_0(\tau_0^{(n)}) + \tau_0^{(n)}$, and a new point process $\{T_n^*; n \ge 1\}$ by

$$T_1^* = Z_0(0), \qquad T_2^* = Z_0\big(\tau_0^{(1)}\big) + T_1^*, T_2^* = Z_0\big(\tau_0^{(2)}\big) + T_2^*, \dots$$

and its counting process $\{\Phi^*(t); t \ge 0\}$ by $\Phi^*(t) = \#\{n: T_n^* \in (0, t]\}$. Note that, if $\overline{\Phi X} > c$ a.s., τ_n^* and hence T_n^* are finite. However, if $\overline{\Phi X} > c$ does not hold with probability 1, T_n^* may be infinite. We have

$$\{Z_a < x\} = \big\{T^*_{\Phi^*(a)+1} - a < x\big\} \tag{4.4}$$

for $0 \le a < \infty$; that is, $\{Z_a < x\}$ means the event that the forward residual time of the point process $\{T_n^*\}$ measured from time a is less than x. This leads to the following reformulation of (4.1) in terms of $\{T_n^*\}$.

LEMMA 4.1. *The condition* (4.1) *holds if and only if, for each* $x \ge 0$,

$$\lim_{a \to \infty} P\big(T^*_{\Phi^*(a)+1} - a < x\big) = \lim_{a \to \infty} P_0\big(T^*_{\Phi^*(a)+1} - a < x | X_0\big) \quad a.s. \tag{4.5}$$

Thus, it becomes clear that some mixing property of the original marked point process $\{(T_n, X_n)\}$ should be sufficient for the validity of (4.1) if $\overline{\Phi X} > c$ a.s. For example, an immediate consequence of Lemma 4.1 is that the bound (4.3) holds for a large class of recurrent marked point processes, where $\{(T_n, X_n)\}$ is called a recurrent marked point process if, with respect to its Palm measure P_0, the pairs $(T_n - T_{n-1}, X_n)$ form a sequence of independent identically distributed random variables. Furthermore, the distribution of a real-valued random variable is said to be arithmetic if it is concentrated on a set of the form $\{0, \pm\delta, \pm 2\delta, \dots\}$ for some $\delta > 0$.

COROLLARY 4.1. *Assume that* $\{(T_n, X_n)\}$ *is a recurrent marked point process with* $\lambda E_0(X_0) > c$. *Then,* (4.2) *and* (4.3) *hold if the distribution of* $T_n - T_{n-1} - X_n$ *is nonarithmetic. Furthermore, we have*

$$p(x) = \frac{1}{E_0(Z_0)} \int_0^x P_0(Z_0 > u)\, du \quad \text{for every } x \ge 0. \tag{4.6}$$

PROOF. From our assumptions it follows that $\{T_n^*\}$ in (4.4) is a nonmarked recurrent point process (or, in other words, a renewal process) that is delayed with respect to P and $P_0(\cdot|X_0)$, respectively, where the $T_n^* - T_{n-1}^*$ have a nonarithmetic distribution. Thus, from well-known results of renewal theory [see, e.g., Chapter XI of Feller (1971)] we get (4.5), and, by Lemma 4.1 and Theorem 4.1, (4.2) follows. Because $T_n^* - T_{n-1}^*$ has the same distribution as Z_0 with respect to P_0, (4.6) is a well-known fact obtained by the key renewal theorem. □

REMARK 4.2. By (4.6), (4.2) gives the relationship

$$(4.7)\quad P(Z_0 < x) = \frac{\lambda}{c}\int_0^x \overline{F}(u)\,du + \frac{c - \lambda E_0(X_0)}{cE_0(Z_0)}\int_0^x P_0(Z_0 > u)\,du.$$

We note that, in some cases, Wiener–Hopf techniques can be applied to determine the distribution $P_0(Z_0 < x)$ and, consequently, by using (4.7), we can determine $P(Z_0 < x)$.

REMARK 4.3. In a similar way, we can prove that the formulas (4.2) and (4.3) hold for further classes of marked point processes. For example, if $\{T_n\}$ is a Markov renewal process or, in particular, a Markov-modulated Poisson process, then $\{T_n^*\}$ is again a Markov renewal process and, consequently, (4.5) can be proved by using a corresponding limit theorem [see, e.g., Chapter 10.6 of Çinlar (1975)], which implies (4.2) and (4.3).

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