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A STOCHASTIC NAVIER–STOKES EQUATION FOR THE VORTICITY OF A TWO-DIMENSIONAL FLUID[1]

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The Navier–Stokes equation for the vorticity of a viscous and incompressible fluid in $\mathbf{R}^2$ is analyzed as a macroscopic equation for an underlying microscopic model of randomly moving vortices. We consider N point vortices whose positions satisfy a stochastic ordinary differential equation on $\mathbf{R}^{2N}$, where the fluctuation forces are state dependent and driven by Brownian sheets. The state dependence is modeled to yield a short correlation length ε between the fluctuation forces of different vortices. The associated signed point measure-valued empirical process turns out to be a weak solution to a stochastic Navier–Stokes equation (SNSE) whose stochastic term is state dependent and small if ε is small. Thereby we generalize the well known approach to the Euler equation to the viscous case. The solution is extended to a large class of signed measures conserving the total positive and negative vorticities, and it is shown to be a weak solution of the SNSE. For initial conditions in $L_2(\mathbf{R}^2, dr)$ the solutions are shown to live on the same space with continuous sample paths and an equation for the square of the L_2-norm is derived. Finally we obtain the macroscopic NSE as the correlation length $\varepsilon \to 0$ and $N \to \infty$ (macroscopic limit), where we assume that the initial conditions are sums of N point measures. As a corollary to the above results we obtain the solution to a bilinear stochastic partial differential equation which can be interpreted as the temperature field in a stochastic flow.

1. Introduction: macroscopic, microscopic and mezoscopic models. Our aim is to model the time evolution of the vorticity of a two-dimensional incompressible fluid (where for a rigid body the vorticity is twice the angular velocity). The restriction to two dimensions is natural in applications like oceanography, where the depth is considered to be negligible in comparison to its planar extension. Although for applications in oceanography one should include the action of the Coriolis force on the vorticity distribution, we will neglect its contribution here since we want to be conceptual. Moreover, we believe that it is fairly easy to include the Coriolis contribution into our models, since it acts in the form of an external force on the system. Under the

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above assumptions we obtain a *macroscopic equation* for the distribution of vorticity in a two-dimensional fluid:

$$
(1.1)\qquad \begin{aligned}
&\frac{\partial}{\partial t}X(r,t) = \nu\Delta X(r,t) - \nabla\cdot(U(r,t)X(r,t)),\\
&X(r,t) = \operatorname{curl} U(r,t) = \frac{\partial U_2}{\partial r_1} - \frac{\partial U_1}{\partial r_2}, \qquad \nabla\cdot U \equiv 0.
\end{aligned}
$$

Here $U(r,t)$ is the velocity field, $r \in \mathbf{R}^2$, $\nu \geq 0$ is the kinematic viscosity (or in the oceanographic setting, the eddy diffusion coefficient), Δ is the Laplacian, ∇ is the gradient and $\cdot$ denotes the scalar product on $\mathbf{R}^2$. If $\nu > 0$, we obtain the Navier–Stokes equation for the vorticity. If the fluid is inviscid (or ideal), that is, $\nu = 0$, we obtain the Euler equation. Note that by the incompressibility condition $\nabla\cdot U = 0$ we obtain

$$
(1.2)\qquad U(r,t) = \int(\nabla^{\perp}g)(r-q)X(q,t)\,dq,
$$

where $g(|r|) := (1/2\pi)\ln(|r|)$ with $|r|^2 = r_1^2 + r_2^2$ and $\nabla^{\perp} = (-\partial/\partial r_2, \partial/\partial r_1)^T$ with T denoting the transpose; $\int(\ )\,dq$ denotes integration over $\mathbf{R}^2$ with respect to the Lebesgue measure. As a consequence we can obtain the velocity field U, which satisfies the standard Navier–Stokes equation from the vorticity distribution.

There is an extensive literature on the (numerical) solution of (1.1) by the so-called (random) point vortex method (cf. Chorin [3, 4], Long [22], Puckett [29] and the references therein). A theoretical model related to the point vortex model has been analyzed by Marchioro and Pulvirenti [23], which is a special case of the following more general model.

Let $0 < \delta \leq 1$ and $g_\delta(|r|) \equiv g(|r|)$, for $\delta \leq |r| \leq 1/\delta$. Let $g_\delta(s)$ be at least twice continuously differentiable with bounded derivatives up to order 2, and let $|g_\delta'(s)| \leq |g'(s)|$ and $|g_\delta''(s)| \leq |g''(s)|$, for $s > 0$. Set

$$
K_\delta(r) := \nabla^{\perp}g_\delta(|r|).
$$

We may assume without loss of generality $g'(0) = 0$, which implies $K_\delta(0) = 0$. Thus we have the smoothed Navier–Stokes equation (NSE):

$$
(1.3)\qquad \begin{aligned}
&\frac{\partial}{\partial t}X(r,t) = \nu\Delta X(r,t) - \nabla\cdot(U_\delta(r,t)X(r,t)),\\
&U_\delta(r,t) := \int K_\delta(r-q)X(q,t)\,dq.
\end{aligned}
$$

Consider N point vortices with intensities $a_i \in \mathbf{R}$ and let r^i be the position of the ith vortex. Abbreviate $r_N := (r^1,\ldots,r^N) \in \mathbf{R}^{2N}$. Assume that the positions satisfy the stochastic ordinary differential equation (SODE)

$$
(1.4)\qquad dr^i(t) = \sum_{j=1}^{N} a_j K_\delta(r^i - r^j)\,dt + \sqrt{2\nu}\,dm^i(r_N,t), \qquad i = 1,\ldots,N.
$$

The $m^i(r_N, t)$ are $\mathbf{R}^2$-valued square-integrable continuous martingales ($i = 1, \dots, N$), which may depend on the positions of the vortices. We will call (1.4) a *microscopic model* for the vorticity and (1.1) a *macroscopic model*. Let us for the moment assume that for suitably adapted square-integrable initial conditions (1.4) has a unique (Itô) solution $r_N(t) = (r^1(t), \dots, r^N(t))$. Set

$$\mathscr{X}_N(t) := \sum_{i=1}^{N} a_i \delta_{r^i(t)}, \tag{1.5}$$

where $r^i(t)$ are the solutions of (1.4) and δ_r is the point measure concentrated in r. We will call $\mathscr{X}_N(t)$ the empirical process associated with the SODE (1.4). Let $L_p(\mathbf{R}^2, dr)$ be the standard L_p-spaces of real-valued functions on $\mathbf{R}^2$ with $p \in [1, \infty]$, where dr is the Lebesgue measure. Set $\mathbf{H}_0 := L_2(\mathbf{R}^2, dr)$ and denote by $\langle \cdot, \cdot \rangle_0$ and $\|\cdot\|_0$ the standard scalar product and its associated norm on $\mathbf{H}_0$. Further let $\langle \cdot, \cdot \rangle$ be the extension of $\langle \cdot, \cdot \rangle_0$ to a duality between distributions and smooth functions. The following facts have been established by Marchioro and Pulvirenti [23]. Suppose for the initial condition in (1.1), $X(0) \in L_1(\mathbf{R}^2, dr) \cap L_\infty(\mathbf{R}^2, dr)$.

A1. Let $\nu = 0$ [i.e., (1.4) is deterministic] and let $\mathscr{X}$ be the solution of the Euler equation (1.1) with $\nu = 0$. Then there is a sequence $K_{\delta(N)}(r) \to K(r) := \nabla^\perp g(r)$, as $N \to \infty$ such that

$$\langle \mathscr{X}_N(t), \varphi \rangle \to \langle \mathscr{X}(t), \varphi \rangle \quad \text{as } N \to \infty, \tag{1.6}$$

that is, $\mathscr{X}_N(t)$ is "approximately" a weak solution of the Euler equation, where φ is sufficiently smooth with compact support. If one chooses $K(r)$ instead of $K_\delta(r)$ in (1.4) and assumes that (1.4) has a unique solution for suitable initial values, we obtain directly that $\mathscr{X}_N(t)$ is a weak solution of the Euler equation.

A2. Let $\nu > 0$. Choose $m^i(r_N, t) := \beta^i(t)$, where $\beta^i(t)$ are i.i.d. $\mathbf{R}^2$-valued standard Wiener processes. In particular, the martingales are state independent. Further assume that half of the intensities a_j are positive and equal to some a^+/N, $a^+ > 0$, and the other half are negative and equal to $-a^-/N$, $a^- > 0$. Let X be the solution of the Navier–Stokes equation (1.1) with $\nu > 0$. Again with the same sequence $K_{\delta(N)}(r)$ and φ as in A1, $\langle E\mathscr{X}_N(0), \varphi \rangle \to \langle X(0), \varphi \rangle$ as $N \to \infty$ implies, for any $t > 0$,

$$\langle E\mathscr{X}_N(t), \varphi \rangle \to \langle X(t), \varphi \rangle \quad \text{as } N \to \infty, \tag{1.7}$$

where E denotes the mathematical expectation, that is, $E(\cdot) = \int (\cdot)\, dP$, with P being the probability measure from an underlying probability space $(\Omega, \mathscr{F}, P)$.

The key to understanding the relation between (1.4), (1.5) and (1.1) is the Itô formula. Let us abbreviate

$$U_{\delta, N}(r, t) := \int K_\delta(r - q) \mathscr{X}_N(dq, t). \tag{1.8}$$

Denote by $\langle\langle m_k^i(r_N,t), m_l^j(r_N,t)\rangle\rangle$ the mutual quadratic variation process of the one-dimensional components of $m^i(r_N,t)$ and $m^j(r_N,t)$, $k,l \in \{1,2\}$, $i,j \in \{1,\dots,N\}$ (cf. Metivier and Pellaumail [24]). For $m \in \mathbf{N}$, let $C_b^m(\mathbf{R}^2,\mathbf{R})$ be the set of bounded and Lebesgue integrable functions from $\mathbf{R}^2$ into $\mathbf{R}$ which are m times continuously differentiable in all variables with bounded and Lebesgue integrable derivatives. Abbreviate $r = (r_1, r_2)$. If $\varphi \in C_b^2(\mathbf{R}^2,\mathbf{R})$, the Itô formula yields

$$
\begin{aligned}
d\langle \mathscr{X}_N(t), \varphi\rangle &= \langle \mathscr{X}_N(t), (U_\delta \cdot \nabla)\varphi\rangle\, dt \\
&\quad + \nu \sum_{i=1}^N a_i \sum_{k,l=1}^2 \frac{\partial^2}{\partial r_k\, \partial r_l}\varphi(r^i(t))\, d\langle\langle m_k^i(r_N,t), m_l^i(r_N,t)\rangle\rangle \\
&\quad + \sqrt{2\nu}\sum_{i=1}^N a_i\big(\nabla\varphi(r^i(t))\big)\cdot dm^i(r_N,t).
\end{aligned}
\tag{1.9}
$$

If $\nu = 0$, then (1.9) is a weak form of the (smoothed) Euler equation (1.1) (with K_δ instead of K). However, if $\nu > 0$ and we choose, following Marchioro and Pulvirenti [23], the i.i.d. standard Wiener processes $\beta^i(t)$ for the $m^i(r_N,t)$, then (1.9) becomes

$$
\begin{aligned}
d\langle \mathscr{X}_N(t), \varphi\rangle &= \langle \mathscr{X}_N(t), (U_\delta \cdot \nabla)\varphi\rangle\, dt + \langle \mathscr{X}_N(t), \nu\Delta\varphi\rangle\, dt \\
&\quad + \sqrt{2\nu}\sum_{i=1}^N a_i \nabla\varphi(r^i(t))\cdot d\beta^i(t).
\end{aligned}
\tag{1.10}
$$

The difference from the Euler case is twofold. First, although the stochastic term in (1.10) disappears after taking the mathematical expectation on both sides, this mathematical expectation does not satisfy the smoothed version of the NSE (1.1) because of the nonlinearity. Second, note that the motion of each particle is perturbed by its own "fluctuation force" $d\beta^i(t)$, $i = 1,\dots,N$ (cf. Nelson [26] for a justification of this terminology). This introduces a tagging in (1.10), that is, the "name tags" ($i = 1,\dots,N$) are preserved in the stochastic term, whereas in the deterministic terms they disappear. If it were not for the tagging in the representation of the "fluctuation forces" by $d\beta^i(t)$, (1.10) would be some sort of (smoothed) stochastic Navier–Stokes equation (SNSE) whose right-hand side is given by the right-hand side of the (smoothed) NSE plus the fluctuation forces. We easily see that under the Marchioro–Pulvirenti assumptions, $\mathscr{X}_N(t)$ is a signed measure-valued Markov process (cf. Dynkin [9], Chapter 10, Section 6). Analyzing its generator, one can derive a (formal) stochastic partial differential equation (SPDE) for $\mathscr{X}_N(t)$ on the space of signed measures (with $W(t)$ an $\mathbf{H}_0$-valued standard cylindrical Brownian motion; cf. Kotelenez [16]):

$$
d\mathscr{X}_N = [\nu\Delta\mathscr{X}_N - \nabla\cdot(U_{\delta,N}\mathscr{X}_N)]\, dt + \sqrt{\frac{2\nu}{N}}\, F_N(\mathscr{X}_N)\cdot dW,
\tag{1.11}
$$

where

$$
F_N(\mathscr{X}_N) := \sqrt{N}\Big(\sqrt{-\nabla^T\mathscr{X}_N^+\,\nabla} - \sqrt{-\nabla^T\mathscr{X}_N^-\,\nabla}\Big).
$$

For nice functions $\mathscr{X}_N$, $\sqrt{-\nabla^T \mathscr{X}_N^{\pm} \nabla}$ is the positive root of the self-adjoint extension of $-\nabla^T \mathscr{X}_N^{\pm} \nabla$ (via quadratic forms, where $\mathscr{X}_N^{\pm}$ acts as a multiplication operator). A smoother version of such an SPDE was derived and analyzed by Dawson [6] (for the mass distribution of branching Brownian particles, now called a superprocess; cf. also Konno and Shiga [13] for an updated analysis of Dawson's result). The high singularity of the diffusion coefficient in (1.11) (and in [6]) is a consequence of the independence assumption (in spatial coordinates) of the fluctuation forces in (1.4) if $m^i(r_N, t) = \beta^i(t)$. One can expect a smoother diffusion term if the fluctuation forces are spatially correlated (as suggested by Vaillancourt [32] for a different physical model). From our point of view, the correlations must be introduced such that the requirements of the following program are satisfied:

1.1 PROGRAM.

(i) $\mathscr{X}_N(t)$ is the weak solution of a (smoothed) SNSE [as $\mathscr{X}_N(t)$ is the weak solution of the (smoothed) Euler equation (1.3) if $\nu = 0$].

(ii) The intensity and the correlation length of the fluctuation forces should be small and short, respectively.

(iii) The SNSE should be extendible to a large class of signed measures including those with densities so that certain density-valued initial conditions yield density-valued solutions.

(iv) The total positive and negative vorticities are (pathwise) conserved quantities.

(v) As noise intensity and correlation length of the fluctuation forces tend to zero [and K_δ from (1.3) tends to $\nabla^{\perp} g$ simultaneously], the solution of the SNSE should converge to the solution of the NSE (1.1) (macroscopic limit provided the initial conditions of the SNSE converge to the smooth initial conditions of the NSE).

In order to realize this program we now introduce rigorous assumptions. [Requirement (v) has been solved in this paper only in part; cf. Remark 4.5.]

Let $(\Omega, \mathscr{F}, \mathscr{F}_{t \geq 0}, P)$ be a stochastic basis with right continuous filtration. All our stochastic processes are assumed to live on Ω and to be $\mathscr{F}_t$-adapted (including all initial conditions in SODE's and SPDE's). Moreover, the processes are assumed to be $(dP \otimes dt)$-measurable, where dt is the Lebesgue measure on $[0, \infty)$. Let $w_l(r, t)$ be independent Brownian sheets on $\mathbf{R}^2 \times \mathbf{R}_+$, $l = 1, 2$ (cf. Walsh [35] and Kotelenez [17]), with mean zero and variance $t|A|$, where A is a Borel set in $\mathbf{R}^2$ with finite Lebesgue measure $|A|$. Adaptedness for $w_l(r, t)$ means that $\int_A w_l(dp, t)$ is adapted for any Borel set $A \subset \mathbf{R}^2$ with $|A| < \infty$. Set $w(p, t) := (w_1(p, t), w_2(p, t))^T$. Further, let $\varepsilon > 0$ and define correlation functions $\tilde{\Gamma}_\varepsilon \colon \mathbf{R}^4 \to \mathbf{R}_+$ to be bounded Borel-measurable functions which are symmetric in $r, p \in \mathbf{R}^2$ such that the following conditions are satisfied:

$$\int \tilde{\Gamma}_\varepsilon^2(r, p)\, dp = 1 \tag{1.12}$$

and there is a finite positive constant c such that, for any $r, q \in \mathbf{R}^2$,

$$\int \tilde{\Gamma}_\varepsilon(r,p)\tilde{\Gamma}_\varepsilon(q,p)\,dp = c\sqrt{\varepsilon}\cdot\tilde{\Gamma}_{2\varepsilon}(r,q). \tag{1.13}$$

There are finite positive constants c, c_ε such that, defining

$$\varrho(r,q) := (c_\varepsilon|r-q|) \wedge 1, \tag{1.14}$$

we have

$$\int\left(\tilde{\Gamma}_\varepsilon(r,p) - \tilde{\Gamma}_\varepsilon(q,p)\right)^2 dp \le c\varrho^2(r,q), \tag{1.15}$$

where $\wedge$ denotes the minimum of two numbers. Let us give a particular correlation function, where (1.12)–(1.15) can be verified.

1.2. Example. Set $\tilde{\Gamma}_\varepsilon(r,p) := ((1/2\pi\varepsilon)\exp(-|r-p|^2/2\varepsilon))^{1/2}$. Equation (1.12) is obviously satisfied. The Chapman–Kolmogorov equation implies

$$\int \tilde{\Gamma}_\varepsilon(r,p)\tilde{\Gamma}_\varepsilon(q,p)\,dp = \exp\left(\frac{-|r-q|^2}{8\varepsilon}\right). \tag{1.16}$$

Therefore, $\int[\tilde{\Gamma}_\varepsilon(r,p) - \tilde{\Gamma}_\varepsilon(q,p)]^2\,dp = 2(1-\exp(-|r-q|^2/8\varepsilon)) \le 2(1 \wedge |r-q|^2/8\varepsilon)$. Hence if we set $\varrho(r,q) := 1 \wedge (|r-q|/\sqrt{8\varepsilon})$, we easily verify (1.14), (1.15) and that $\varrho(r,q)$ is a metric (using Minkowski's inequality).

Apparently, one can get a more general class of $\tilde{\Gamma}_\varepsilon(r,p)$ satisfying (1.12)–(1.15) than in the above example by taking $\tilde{\Gamma}_\varepsilon(r,q) := \sqrt{p(\varepsilon,r,q)}$, where $p(\varepsilon,r,q)$ is the transition density of an $\mathbf{R}^2$-valued diffusion process (at time $t=\varepsilon$), whose generator is a strictly elliptic (second order) operator with smooth coefficients.

Set

$$\hat{\Gamma}_\varepsilon(r,p) := \begin{pmatrix} \tilde{\Gamma}_\varepsilon(r,p) & 0 \\ 0 & \tilde{\Gamma}_\varepsilon(r,p) \end{pmatrix}. \tag{1.17}$$

If $q(t)$ is an $\mathbf{R}^2$-valued adapted stochastic process, then (1.12) and Walsh [35] imply that $\int_0^t\int\hat{\Gamma}_\varepsilon(q(s),p)w(dp,ds)$ is an $\mathbf{R}^2$-valued square-integrable continuous martingale. Therefore, we may choose $m^i(r_N,t) := \int\hat{\Gamma}_\varepsilon(r^i,p)w(dp,t)$ and (1.4) becomes

$$dr^i(t) := \sum_{j=1}^N a_j K_\delta(r^i - r^j)\,dt + \sqrt{2\nu}\int\hat{\Gamma}_\varepsilon(r^i,p)w(dp,dt), \qquad i=1,\dots,N. \tag{1.18}$$

1.3. Remarks. (i) Let $\{\tilde{\phi}_n\}_{n\in\mathbf{N}}$ be a complete orthonormal system (CONS) in $L_2(\mathbf{R}^2, dr)$ and set

$$\phi_n := \begin{pmatrix} \tilde{\phi}_n & 0 \\ 0 & \tilde{\phi}_n \end{pmatrix}.$$

Then

$$\int \hat{\Gamma}_\varepsilon(r,p) w(dp,t) = \sum_{n=1}^{\infty} \int \hat{\Gamma}_\varepsilon(r,q)\phi_n(q)\,dq\,\beta^n(t), \tag{1.19}$$

where $\beta^n(t) := \int \phi_n(q) w(dq,t)$ are $\mathbf{R}^2$-valued i.i.d. standard Wiener processes. Hence (1.18) can be treated as a $2N$-dimensional ordinary Itô equation which is driven by infinitely many i.i.d. $\mathbf{R}^2$-valued standard Wiener processes. It is well known that an $L_2(\mathbf{R}^2, dr)$-valued standard cylindrical Brownian motion $W(t)$ can be visualized as the (weak) limit of $\sum_{n=1}^{\infty} \tilde{\beta}_n(t)\tilde{\phi}_n$ (cf. Kotelenez [17]), where $\tilde{\beta}_n(t)$ is without loss of generality the first component of $\beta_n(t)$.

(ii) We now see that (1.15) is a Lipschitz assumption on the stochastic coefficient in (1.18). If $r, q \in \mathbf{R}^2$, we obtain that

$$\begin{aligned}&\sum_{n=1}^{\infty} \left[\int \left(\tilde{\Gamma}_\varepsilon(r,p) - \tilde{\Gamma}_\varepsilon(q,p)\right)^2 \tilde{\phi}_n(p)\,dp\right]^2 \\ &\quad = \int \left[\tilde{\Gamma}_\varepsilon(r,p) - \tilde{\Gamma}_\varepsilon(q,p)\right]^2 dp \le c\varrho^2(r,q).\end{aligned} \tag{1.20}$$

(iii) Assume there are two $\mathbf{R}^2$-valued adapted stochastic processes $q^1(t)$ and $q^2(t)$. Then we easily see that the $\int_0^t \int \hat{\Gamma}_\varepsilon(q^i(s), p) w(dp, ds)$ are $\mathbf{R}^2$-valued square-integrable continuous martingales ($i = 1, 2$) and their mutual quadratic variation is given by

$$\begin{aligned}&\left\langle\!\left\langle \int_0^t \int \tilde{\Gamma}_\varepsilon(q^1(s),p) w_k(dp,ds), \int_0^t \int \tilde{\Gamma}_\varepsilon(q^2(s),p) w_l(dp,ds) \right\rangle\!\right\rangle \\ &\quad = \int_0^t \int \tilde{\Gamma}_\varepsilon(q^1(s),p)\tilde{\Gamma}_\varepsilon(q^2(s),p)\,dp\,ds \cdot \delta_{k,l},\end{aligned} \tag{1.21}$$

$k, l = 1, 2$ with $\delta_{k,l} = 1$, if $k = l$, and 0, otherwise. Moreover, assuming the setup of Example 1.2, (1.16) implies that correlations are negligible if $|q_1(s) - q_2(s)|^2 \gg \varepsilon$ and that they are observable if $|q_1(s) - q_2(s)|^2 \sim \varepsilon$. In other words, ε is the (short) correlation length of requirement (ii) of Program 1.1. For metric spaces $\mathbf{M}_1, \mathbf{M}_2$, $C(\mathbf{M}_1, \mathbf{M}_2)$ is the space of continuous functions from $\mathbf{M}_1$ into $\mathbf{M}_2$. Let us endow $\mathbf{R}^2$ with the metric ϱ from (1.14) and $\mathbf{R}^{2N}$ with $\varrho_N(r_N, q_N) := \max_{1 \le i \le N} \varrho(r^i, q^i)$. To indicate this choice of the metric on $\mathbf{R}^2$ (resp., $\mathbf{R}^{2N}$), we will write $(\mathbf{R}^2, \varrho)$ [resp., $(\mathbf{R}^{2N}, \varrho_N)$] and just $\mathbf{R}^2$ (resp., $\mathbf{R}^{2N}$) if we use the usual Euclidean metric.

1.4. LEMMA. *To each $\mathscr{F}_0$-adapted initial condition $r_N(0) \in (\mathbf{R}^{2N}, \varrho_N)$, (1.18) has a unique $\mathscr{F}_t$-adapted solution $r_N(\cdot) \in C([0,\infty); \mathbf{R}^{2N})$ a.s., which is an $\mathbf{R}^{2N}$-valued Markov process.*

PROOF. (i) Let $q_{N,l}(\cdot) = (q_l^1(\cdot), \dots, q_l^N(\cdot))$ be $\mathbf{R}^{2N}$-valued adapted $(dt \otimes dP)$-measurable stochastic processes, $l = 1, 2$. Set, for $l = 1, 2$,

$$Q_{N,l}(t) := \sum_{i=1}^{N} a_i \delta_{q_l^i(t)}$$

and

$$\hat{q}_l^i(t) := q_l^i(0) + \int_0^t \int K_\delta(q_l^i(s) - p) Q_{N,l}(dp, s)\, ds + \int_0^t \int \hat{\Gamma}_\varepsilon(q_l^i(s), p) w(dp, ds).$$

(ii) By the smoothness and boundedness of K_δ, (1.14) and the Cauchy–Schwarz inequality, we obtain

$$\varrho^2\left(\int_0^t \int K_\delta(q_1^i(s) - p) Q_{N,1}(dp, s)\, ds, \int_0^t \int K_\delta(q_2^i(s) - p) Q_{N,2}(dp, s)\, ds\right) \le c_T \bar{c}_\delta \cdot N \int_0^t \varrho_N^2(q_{N,1}(s), q_{N,2}(s))\, ds.$$

(iii) Doob's inequality, (1.15), (1.17) and (1.21) imply

$$E \sup_{0 \le t \le T} \left| \int_0^t \int \left[\hat{\Gamma}_\varepsilon(q_1^i(s), p) - \hat{\Gamma}_\varepsilon(q_2^i(s), p)\right] w(dp, ds)\right|^2 \le 4c \int_0^T E \varrho^2(q_1^i(s), q_2^i(s))\, ds.$$

(iv) Hence the existence of a unique continuous solution follows from the contraction mapping principle. The Markov property follows as for Itô equations with perturbations by finitely many Wiener processes (cf. Dynkin [9], Chapter 11, Section 2). □

If μ is a finite (signed) Borel measure on $\mathbf{R}^2$, we set

$$\mu \int \hat{\Gamma}_\varepsilon(\cdot, p) w(dp, t) := \int \hat{\Gamma}_\varepsilon(\cdot, p) w(dp, t) \mu(dr),$$

that is, $\int \hat{\Gamma}_\varepsilon(\cdot, p) w(dp, t)$ is treated as a density with respect to μ.

If μ itself has a density with respect to the Lebesgue measure, we will denote this density also by μ (i.e., $\langle \varphi, \mu \rangle = \langle \varphi, \mu \rangle_0$), and the above expression reduces to multiplication between μ and the stochastic integral. Further, μ^+ and μ^- are the Jordan decompositions of the finite signed measure μ and $|\mu| := \mu^+ + \mu^-$ is the total variation. If b^+, b^- are nonnegative numbers and A is a Borel set in $\mathbf{R}^2$, we will write

$$\mu^\pm(A) = b^\pm \quad \text{if and only if } \mu^+(A) = b^+ \text{ and } \mu^-(A) = b^-.$$

Let a^+ and a^- be nonnegative numbers such that $a := a^+ + a^- > 0$. Set

$$\mathbf{M} := \{\mu: \mu \text{ is a finite signed Borel measure on } \mathbf{R}^2,\ \mu^{\pm}(\mathbf{R}^2) = a^{\pm}\}.$$

The number $a, a^{\pm}$ will be fixed for the rest of the paper. The following SNSE on $\mathbf{M}$ will be analyzed in this paper according to Program 1.1:

$$\begin{aligned} d\mathscr{X}(t) &= \left[\nu\Delta\mathscr{X} - \nabla\cdot\left(\tilde{U}_\delta\mathscr{X}\right)\right]dt \\ &\qquad - \sqrt{2\nu}\,\nabla\cdot\left(\mathscr{X}\int\hat{\Gamma}_\varepsilon(\cdot,p)\right)w(dp,dt), \qquad (1.22)\\ \tilde{U}_\delta(r,t) &:= \int K_\delta(r-q)\mathscr{X}(dq,t), \end{aligned}$$

(1.23) $\mathscr{X}^{\pm}(\mathbf{R}^2,t) = \mathscr{X}^{\pm}(\mathbf{R}^2,0) = a^{\pm}$ a.s. (conservation of vorticity).

[The assumption $\mathscr{X}^{\pm}(\mathbf{R}^2,0,\omega) = a^{\pm}$ for all ω has been made to avoid a cumbersome notation when working with a variant of the Wasserstein metric on the space of finite signed measures on $\mathbf{R}^2$.]

1.5. LEMMA. *Let $\mathscr{X}_N(t)$ be the empirical process associated with* (1.18) *such that $\sum_{a_j\ge 0} a_j = a^+$ and $\sum_{a_j<0} a_j = -a^-$. Then $\mathscr{X}_N(t)$ is a weak solution of* (1.22) *and* (1.23).

PROOF. (i) Equations (1.9), (1.12) and (1.21) for $q^1(t) \equiv q^2(t) \equiv r^i(t)$ yield, with $\varphi \in C^2(\mathbf{R}^2,\mathbf{R})$, $\mathscr{X}(t) := \mathscr{X}_N(t)$ and $U_\delta := U_{\delta,N}$,

$$\begin{aligned} d\langle\mathscr{X}(t),\varphi\rangle &= \left[\langle\mathscr{X}(t),\nu\Delta\varphi\rangle + \langle\mathscr{X}(t),(U_\delta\cdot\nabla)\varphi\rangle\right]dt \\ &\qquad + \sqrt{2\nu}\left\langle\mathscr{X}(t),\int\hat{\Gamma}_\varepsilon(\cdot,p)w(dp,dt)\cdot\nabla\varphi\right\rangle, \end{aligned} \qquad (1.24)$$

which is just the weak form of (1.22).

(ii) The conservation of total positive and negative vorticities required by (1.23) follows from the construction. □

Note that the empirical process $\mathscr{X}_N(t)$ reduces the detailed information provided by the microscopic model (1.18) to the information which is relevant for the problem under investigation. The process $\mathscr{X}_N(t)$ describes just how much vorticity is in a two-dimensional domain B at a given time t [and not where the single vortices are, as done in (1.18)]. By Lemma 1.5, $\mathscr{X}_N(t)$ satisfies (1.22), which is (1.3) plus the fluctuation term which we derived from the microscopic model (1.18). We may therefore call (1.22) a *mezoscopic model* for the vorticity. [Accordingly, (1.10) and (1.11) are different microscopic and mezoscopic models, respectively; cf. van Kampen [33] for the terminology.]

If $\mu, \tilde{\mu} \in \mathbf{M}$, we will call positive Borel measures $Q^{\pm}$ on $\mathbf{R}^4$ joint representations of $(\mu^+,\tilde{\mu}^+)$ [resp., $(\mu^-,\tilde{\mu}^-)$] if $Q^{\pm}(A\times\mathbf{R}^2) = \mu^{\pm}(A)a^{\pm}$ and $Q^{\pm}(\mathbf{R}^2\times B) = \tilde{\mu}^{\pm}(B)a^{\pm}$ for arbitrary Borel sets $A, B \subset \mathbf{R}^2$. The set of all joint representations of $(\mu^+, \tilde{\mu}^+)$ [resp., $(\mu^-, \tilde{\mu}^-)$] will be denoted by $C(\mu^+, \tilde{\mu}^+)$ [resp., $C(\mu^-, \tilde{\mu}^-)$]. For $\mu, \tilde{\mu} \in \mathbf{M}$ and $m = 1, 2$, set

$$\gamma_m(\mu, \tilde{\mu}) := \left[\inf_{Q^+ \in C(\mu^+, \tilde{\mu}^+)} \int\int Q^+(dr, dq)\varrho^m(r, q) + \inf_{Q^- \in C(\mu^-, \tilde{\mu}^-)} \int\int Q^-(dr, dq)\varrho^m(r, q)\right]^{1/m}. \tag{1.25}$$

By the boundedness of ϱ and the Cauchy–Schwarz inequality,

$$\gamma_1(\mu, \tilde{\mu}) \geq \gamma_2^2(\mu, \tilde{\mu}) \geq \frac{1}{a^+ \vee a^-}\gamma_1^2(\mu, \tilde{\mu}), \tag{1.26}$$

where $\vee$ denotes the maximum of two numbers.

After normalizing the measures by $\mu^\pm \to \mu^\pm / a^\pm$, the Kantorovich–Rubinstein theorem implies $\gamma_2(\mu, \tilde{\mu}) = 0$ if and only if $\mu^+ = \tilde{\mu}^+$ and $\mu^- = \tilde{\mu}^-$ (Dudley [8], Chapter 11). The triangle inequality for $\gamma_2(\mu, \tilde{\mu})$ follows as for the Wasserstein metric [which is $\gamma_1(\mu^+/a^+, \tilde{\mu}^+/a^+)$ for μ^+/a^+ and $\tilde{\mu}^+/a^+$]. Hence γ_2 is a metric on $\mathbf{M}$, and $\mathbf{M}$ endowed with γ_2 will be denoted by $(\mathbf{M}, \gamma_2)$. By (1.26), the Prohorov and the Kantorovich–Rubinstein theorems $(\mathbf{M}, \gamma_2)$ is complete (cf. Dudley [8], 11.5.5 and 11.8.2). Moreover, as in the Wasserstein case (cf. De Acosta [7], Appendix, Lemma 4) we obtain that the set of linear combinations of signed point measures from $\mathbf{M}$ is dense in $(\mathbf{M}, \gamma_2)$. For $f \in C(\mathbf{R}^2, \mathbf{R})$ we set

$$\|f\|_L := \sup_{\substack{r, q \in \mathbb{R}^2 \\ r \neq q}} \left\{\frac{|f(r) - f(q)|}{\varrho(r, q)}\right\}.$$

Inequality (1.26) and the Kantorovich–Rubinstein theorem imply

$$\gamma_2^2(\mu, \tilde{\mu}) \geq \frac{1}{2(a^+ \vee a^-)} \sup_{\|f\|_L \leq 1} \left|\langle \mu - \tilde{\mu}, f\rangle^2\right|. \tag{1.27}$$

In order to use SPDE techniques, it is convenient to analyze (1.22)–(1.23) also on certain Hilbert spaces. Let I be the identity operator on $\mathbf{H}_0$. Then, for any $\alpha \in \mathbf{R}$, $(I - \nu\Delta)^\alpha$ is defined through the spectral resolution of $(I - \nu\Delta)$ and is self-adjoint. Let $C_0^\infty(\mathbf{R}^2, \mathbf{R})$ be the subspace of $C_b^m(\mathbf{R}^2, \mathbf{R})$ of infinitely often differentiable functions with compact support. Suppose $\varphi, \psi \in C_0^\infty(\mathbf{R}^2, \mathbf{R})$. For $\alpha \geq 0$ set

$$\langle \varphi, \psi\rangle_\alpha := \left\langle (I - \nu\Delta)^{\alpha/2}\varphi, (I - \nu\Delta)^{\alpha/2}\psi\right\rangle_0$$

and $\|\varphi\|_\alpha := \langle \varphi, \varphi\rangle_\alpha^{1/2}$. Let $\mathbf{H}_\alpha$ be the completion of $C_0^\infty(\mathbf{R}^2, \mathbf{R})$ in $\mathbf{H}_0$ with respect to $\|\cdot\|_\alpha$, identify $\mathbf{H}_0$ with its strong dual $\mathbf{H}_0^*$ and denote by $\mathbf{H}_{-\alpha}$ the strong dual of $\mathbf{H}_\alpha$. The norms $\|\cdot\|_{-\alpha}$ on $\mathbf{H}_{-\alpha}$ are Hilbert norms, and we easily see that if $\varphi, \psi \in \mathbf{H}_0$,

$$\langle \varphi, \psi\rangle_{-\alpha} = \left\langle (I - \nu\Delta)^{-\alpha/2}\varphi, (I - \nu\Delta)^{-\alpha/2}\psi\right\rangle_0$$

(cf. Kotelenez [15]). Hence we have the sequence of Hilbert spaces

$$\mathbf{H}_\alpha \subset \mathbf{H}_\beta \subset \mathbf{H}_0 = \mathbf{H}_0^* \subset \mathbf{H}_{-\beta} \subset \mathbf{H}_{-\alpha}, \tag{1.28}$$

for $0 \le \beta \le \alpha$, with dense continuous inclusions. For $\varphi \in C_b^m(\mathbf{R}^2, \mathbf{R})$, we set

$$|||\varphi|||_m := \max_{0 \le |j| \le m} \sup_{r \in \mathbb{R}^2} \left| \frac{\partial^j}{\partial r_1^{j_1} \partial r_2^{j_2}} \varphi(r) \right|,$$

where j is a multiindex from $(\mathbf{N} \cup \{0\}) \times (\mathbf{N} \cup \{0\})$ with $j = (j_1, j_2)$ and $|j| := j_1 + j_2$. By imbedding (cf. Triebel [31], 2.8),

$$\mathbf{H}_\alpha \subset C_b^m(\mathbf{R}^2, \mathbf{R}),$$

if $1 \le m + 1 < \alpha$ with continuous inclusion. Since $|||\varphi|||_1 \ge c\|\varphi\|_L$, with $c \in (0, \infty)$, that is, since $|||\cdot|||_1$ is stronger than $\|\cdot\|_L$, we finally obtain

$$(\mathbf{M}, \gamma_2) \subset \mathbf{H}_{-\alpha}, \qquad \alpha > 2, \tag{1.29}$$

with continuous inclusion. In particular, for $\alpha > 2$ there is a $c > 0$ such that, for $\mu, \tilde{\mu} \in \mathbf{M}$,

$$\|\mu - \tilde{\mu}\|_{-\alpha} \le c\gamma_2(\mu, \tilde{\mu}). \tag{1.30}$$

Suppose $\alpha > 2$. By (1.28), $\mathbf{H}_0^*$ can be defined by $\mathbf{H}_0^* = \{\mathscr{F} \in \mathbf{H}_{-\alpha}: \|\mathscr{F}\|_0 < \infty\}$. The identification $\mathbf{H}_0 = \mathbf{H}_0^*$ implies that for $\mathscr{F} \in \mathbf{M}$ with $\|\mathscr{F}\|_0 < \infty$ we may write $\mathscr{F}(dr) = f(r)\,dr$ and identify $\mathscr{F}$ with f, which is the density of $\mathscr{F}$ with respect to dr.

All assumptions made on (1.22) and (1.23) will be used throughout the rest of the paper without mentioning them (unless we change them, e.g., in $K_\delta \to \nabla^\perp g$). Let us make some final remarks on notation. We will, for example, use $c_\delta := |||K_\delta|||_1 := |||K_\delta^1|||_1 + |||K_\delta^2|||_1$, where K_δ^l are the one-dimensional components of K_δ. If $\mathbf{B}$ and $\tilde{\mathbf{B}}$ are some normed vector spaces, $\mathscr{L}(\mathbf{B}, \tilde{\mathbf{B}})$ will be the space of linear bounded operators from $\mathbf{B}$ into $\tilde{\mathbf{B}}$ and $\|\cdot\|_{\mathscr{L}(\mathbf{B}, \tilde{\mathbf{B}})}$ will be the usual operator norm on $\mathscr{L}(\mathbf{B}, \tilde{\mathbf{B}})$. For $\mathbf{B} = \tilde{\mathbf{B}}$ we will just write $\mathscr{L}(\mathbf{B})$ instead of $\mathscr{L}(\mathbf{B}, \mathbf{B})$.

We next state the main results here and give the proofs in the corresponding sections.

If $(\hat{\mathbf{M}}, \hat{\gamma})$ is some metric space and $p \ge 1$, $L_p(\Omega; \hat{\mathbf{M}})$ is the metric space of $\hat{\mathbf{M}}$-valued p-integrable random variables with metric $(E\hat{\gamma}^p(\xi, \eta))^{1/p}$ for $\xi, \eta \in L_p(\Omega; \hat{\mathbf{M}})$. Set

$$\mathbf{M}_d := \{\mu \in \mathbf{M}: \mu \text{ is a finite linear combination of point measures on } \mathbf{R}^2\},$$

$$\tilde{\mathscr{M}}_0 := L_2(\Omega; \mathbf{M}_d),$$

$$\mathscr{M}_0 := L_2(\Omega; \mathbf{M}),$$

$$\mathscr{M}_{[0,T]} := L_2(\Omega; C([0,T]; \mathbf{M})).$$

Note that $\mathscr{M}_0$ and $\mathscr{M}_{[0,T]}$ are complete metric spaces, since $\mathbf{M}$ is complete, where the metric on $\mathscr{M}_{[0,T]}$ is given by $[E \sup_{0 \le t \le T} \gamma_2^2(\mu_t, \tilde{\mu}_t)]^{1/2}$ for $\mu_t, \tilde{\mu}_t \in \mathscr{M}_{[0,T]}$.

Set $\mathscr{X}_\varepsilon(t, \mathscr{X}_N(0)) := \mathscr{X}_N(t)$, where $\mathscr{X}_N(t)$ is the empirical process associated with (1.18).

The following Theorems 1.6 and 1.7 are on existence for (1.22) and (1.23). Their proofs are given in Section 2.

1.6. THEOREM. *The map* $\mathscr{X}_N(0) \mapsto \mathscr{X}_\varepsilon(\cdot, \mathscr{X}_N(0))$ *from* $\tilde{\mathscr{M}}_0$ *into* $\mathscr{M}_{[0,T]}$ *extends uniquely to a map* $\mathscr{X}_0 \mapsto \mathscr{X}_\varepsilon(\cdot, \mathscr{X}_0)$ *from* $\mathscr{M}_0$ *into* $\mathscr{M}_{[0,T]}$. *Moreover, for any* $\mathscr{X}_0, \mathscr{Y}_0 \in \mathscr{M}_0$,

$$E \sup_{0 \le t \le T} \gamma_2^2(\mathscr{X}_\varepsilon(t, \mathscr{X}_0), \mathscr{X}_\varepsilon(t, \mathscr{Y}_0)) \le \exp\big(c\big[c_\varepsilon + c_\delta^2(a^2 + 1) + 1\big]\big) E\gamma_2^2(\mathscr{X}_0, \mathscr{Y}_0). \tag{1.31}$$

1.7. THEOREM. *For any* $\varphi \in C_b^3(\mathbf{R}^2; \mathbf{R})$ *and* $\mathscr{X}_0 \in \mathscr{M}_0$, $\langle \mathscr{X}_\varepsilon(t, \mathscr{X}_0), \varphi \rangle$ *satisfies* (1.24), $0 \le t < \infty$.

Theorems 1.8–1.10 are on $\mathbf{H}_0$-valued solutions of (1.22) and (1.23) and on uniqueness. Their proofs are given in Section 3.

1.8. THEOREM. *Suppose* $\mathscr{X}_\varepsilon(0, dr) = X_\varepsilon(0, r)\, dr$ *and* $E\|X_\varepsilon(0)\|_0^{2n} < \infty$, *for some* $n \ge 1$. *Then* $\mathscr{X}_\varepsilon(t, dr) = X_\varepsilon(t, r)\, dr$, $X_\varepsilon(t, \omega)$ *is* $2n$*-integrable over* $[0, T] \times \Omega$ *with values in* $\mathbf{H}_0$ *for any* $T > 0$, $X_\varepsilon(t)$ *is adapted and, for any* $t > 0$,

$$E\| X_\varepsilon(t) \|_0^{2n} \le 2^{n-1} \exp(c(\varepsilon, \nu, \delta, n)t) E\| X_\varepsilon(0) \|_0^{2n}, \tag{1.32}$$

where $c(\varepsilon, \nu, \delta, n)$ *is a finite constant given by* (3.11). *Moreover,* $X_\varepsilon(t)$ *is a weak solution of* (1.22).

1.9. THEOREM. *Suppose* $E\|X_\varepsilon(0)\|_0^{4n} < \infty$, *for some* $n \ge 1$. *Then*:

(i) *For any* $t \ge 0$ [*with* $*$ *denoting the convolution of two functions from* $L_1(\mathbf{R}^2, dr)$ *and* $W_\varepsilon(r, t) := \int \hat{\Gamma}_\varepsilon(r - p) w(dp, t)$],

$$\begin{aligned} \| X_\varepsilon(t) \|_0^{2n} &= \| X_\varepsilon(0) \|_0^{2n} + \frac{\nu n}{\varepsilon} \int_0^t \| X_\varepsilon(s) \|_0^{2n}\, ds \\ &\quad - n \int_0^t \| X_\varepsilon(s) \|_0^{2(n-1)} \big\langle X_\varepsilon^2(s), (\nabla \cdot K_\delta) * X_\varepsilon(s) \big\rangle_0\, ds \\ &\quad - n\sqrt{2\nu} \int_0^t \| X_\varepsilon(s) \|_0^{2(n-1)} \big\langle X_\varepsilon^2(s), \nabla \cdot dW_\varepsilon(s) \big\rangle_0 \\ &\quad + 2n(n-1)\nu \int_0^t \| X_\varepsilon(s) \|_0^{2(n-2)} \big[\big\langle X_\varepsilon^2(s), \nabla \cdot dW_\varepsilon(s) \big\rangle_0\big]. \end{aligned} \tag{1.33}$$

(ii) *We have*

$$X_\varepsilon(\cdot) \in C([0, \infty); \mathbf{H}_0) \quad a.s. \tag{1.34}$$

(iii) *There is a finite constant* $\bar{c}(\varepsilon, \nu, \delta, n)$ *such that, for any* $T \ge 0$,

$$E \sup_{0 \le t \le T} \| X_\varepsilon(t) \|_0^{2n} \le \exp(\bar{c}(\varepsilon, \nu, \delta, n, T) \cdot T) \big(E\| X_\varepsilon(0) \|_0^{4n}\big)^{1/2}, \tag{1.35}$$

where $\bar{c}(\varepsilon, \nu, \delta, n, T) := 3c(\varepsilon, \nu, \delta, n)T + nc_b + 1$ *with* $c_b \ge 1$ *the constant from the Burkholder–Davies–Gundy inequality.*

1.10. THEOREM. *Suppose $K_\delta \equiv 0$ and $X_0 \in \mathbf{H}$ with $E\|X_0\|_0^2 < \infty$. Suppose there is a weak solution $Y_\varepsilon(\cdot, X_0)$ of* (1.22) *such that, for all $T > 0$, $\int_0^T E\|Y_\varepsilon(s)\|_0^2\, ds < \infty$ and $Y_\varepsilon(0, X_0) = X_0$. Then, for all $t \geq 0$,*

$$(*) \qquad \int_0^T E\| Y_\varepsilon(s) - X_\varepsilon(s)\|_0^2\, ds = 0,$$

where $X_\varepsilon(\cdot, X_0)$ is the $\mathbf{H}_0$-valued solution of (1.22) *from Theorems* 1.6, 1.7 *and* 1.9 *with $X_\varepsilon(0, X_0) = X_0$.*

Finally, we obtain a macroscopic limit theorem, whose proof is given in Section 4.

Set

$$(1.36) \quad \Lambda_N := \left\{ r_N \in \mathbf{R}^{2N} \colon \exists\, (i,j),\, 1 \leq i < j \leq N, \text{ such that } r^i = r^j \right\}.$$

1.11. THEOREM. *For each $N \in \mathbf{N}$ suppose $r_N(0) \notin \Lambda_N$ a.s. Let $\varphi \in C_b(\mathbf{R}^2, \mathbf{R})$ and suppose $E\langle \mathscr{X}_N(0), \varphi\rangle \to \langle X(0), \varphi\rangle_0$, as $N \to \infty$. Then there is a sequence $\delta(N) \to 0$, as $N \to \infty$, such that for any $t > 0$,*

$$(1.37) \qquad E\left\langle \mathscr{X}_{\varepsilon, \delta(N)}(t), \varphi\right\rangle \to \left\langle X(t), \varphi\right\rangle_0 \quad \text{as } \varepsilon \to 0 \text{ and } N \to \infty.$$

2. Existence for the SNSE. Let $x_N(t) := x_N(t, x_N(0))$ and $y_N(t) := y_N(t, y_N(0))$ be the solutions of the SODE (1.18) with initial conditions $x_N(0)$ and $y_N(0)$, respectively and let $\mathscr{X}_N(t)$ and $\mathscr{Y}_N(t)$ be the empirical processes associated with $x_N(t)$ and $y_N(t)$, respectively. The following lemma allows us to extend the solution $\mathscr{X}(t, \mathscr{X}_N(0)) := \mathscr{X}_N(t)$ of (1.24) from discrete initial conditions to arbitrary (adapted) initial conditions in $\mathbf{M}$.

2.1. LEMMA. *For any $T > 0$ there is a $c > 0$ such that, for all $N \in \mathbf{N}$,*

$$(2.1) \qquad \begin{aligned} &E \sup_{0 \leq t \leq T} \gamma_2^2(\mathscr{X}_N(t), \mathscr{Y}_N(t)) \\ &\quad \leq \exp\left(c\left[c_\varepsilon + c_\delta^2(a^2 + 1) + 1\right]\right) E\gamma_2^2(\mathscr{X}_N(0), \mathscr{Y}_N(0)) \end{aligned}$$

with c_ε from (1.14) *and $c_\delta := |||\, K_\theta \,|||_1$.*

PROOF. (i) Since (x_N, y_N) is an $\mathbf{R}^{4N}$-valued Markov process, we may first consider deterministic initial conditions $(\xi, \eta) \in \mathbf{R}^{4N}$ and then average over the distribution of $(x_N(0), y_N(0))$. The empirical processes associated with $x_N(t, \xi)$ and $y_N(t, \eta)$ will be denoted by $\mathscr{X}_N(t, \xi)$ and $\mathscr{Y}_N(t, \eta)$, respectively. Consider the two $\mathbf{R}^2$-valued Itô equations with deterministic initial conditions $r, q \in \mathbf{R}^2$:

$$(2.2\text{a}) \qquad \begin{aligned} dz(t, \mathscr{X}_N(t)) &= \int K_\delta(z(t) - p)\mathscr{X}_N(dp, t)\, dt \\ &\quad + \int \hat{\Gamma}_\varepsilon(z(t), p) w(dp, dt), \qquad z(0, \mathscr{X}_N, (0)) = r; \end{aligned}$$

$$\text{(2.2b)}\qquad \begin{aligned} dz(t,\mathscr{Y}_N(t)) &= \int K_\delta(z(t)-p)\mathscr{Y}_N(dp,t)\,dt \\ &\quad + \int \hat{\Gamma}_\varepsilon(z(t),p)w(dp,dt), \qquad z(0,\mathscr{Y}_N(0)) = q. \end{aligned}$$

Clearly, (2.2) has unique continuous solutions, which follows as in Lemma 1.4. We set $r(t) := z(t, \mathscr{X}_N(t,\xi))$ with $r(0) = r$ and $q(t) := z(t,\mathscr{Y}_N(t,\eta))$ with $q(0) = q$. Note that if, for example, $r = \xi^i$ (the ith two-dimensional component of ξ), then $r(t)$ is the position of the ith vortex starting at ξ^i. This fact leads to the following observation. Assume $f \in C_b(\mathbf{R}^4, \mathbf{R})$ and $Q \in C(\mathscr{X}_N(0,\xi), \mathscr{Y}_N(0,\eta))$. Then

$$\text{(2.3)}\qquad \int\int Q_t(dr,dq)f(r,q) := \int\int Q(dr,dq)f(r(t),q(t))$$

defines $Q_t(dr,dq) \in C(\mathscr{X}_N(t,\xi),\mathscr{Y}_N(t,\eta))$.

(ii) As in Lemma 1.4 we obtain

$$\begin{aligned} &E \sup_{0\le t\le T} \varrho^2\left(\int_0^t \int \left[\hat{\Gamma}_\varepsilon(r(s),p) - \hat{\Gamma}_\varepsilon(q(s),p)\right]w(dp,ds),0\right) \\ &\qquad \le cc_\varepsilon \int_0^T E\varrho^2(r(s),q(s))\,ds. \end{aligned}$$

The smoothness of K_δ and the conservation of vorticity (1.23) imply

$$\begin{aligned} &\varrho\left(\int_0^T \int |K_\delta(r(s)-p) - K_\delta(q(s)-p)|\cdot|\mathscr{X}_N|(dp,s,\xi)\,ds,0\right) \\ &\qquad \le c_\delta a \int_0^T \varrho(r(s),q(s))\,ds. \end{aligned}$$

Further,

$$\begin{aligned} &\int_0^T \left|\int [\mathscr{X}_N(dp,s,\xi) - \mathscr{Y}_N(dp,s,\eta)]K_\delta(q(s)-p)\right| ds \\ &\quad = \sum_{(+,-)} \int_0^T \left|\int\int \frac{1}{a^\pm} Q_s^\pm(dp,d\tilde{p})[K_\delta(q(s)-p) - K_\delta(q(s)-\tilde{p})]\right| ds \end{aligned}$$

[with $Q_s^\pm \in C(\mathscr{X}_N^\pm(s,\xi),\mathscr{Y}_N^\pm(s,\eta))$ arbitrary, where by definition $(1/a^\pm)Q^\pm \equiv 0$ if $a^\pm = 0$]

$$\le c_\delta \sum_{(+,-)} \int_0^T \int\int \frac{1}{a^\pm} Q_s^\pm(dp,d\tilde{p})\varrho(p,\tilde{p}).$$

Hence by the arbitrariness of Q_s, (2.3) and (1.26),

$$\begin{aligned} &\int_0^T \left|\int [\mathscr{X}_N(dp,s,\xi) - \mathscr{Y}_N(dp,s,\eta)]K_\delta(q(s)-p)\right| ds \\ &\qquad \le \int_0^T c_\delta \gamma_2(\mathscr{X}_N(s,\xi),\mathscr{Y}_N(s,\eta))\,ds. \end{aligned}$$

The preceding estimates, the Cauchy–Schwarz inequality and the Gronwall lemma imply

$$E \sup_{0 \le t \le T} \varrho^2(r(t), q(t)) \le \exp\big(c[c_\varepsilon + c_\delta^2 a^2]\big) \times \left[\varrho^2(r,q) + c_\delta^2 \int_0^T E\gamma_2^2(\mathscr{X}_N(s,\xi), \mathscr{Y}_N(s,\eta))\, ds\right]. \tag{2.4}$$

Hence,

$$E \sup_{0 \le t \le T} \gamma_2^2(\mathscr{X}_N(t,\xi), \mathscr{Y}_N(t,\eta)) \le \exp\big(c[c_\varepsilon + c_\delta^2 a^2]\big)\Big[\gamma_2^2(\mathscr{X}_N(0,\xi), \mathscr{Y}_N(0,\eta)) + c_\delta^2 \int_0^T E\gamma_2^2(\mathscr{X}_N(s,\xi), \mathscr{Y}_N(s,\eta))\, ds\Big].$$

The Gronwall lemma and averaging over the distribution of $(x_N(0), y_N(0))$ implies (2.1). □

2.2. PROOF OF THEOREM 1.6. Since by (2.1), $\mathscr{X}_N(0) \mapsto \mathscr{X}_\varepsilon(\cdot, \mathscr{X}_N(0))$ is uniformly continuous, we can extend it by continuity to all $\mathscr{X}_0 \in \mathscr{M}_0$ by the density of $\tilde{\mathscr{M}}_0$ in $\mathscr{M}_0$. Inequality (1.31) follows immediately from (2.1). □

2.3. PROOF OF THEOREM 1.7. (i) Note that, by the choice of φ, $\|\Delta\varphi\|_L < \infty$ and $\|(\partial/\partial r_l)\varphi\|_L < \infty$, $l = 1, 2$. So the right-hand side of (1.24) is defined for $\mathscr{X}_\varepsilon(t, \mathscr{X}_0)$.

(ii) Set $f_N(t) := \mathscr{X}_\varepsilon(t, \mathscr{X}_0) - \mathscr{X}(t, \mathscr{X}_N(0))$. Then

$$E\left(\int_0^t \left\langle f_N(s), \int \tilde{\Gamma}_\varepsilon(\cdot, p) w(dp, ds) \cdot \nabla\varphi \right\rangle\right)^2 = \sum_{l=1}^{2} E\int_0^t \int\int f_N(s, dr) f_N(s, dq) \times \int \tilde{\Gamma}_\varepsilon(q,p)\tilde{\Gamma}_\varepsilon(r,p)\, dp \frac{\partial}{\partial r_l}\varphi(r) \frac{\partial}{\partial q_l}\varphi(q)\, ds. \tag{$*$}$$

Since, for any r,

$$\left|\int\left[\tilde{\Gamma}_\varepsilon(q,p) - \tilde{\Gamma}_\varepsilon(\tilde{q},p)\right]\tilde{\Gamma}_\varepsilon(r,p)\, dp\right| \le \left(\int\left(\tilde{\Gamma}_\varepsilon(q,p) - \tilde{\Gamma}_\varepsilon(\tilde{q},p)\right)^2 dp\right)^{1/2} \le c\rho(q,\tilde{q}) \quad \text{by (1.15)},$$

we obtain that the right-hand side of $(*)$ tends to zero as $N \to \infty$ as a consequence of (1.31).

(iii) Because $\sup_p \|K_\delta(\cdot - p)\|_L \le c < \infty$, the analogue to step (ii) also holds for the deterministic integrals for the right-hand side of (1.24). □

To avoid cumbersome notation, we will assume for the rest of this paper that $\hat{\Gamma}_\varepsilon$ in (1.17) and (1.18) is the kernel from Example 1.2.

3. Smoothness. Let $\lambda > 0$ and set $R_\lambda := \lambda^3(\lambda - \nu\Delta)^{-3}$. Note that, for $f \in \mathbf{H}_0$, $(\lambda - \nu\Delta)^{-1}f = \int_0^\infty e^{-\lambda t}T(t)f\,dt$; that is, $(\lambda - \nu\Delta)^{-1}$ is the resolvent of $\nu\Delta$ at $\lambda > 0$. Since $\lambda(\lambda - \nu\Delta)^{-1}(I - \nu\Delta) = \lambda(\lambda - \nu\Delta)^{-1}(I - \lambda) + \lambda$, we have that $\|\cdot\|_{-6}$ and $\|\cdot\|_{-6,\lambda}$ are equivalent norms, where for $f \in \mathbf{H}_0$, $\|f\|_{-6,\lambda} = \|R_\lambda f\|_0$. Let $\mathscr{X}_\varepsilon(t)$ be the solution of (1.22) as constructed in the last section and set

$$X_{\varepsilon,\lambda} := R_\lambda \mathscr{X}_\varepsilon.$$

Since $\mathbf{M} \subset \mathbf{H}_{-\alpha}$, for all $\alpha > 2$, $X_{\varepsilon,\lambda} \in \mathbf{H}_2 \subset \mathbf{H}_0$, whence on $\mathbf{H}_0$,

$$\begin{aligned} X_{\varepsilon,\lambda}(t) &= X_{\varepsilon,\lambda}(0) + \int_0^t \nu\Delta X_{\varepsilon,\lambda}(s)\,ds \\ &\quad - \int_0^t \nabla\cdot R_\lambda(\mathscr{X}_\varepsilon(s) K_\delta * \mathscr{X}_\varepsilon(s))\,ds \\ &\quad - \sqrt{2\nu}\int_0^t \nabla\cdot R_\lambda(\mathscr{X}_\varepsilon(s)\,dW_\varepsilon(s)) \end{aligned} \tag{3.1}$$

with $W_\varepsilon(r,t) = \int \hat{\Gamma}_\varepsilon(r-p)w(dp,t)$ and $(K_\delta * \mu)(r) = \int K_\delta(r-p)\mu(dp)$, $\mu \in \mathbf{M}$. In what follows we will assume, for any $\lambda > 0$, $\|X_{\varepsilon,\lambda}(0)\|_0 < \infty$ a.s. Let $n \in \mathbf{N}$. Then Itô's formula yields

$$\begin{aligned} \|X_{\varepsilon,\lambda}(t)\|_0^{2n} &= \|X_{\varepsilon,\lambda}(0)\|_0^{2n} \\ &\quad + \int_0^t 2\nu n\|X_{\varepsilon,\lambda}(s)\|_0^{2(n-1)}\langle X_{\varepsilon,\lambda}(s), \Delta X_{\varepsilon,\lambda}(s)\rangle_0\,ds \\ &\quad - \int_0^t 2n\|X_{\varepsilon,\lambda}(s)\|_0^{2(n-1)}\langle X_{\varepsilon,\lambda}(s), \nabla R_\lambda(\mathscr{X}_\varepsilon(s)K_\delta * \mathscr{X}_\varepsilon(s))\rangle_0\,ds \\ &\quad - \int_0^t 2n\sqrt{2\nu}\|X_{\varepsilon,\lambda}(s)\|_0^{2(n-1)}\langle X_{\varepsilon,\lambda}(s), \nabla R_\lambda(\mathscr{X}_\varepsilon(s)\,dW_\varepsilon(s))\rangle_0 \\ &\quad + \int_0^t 2\nu n\|X_{\varepsilon,\lambda}(s)\|_0^{2(n-1)}[\nabla R_\lambda(\mathscr{X}_\varepsilon(s)\,dW_\varepsilon(s))] \\ &\quad + \int_0^t 4\nu n(n-1)\|X_{\varepsilon,\lambda}(s)\|_0^{2(n-2)} \\ &\qquad \times\left[\langle X_{\varepsilon,\lambda}(s), \nabla R_\lambda(\mathscr{X}_\varepsilon(s)\,dW_\varepsilon(s))\rangle_0\right], \end{aligned} \tag{3.2}$$

where [ ] denotes the quadratic variation (on $\mathbf{H}_0$, resp. $\mathbf{R}$).

In what follows we will assume that, for some given $n \in \mathbf{N}$ and $\lambda > 0$ (and hence by the equivalence of norms for all $\lambda > 0$), $E\|X_{\varepsilon,\lambda}(0)\|_0^{2n} < \infty$.

3.1. LEMMA.

$$(3.3)\quad \left|\left[\nabla R_\lambda(\mathscr{X}_\varepsilon(s)\,dW_\varepsilon(s))\right] + \langle X_{\varepsilon,\lambda}(s), \Delta X_{\varepsilon,\lambda}(s)\rangle_0\,ds\right| \le \frac{2}{\varepsilon}\|R_\lambda|\mathscr{X}_\varepsilon|(s)\|_0^2\,ds.$$

PROOF. (i)

$$\begin{aligned} E&\left\|\int_0^t R_\lambda\,\nabla\left(\mathscr{X}_\varepsilon(s)\int\hat{\Gamma}_\varepsilon(\cdot - p)w(dp,ds)\right)\right\|_0^2 \\ &= \lambda^6 E\int_0^t\int_0^\infty\int_0^\infty\int_0^\infty \exp(-\lambda(u_1+\cdots+u_6))\int\int\int G(u_1+\cdots+u_6, r-q) \\ &\qquad\times\left[\sum_{l=1}^2 \frac{\partial}{\partial q_l}\left(\mathscr{X}_\varepsilon(s,dq)\tilde{\Gamma}_\varepsilon(q-p)\right)\right. \\ &\qquad\qquad\left.\times\frac{\partial}{\partial r_l}\left(\mathscr{X}_\varepsilon(s,dr)\tilde{\Gamma}_\varepsilon(r-p)\right)\right] dp\,du_1\cdots du_6\,ds \\ &= -\lambda^6 E\int_0^t\int_0^\infty\int_0^\infty\int_0^\infty \exp(-\lambda(u_1+\cdots+u_6)) \\ &\qquad\times\int\int(\Delta_q G(u_1+\cdots+u_6, r-q))g_\varepsilon(r-q) \\ &\qquad\qquad\times\mathscr{X}_\varepsilon(s,dq)\mathscr{X}_\varepsilon(s,dr)\,du_1\cdots du_6\,ds \end{aligned}$$

with $g_\varepsilon(r-q) := \exp(-|r-q|^2/8\varepsilon)$.

(ii) Denoting the left-hand side of (3.3) by $|B_1(s,\varepsilon,\lambda)\,ds|$, we obtain

$$\begin{aligned} |B_1(s,\varepsilon,\lambda)\,ds| &\le \lambda^6\int_0^t\int_0^\infty\int_0^\infty\int_0^\infty \exp(-\lambda(u_1+\cdots+u_6)) \\ &\qquad\times\int\int\Big|(\Delta_q G(u_1+\cdots+u_6, r-q)) \\ &\qquad\qquad\times(1-g_\varepsilon(r-q))\mathscr{X}_\varepsilon(s,dq)\mathscr{X}_\varepsilon(s,dr)\,du_1\cdots du_6\,ds\Big|. \end{aligned}$$

(iii)

$$\begin{aligned} \left|(\Delta_q G(u,r-q))(1-g_\varepsilon(r-q))\right| &\le \left|\frac{-1}{\nu u} + \frac{|r-q|^2}{4\nu^2u^2}\right| G(u,r-q)\frac{|r-q|^2}{8\varepsilon} \\ &\qquad\left[\text{since } |1-g_\varepsilon(r)| \le \frac{|r|^2}{8\varepsilon}\right] \\ &\le \frac{2}{\varepsilon}\left|-\frac{|r-q|^2}{8\nu u} + \frac{2|r-q|^4}{64\nu^2u^2}\right| \exp\left(-\frac{|r-q|^2}{8\nu u}\right), \\ G(2u,r-q) &\le \frac{2}{\varepsilon}G(2u,r-q). \end{aligned}$$

(iv) The previous calculations and the change of variables $2u_i = v_i$, $i = 1,\dots,6$, imply

$$|B_1(s,\varepsilon,\lambda)| \le \frac{2}{\varepsilon}\| R_{\lambda/2}|\mathscr{X}_\varepsilon|(s)\|_0^2.$$

(v) Using standard estimates for the norm of the resolvent $(\mu - \nu\Delta)^{-1}$ (cf. Davies [5], page 48) we obtain $\|R_{\lambda/2}R_\lambda^{-1}\|_{\mathscr{L}(\mathbb{H}_0)} \le 1$ and, by unique extendibility to $\mathbf{H}_{-6}$, $\|f\|_{-6,\lambda/2} \le \|f\|_{-6,\lambda}$, for all $f \in \mathbf{H}_{-6}$. From this we obtain (3.3). □

For sufficiently smooth functions f and F we obtain, for $l = 1,2$,

$$2\left\langle f, \frac{\partial}{\partial r_l}(f\cdot F)\right\rangle_0 = \left\langle f^2, \frac{\partial}{\partial r_l}F\right\rangle_0, \tag{3.4}$$

where $f\cdot F$ is pointwise multiplication. This implies

$$2\langle X_{\varepsilon,\lambda}(s), \nabla\cdot(X_{\varepsilon,\lambda}(s)K_\delta * \mathscr{X}_\varepsilon(s))\rangle_0 = \langle X_{\varepsilon,\lambda}^2(s), (\nabla\cdot K_\delta) * \mathscr{X}_\varepsilon(s)\rangle_0. \tag{3.5}$$

Set $B_2(s,\varepsilon,\lambda) := \langle X_{\varepsilon,\lambda}(s), \nabla(X_{\varepsilon,\lambda}(s)K_\delta * \mathscr{X}_\varepsilon(s)) - \nabla R_\lambda(\mathscr{X}_\varepsilon(s)K_\delta * \mathscr{X}_\varepsilon(s))\rangle_0$.

3.2. Lemma.

$$|B_2(s,\varepsilon,\lambda)| \le |||K_\delta|||_1 \cdot 3a\|R_\lambda|\mathscr{X}_\varepsilon|(s)\|_0^2. \tag{3.6}$$

Proof. (i) Let $f\in\mathbf{M}$, $f_\lambda := R_\lambda f$ and F be a sufficiently smooth function on $\mathbf{R}^2$ with values in $\mathbf{R}^2$. Then

$$\begin{aligned}&\int f_\lambda(r)\int(\nabla_r G(u,r-q))f(dq)(F(q)-F(r))\,dr\\ &\quad= \int f_\lambda(r)\sum_{l=1}^{2}\frac{(r_l-q_l)}{4\nu u}G(u,r-q)\left(\frac{F_l(q)-F_l(r)}{q_l-r_l}\right)(q_l-r_l)f(dq)\,dr\\ &\quad\le 2\,|||F|||_1\int|f_\lambda(r)|G(2u,r-q)|f|(dq)\,dr\end{aligned}$$

[similarly to step (iii) in the proof of Lemma 3.1].

(ii) Clearly,

$$\left|\int f_\lambda(r)\int G(u,r-q)f(dq)(\nabla_r\cdot F)(r)\,dq\,dr\right| \le |||F\|_1\int|f_\lambda(r)|G(u,r-q)|f|(dq)\,dr,$$

where $|||F|||_1 := |||F_1|||_1 + |||F_2|||_1$.

(iii) Again as in steps (iv) and (v) of the proof of Lemma 3.1, we obtain (3.6) from the previous steps as well as from $\mathscr{X}_\varepsilon(s)\in\mathbf{M}$ and $|R_\lambda\mathscr{X}_\varepsilon|(s) \le R_\lambda|\mathscr{X}_\varepsilon|(s)$. □

In order to estimate the last term in (3.2) we proceed as follows. By (1.19) and (3.4),

$$2\langle X_{\varepsilon,\lambda}(s), \nabla(X_{\varepsilon,\lambda}(s)\,dW_\varepsilon(s))\rangle_0 = \langle X^2_{\varepsilon,\lambda}(s), \nabla\cdot dW_\varepsilon(s)\rangle_0. \tag{3.7}$$

3.3. LEMMA.

$$\left[\langle X^2_{\varepsilon,\lambda}(s), \nabla\cdot dW_\varepsilon\rangle_0\right] \le \frac{1}{2\varepsilon}\|X_{\varepsilon,\lambda}(s)\|_0^4\,ds. \tag{3.8}$$

PROOF. Set $f := X^2_{\varepsilon,\lambda}(s)$. Then

$$\begin{aligned}
&E\langle f, \nabla\cdot dW_\varepsilon(s)\rangle_0^2 \\
&= \sum_{l=1}^{2} E\int\int\int f(r)f(q)\frac{\partial}{\partial r_l}\tilde{\Gamma}_\varepsilon(r-p)\frac{\partial}{\partial q_l}\tilde{\Gamma}_\varepsilon(q-p)\,dp\,dq\,dr\,ds \\
&\le \sum_{l=1}^{2} E\int\int f(r)f(q)\left(\int\left(\frac{\partial}{\partial r_l}\tilde{\Gamma}_\varepsilon(r-p)\right)^2 dp\right)^{1/2} \\
&\quad\times\left(\int\left(\frac{\partial}{\partial q_l}\tilde{\Gamma}_\varepsilon(q-p)\right)^2 dp\right)^{1/2} dq\,dr\,ds \\
&\le \frac{1}{2\varepsilon}E\left(\int f(r)\,dr\right)^2 ds.
\end{aligned}$$

□

3.4. LEMMA.

$$\begin{aligned}
&\left[\langle X_{\varepsilon,\lambda}(s), \nabla R_\lambda(\mathscr{X}_\varepsilon(s)\,dW_\varepsilon(s))\rangle_0\right] \\
&\quad\le \frac{1}{\varepsilon}\|X_{\varepsilon,\lambda}(s)\|_0^2\left\{64\|R_\lambda|\mathscr{X}_\varepsilon|(s)\|_0^2 + \frac{1}{4}\|X_{\varepsilon,\lambda}(s)\|_0^2\right\} ds.
\end{aligned} \tag{3.9}$$

PROOF. (i) Equation (3.7) implies

$$\langle X_{\varepsilon,\lambda}(s), (\nabla X_{\varepsilon,\lambda}(s))\cdot dW_\varepsilon(s)\rangle_0 = -\tfrac{1}{2}\langle X^2_{\varepsilon,\lambda}(s), \nabla\cdot dW_\varepsilon(s)\rangle_0.$$

(ii) From the definition of the quadratic variation and (i), we obtain

$$\begin{aligned}
&\left[\langle X_{\varepsilon,\lambda}(s), \nabla R_\lambda(\mathscr{X}_\varepsilon(s)\,dW_\varepsilon(s))\rangle_0\right] \\
&\quad\le 2\left[\langle X_{\varepsilon,\lambda}(s), \nabla\cdot R_\lambda(\mathscr{X}_\varepsilon(s)\,dW_\varepsilon(s)) - (\nabla X_{\varepsilon,\lambda}(s))\cdot dW_\varepsilon(s)\rangle_0\right] \\
&\qquad+\tfrac{1}{2}\left[\langle X^2_{\varepsilon,\lambda}(s), \nabla\cdot dW_\varepsilon(s)\rangle_0\right].
\end{aligned}$$

(iii) We have

$$\begin{aligned}
&\nabla\cdot R_\lambda(\mathscr{X}_\varepsilon(s)\,dW_\varepsilon(s)) - (\nabla X_{\varepsilon,\lambda}(s))\cdot dW_\varepsilon(s) \\
&\quad= \sum_{l=1}^{2}\lambda^3\int_0^\infty\int_0^\infty\int_0^\infty -\exp(\lambda(u_1+u_2+u_3))\int\frac{\partial}{\partial r_l}G(u_1+u_2+u_3, r-q) \\
&\qquad\times\mathscr{X}_\varepsilon(s,dq)\int\tilde{\Gamma}_\varepsilon(q-p)w_l(dp,ds)\,du_1du_2du_3
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{l=1}^{2} \lambda^3 \int_0^\infty \int_0^\infty \int_0^\infty \exp(-\lambda(u_1+u_2+u_3)) \\
&\quad \times \int \left(\frac{\partial}{\partial r_l} G(u_1+u_2+u_3, r-q) \right) \\
&\quad \times \mathscr{X}_\varepsilon(s, dq) \int \tilde{\Gamma}_\varepsilon(r-p) w_l(dp, ds)\, du_1 du_2 du_3 \\
&= \sum_{l=1}^{2} \lambda^3 \int_0^\infty \int_0^\infty \int_0^\infty \exp(-\lambda(u_1+u_2+u_3))\, du_1\, du_2\, du_3 \\
&\quad \times \int G(2(u_1+u_2+u_3), r-q) \mathscr{X}_\varepsilon(s, dq) \\
&\quad \times \left\{ -\frac{8(r_l-q_l)}{8\nu(u_1+u_2+u_3)} \exp\left(-\frac{|r-q|^2}{8\nu(u_1+u_2+u_3)} \right) \right. \\
&\qquad \left. \times \int \left(\tilde{\Gamma}_\varepsilon(q-p) - \tilde{\Gamma}_\varepsilon(r-p) \right) w_l(dp, ds) \right\}.
\end{aligned}
$$

(iv) Hence, since $\sum_{l=1}^{2} |r_l - q_l| \le \sqrt{2}\, |r-q|^2 / |r-q|$,

$$
\begin{aligned}
&\left[\nabla \cdot R_\lambda(\mathscr{X}_\varepsilon(s)\, dW_\varepsilon(s)) - (\nabla X_{\varepsilon,\lambda}(s)) \cdot dW_\varepsilon(s) \right] \\
&\quad \le 128 \lambda^6 \int_0^\infty \int_0^\infty \int_0^\infty \exp(-\lambda(u_1 + \cdots + u_6))\, du_1 \cdots du_6 \\
&\quad \times \int \int G(2(u_1+u_2+u_3), r-q) |\mathscr{X}_\varepsilon|(s, dq) \\
&\quad \times G(2(u_4+u_5+u_6), r-\tilde{q}) |\mathscr{X}_\varepsilon|(s, d\tilde{q}) \\
&\quad \times \left| \int \left(\frac{\tilde{\Gamma}_\varepsilon(q-p) - \tilde{\Gamma}_\varepsilon(r-p)}{|r-q|} \right) \left(\frac{\tilde{\Gamma}_\varepsilon(\tilde{q}-p) - \tilde{\Gamma}_\varepsilon(r-p)}{|r-\tilde{q}|} \right) dp \right| dr\, ds.
\end{aligned}
$$

(v) By Example 1.2,

$$
\int \frac{\left(\tilde{\Gamma}_\varepsilon(q-p) - \tilde{\Gamma}_\varepsilon(r-p) \right)^2}{|r-q|^2}\, dp = \frac{2\left(1 - \exp(-|r-q|^2/8\varepsilon\right)}{|r-q|^2} \le \frac{1}{4\varepsilon}.
$$

Hence, by the Cauchy–Schwarz inequality the right-hand side in step (iv) is bounded above by

$$
\frac{32}{\varepsilon} \| R_{\lambda/2} |\mathscr{X}_\varepsilon|(s) \|_0^2\, ds \le \frac{32}{\varepsilon} \| R_\lambda |\mathscr{X}_\varepsilon|(s) \|_0^2\, ds.
$$

(vi) By Metivier and Pellaumail ([24], Chapter 2.4.2) and the last step,

$$
\begin{aligned}
&\left[\langle X_{\varepsilon,\lambda}(s), \nabla \cdot R_\lambda(\mathscr{X}_\varepsilon(s)\, dW_\varepsilon(s)) - (\nabla X_{\varepsilon,\lambda}(s)) \cdot dW_\varepsilon(s) \rangle_0 \right] \\
&\quad \le \| X_{\varepsilon,\lambda}(s) \|_0^2 \frac{32}{\varepsilon} \| R_\lambda |X_\varepsilon|(s) \|_0^2\, ds.
\end{aligned}
$$

This together with step (ii) and (3.8) implies (3.9). □

3.5. LEMMA. *Suppose* $a^- = 0$ *and* $E\|X_{\varepsilon,\lambda}(0)\|_0^{2n} < \infty$, *for some* $n \geq 1$. *Then for any* $t \geq 0$,

$$E\| X_{\varepsilon,\lambda}(t)\|_0^{2n} \leq \exp(c(\varepsilon, \nu, \delta, n)t) E\| X_{\varepsilon,\lambda}(0)\|_0^{2n} \tag{3.10}$$

with

$$c(\varepsilon, \nu, \delta, n) := \frac{260\nu(n-1)}{\varepsilon} + \frac{4\nu n}{\varepsilon} + 7a ||| K_\delta |||_1 n. \tag{3.11}$$

PROOF. Note that we can first stop at an arbitrary $M < \infty$ such that $\sup_{s \geq 0} \|X_{\varepsilon,\lambda}(s \wedge \tau_M)\|_0^{2n} \leq M$ and [since $X_{\varepsilon,\lambda}(t)$ has a.s. continuous sample paths in $\mathbf{H}_0$] $P\{\lim_{M\to\infty} \tau_M = \infty\} = 1$. Then the previous lemmas imply

$$\begin{aligned} E\| X_{\varepsilon,\lambda}(t \wedge \tau_M)\|_0^{2n} &\leq E\| X_{\varepsilon,\lambda}(0)\|_0^{2n} \\ &\leq ||| K_\delta |||_1 an \int_0^t E\| X_{\varepsilon,\lambda}(s \wedge \tau_M)\|_0^{2n}\, ds \\ &\quad + \left(\frac{260\nu n(n-1)}{\varepsilon} + \frac{4\nu n}{\varepsilon} + 6a\, ||| K_\delta |||_1 n \right) \\ &\quad \times \int_0^t E\|X_{\varepsilon,\lambda}(s \wedge \tau_M)\|_0^{2n}\, ds. \end{aligned}$$

The Gronwall lemma implies (3.10). □

3.6. LEMMA. *Suppose* $a^- = 0$ *and* $E\|X_\varepsilon(0)\|_0^{2n} < \infty$, *for some* $n \geq 1$. *Then for any* $t \geq 0$, $\mathscr{X}_\varepsilon(t, dr) = X_\varepsilon(t, r)\, dr$ *and*

$$E\|X_\varepsilon(t)\|_0^{2n} \leq \exp(c(\varepsilon, \nu, \delta, n)t) \cdot E\| X_\varepsilon(0)\|_0^{2n}, \tag{3.12}$$

where $c(\varepsilon, \nu, \delta, n)$ *is given by* (3.11).

The proof follows from Fatou's lemma. □

We will now derive the same estimates for the signed measure case. Set

$$F(s) := K_\delta * \mathscr{X}_\varepsilon(s).$$

We easily see that the positive and negative components of the empirical process $\mathscr{X}_N(t)$ satisfy the "bilinear" equation

$$d\mathscr{Y}(t) = \{\nu \Delta \mathscr{Y}(t) - \nabla(\mathscr{Y}(t)F(t))\}\, dt - \sqrt{2\nu}\, \nabla \cdot (\mathscr{Y}(t)\, dW_\varepsilon(t)), \tag{3.13}$$

where $\mathscr{Y}(t, \mathscr{X}_N^+(0)) = \mathscr{X}_N^+(t)$ and $\mathscr{Y}(t, \mathscr{X}_N^-(0)) = \mathscr{X}_N^-(t)$. Since the extended process $\mathscr{X}_\varepsilon(t)$ is obtained by extending both the positive and negative components [see (1.25)], we obtain $\mathscr{Y}(t, \mathscr{X}_\varepsilon^+(0)) = \mathscr{X}_\varepsilon^+(t, \mathscr{X}_\varepsilon(0))$ and $\mathscr{Y}(t, \mathscr{X}_\varepsilon^-(0)) = \mathscr{X}_\varepsilon^-(t, \mathscr{X}_\varepsilon(0))$ in the general case as well. On the other hand, only the smoothness of K_δ and $\tilde{\Gamma}_\varepsilon$ and $\mathscr{X}_\varepsilon(s) \in \mathbf{M}$ were used in the derivation of (3.10) and (3.12), whence the same estimates also hold for $\mathscr{X}_\varepsilon^\pm(t)$. Since $\langle X_\varepsilon^+(s), X_\varepsilon^-(s)\rangle_0 = 0$, we obtain $\|X_\varepsilon(t)\|_0^2 = \|X_\varepsilon^+(t)\|_0^2 + \|X_\varepsilon^-(t)\|_0^2$.

3.7. PROOF OF THEOREM 1.8. (i) The $2n$-integrability of $X_\varepsilon(t)$ as an $\mathbf{H}_0$-valued process follows from the $2n$-integrability of $X_\varepsilon^+(t)$ and $X_\varepsilon^-(t)$, whose integrability properties follow from the corresponding properties of $X_{\varepsilon,\lambda}^{\pm}(t)$ and (3.12). The bound in (1.32) follows from (3.12).

(ii) Since the bilinearity between measures and smooth functions is an extension of the inner product on $\mathbf{H}_0$, $X_\varepsilon(t)$ satisfies (1.22). □

In what follows we will derive an expression for $\|X_\varepsilon(t)\|_0^{2n}$ by Itô's formula, where $X_\varepsilon(t)$ is the density process for (1.22), as derived in Theorem 1.8. The following lemma will be used at various steps in that derivation.

3.8. LEMMA. *Let* $f, g \in \mathbf{H}_0 \cap L_1(\mathbf{R}^2, dr)$ *and set* $g_\lambda := |(\tilde{R}_\lambda - I)g| + |g|$, *where* $\tilde{R}_\lambda = R_\lambda^n$, *for some* $n \ge 1$. *Then, for any* $m \ge 1$,

$$\begin{aligned}(3.14)\qquad &\lim_{\lambda\to\infty} \int\int_0^\infty \int_0^\infty \int_0^\infty \exp(-\lambda(u_1 + \cdots + u_m))\lambda^m \, du_1 \cdots du_m \\ &\qquad \times \int G(u_1 + \cdots + u_m, r - q)|f(q) - f(r)|\, dq\, g_\lambda(r)\, dr = 0.\end{aligned}$$

PROOF. (i) Let $\delta > 0$ be given and denote the multiple integral on the left-hand side of (3.14) by $\Lambda(f, g)$. By change of variables $p := (q - r)/\sqrt{u}$,

$$\begin{aligned}&\int\int G(u, r - q)|f(q) - f(r)| g_\lambda(r)\, dq\, dr \\ &\quad = \int\int G(1, p)\big|f(r + p\sqrt{u}) - f(r)\big|\, dp\, g_\lambda(r)\, dr \\ &\quad = \int\int_{B_L} G(1, p)\big|f(r + p\sqrt{u}) - f(r)\big|\, dp\, g_\lambda(r)\, dr \\ &\qquad + \int\int_{B_L^c} G(1, p)\big|f(r + p\sqrt{u}) - f(r)\big|\, dp\, g_\lambda(r)\, dr \\ &\quad =: A_L(u) + A_L^c(u),\end{aligned}$$

where $B_L = \{p \in \mathbf{R}^2 : |p| \le L\}$, $B_L^c = \mathbf{R}^2 \setminus B_L$, $L > 0$.

(ii) On B_L^c, $G(1, p) \le 2\exp(-L^2/8\nu)G(2, p)$, which implies

$$A_L^c(u) \le 2\exp\left(\frac{-L^2}{8\nu}\right)\{\|f\|_0^2 + \|g\|_0^2\} \le \frac{\delta}{2},$$

for L sufficiently large.

(iii) Let m denote the two-dimensional Lebesgue measure and set

$$F(r, u) := \frac{1}{m(B_{\sqrt{u}L})}\int_{B_{\sqrt{u}L}} |f(r + q) - f(r)|\, dr.$$

We have

$$0 \le \int_{B_L} G(1, p)\big|f(r + p\sqrt{u}) - f(r)\big|\, dp \le \pi L^2 F(r, u) \to 0 \quad \text{as } u \to 0,$$

m-a.e. (m-almost everywhere) by the Lebesgue differentiation theorem.

(iv) To conclude from (iii) that $A_L(v/\lambda) \to 0$, $\lambda \to \infty$, for any $v > 0$, we first set

$$Hf(r) := \sup_{u>0} \frac{1}{m(B_u)} \int_{B_u} |f(r+q)|\, dq,$$

which is the Hardy–Littlewood maximal function for f, and

$$\tilde{H}f(r) := Hf(r) + |f|(r).$$

Let $N \in \mathbf{N}$. Then

$$\int F\left(r, \frac{v}{\lambda}\right) g_\lambda(r)\, dr = \int_{\{\tilde{H}f \geq N\}} F\left(r, \frac{v}{\lambda}\right) g_\lambda(r)\, dr + \int_{\{\tilde{H}f < N\}} F\left(r, \frac{v}{\lambda}\right) g_\lambda(r)\, dr$$
$$=: I_\lambda(v, N) + II_\lambda(v, N).$$

(v) Our assumptions on g imply $g_\lambda \in \mathbf{H}_0 \cap L_1(\mathbf{R}^2, dr)$, and we easily check that both $\{g_\lambda\}$ and $\{g_\lambda^2\}$ are uniformly integrable. Hence, for any $v > 0$, $N \in \mathbf{N}$,

$$II_\lambda(v, N) \to 0 \quad \text{as } \lambda \to \infty.$$

(vi) $I_\lambda(v, N) \leq 2(\int 1_{\{\tilde{H}f \geq N\}}(r) g_\lambda^2(r)\, dr)^{1/2} \|f\|_0 \to 0$, as $N \to \infty$, since $\{g_\lambda^2\}$ is uniformly integrable and $m\{\tilde{H}f \geq N\} \to 0$, and $N \to \infty$ (cf. Folland [11], Theorem 3.17).

(vii) By steps (iv)–(vi) we first choose $N = N(\varepsilon)$, for given $\varepsilon > 0$, such that $I_\lambda(v, N) \leq \varepsilon/2$, for all v, λ, and then choose $\lambda = \lambda(v, N, \varepsilon)$ such that, for $\lambda \geq \lambda(v, N, \varepsilon)$, $II_\lambda(v, N) \leq \varepsilon/2$. This implies, for any $v > 0$, $L > 0$,

$$A_L\left(\frac{v}{\lambda}\right) \to 0 \quad \text{as } \lambda \to \infty.$$

(viii) By change of variables,

$$(*) \quad \begin{aligned} \Lambda(f, g) &= \int_0^\infty \int_0^\infty \int_0^\infty \exp(-(v_1 + \cdots + v_m))\, dv_1 \cdots dv_m \left\{ A_L\left(\frac{v_1 + \cdots + v_m}{\lambda}\right) + A_L^c\left(\frac{v_1 + \cdots + v_m}{\lambda}\right) \right\} \\ &\leq \int_0^\infty \int_0^\infty \int_0^\infty \exp(-(v_1 + \cdots + v_m))\, dv_1 \cdots dv_m \left(A_L\left(\frac{v_1 + \cdots + v_m}{\lambda}\right)\right) + \frac{\delta}{2} \end{aligned}$$

by step (ii) for sufficiently large L. Since $A_L((v_1 + \cdots + v_m)/\lambda) \leq L^2 \pi 2(\|f\|_0^2 + \|g\|_0^2)$, (vii) and Lebesgue's dominated convergence theorem imply that, for any L, the multiple integral in the right-hand side of $(*)$ will be less than $\delta/2$ for $\lambda \geq \lambda(L, \delta) = \lambda(L(\delta), \delta)$. □

3.9. LEMMA. *Let f be a jointly measurable adapted $\mathbf{H}_0$-valued process such that, for any $T > 0$, $\int_0^T E\|f(s)\|_0^4\, ds < \infty$. Then the stochastic integral $\int_0 \langle f^2(s), \nabla \cdot dW_\varepsilon(s)\rangle_0$ defines a real-valued square-integrable continuous mar-*

tingale whose quadratic variation satisfies, for any $0 \le s \le t < \infty$, *the relation*

$$\left[\int_0^t \langle f^2(u), \nabla \cdot dW_\varepsilon(u)\rangle_0\right] - \left[\int_0^s \langle f^2(u), \nabla \cdot dW_\varepsilon(u)\rangle\right]$$

$$(3.15)\qquad = \sum_{l=1}^{2} \int_s^t \int\int f^2(u,r) f^2(u,q) \frac{\partial^2}{\partial r_l\,\partial q_l} g_\varepsilon(r-q)\,dr\,dq$$

$$\le \frac{1}{2\varepsilon}\int_s^t \| f(u)\|_0^4\,du.$$

Proof. We easily check that the quadratic variation is given by the equality in (3.15). From this, the upper bound for its increments follows immediately. Hence the existence of a continuous version of the stochastic integral follows from standard arguments (cf. Metivier and Pellaumail [24], Chapter 1.2.5). □

3.10. Proof of Theorem 1.9. (i) The assumption implies by Lemma 3.9 that the martingale on the right-hand side of (1.33) is a square-integrable continuous martingale. Hence the right-hand side of (1.33) defines a continuous real-valued process.

(ii) We will first replace the martingale and the last quadratic variation integral on the right-hand side of (3.2) by their limits [cf. step (iii) in the proof of Lemma 3.4].

(ii.1) Set $f(s) := X_\varepsilon(s)$, $f_\lambda(s) := X_{\varepsilon,\lambda}(s) := R_\lambda \mathscr{X}_\varepsilon(s)$. Then

$$\langle f_\lambda(s), \nabla \cdot R_\lambda(f(s)\,dW_\varepsilon(s)) - (\nabla f_\lambda(s))\cdot dW_\varepsilon(s)\rangle_0$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty \exp(-\lambda(u_1+u_2+u_3))\,du_1du_2du_3$$

$$\times\Bigg\{\sum_{l=1}^{2}\int\int\left(\frac{\partial}{\partial r_l}G(u_1+u_2+u_3, r-q)\right) f(s,r) f_\lambda(s,r)$$

$$\times \int\big(\tilde{\Gamma}_\varepsilon(q-p) - \tilde{\Gamma}_\varepsilon(r-p)\big) w_l(dp,ds)\,dq\,dr$$

$$-\sum_{l=1}^{2}\int\int \frac{(r_l-q_l)|q-r|}{2\nu(u_1+u_2+u_3)} G(u_1+u_2+u_3, r-q)$$

$$\times[f(s,q) - f(s,r)] f_\lambda(s,r)$$

$$\times\int \frac{\tilde{\Gamma}_\varepsilon(q-p) - \tilde{\Gamma}_\varepsilon(r-p)}{|q-r|} w_l(dp,ds)\,dq\,dr\Bigg\}$$

$$= I_\lambda(ds) + II_\lambda(ds).$$

(ii.2) The spatial and stochastic integrals in $I_\lambda(ds)$ are equal to

$$\int\int\left(\frac{\partial}{\partial r_l}G(u_1+u_2+u_3,r-q)\right)f(s,r)f_\lambda(s,r)\int\hat{\Gamma}_\varepsilon(q-p)w_l(dp,ds)\,dq\,dr$$

$$\left[\text{since} \int\frac{\partial}{\partial r_l}G(u,r-q)\,dq=0\right]$$

$$=-\int\int\left(\frac{\partial}{\partial q_l}G(u_1+u_2+u_3,r-q)\right)f(s,r)f_\lambda(s,r)$$

$$\times\int\hat{\Gamma}_\varepsilon(q-p)w_l(dp,ds)\,dq\,dr$$

$$=\int\int G(u_1+u_2+u_3,r-q)f(s,r)f_\lambda(s,r)$$

$$\times\int\frac{\partial}{\partial q_l}\hat{\Gamma}_\varepsilon(q-p)w_l(dp,ds)\,dq\,dr$$

by (1.19), Fubini's theorem and integration by parts. Hence

$$
\begin{aligned}
I_\lambda(ds)=\int_0^\infty\int_0^\infty\int_0^\infty &\exp(-\lambda(u_1+u_2+u_3))\lambda^3\,du_1du_2du_3\\
&\times\Bigg\{\sum_{l=1}^{2}\int\int G(u_1+u_2+u_3,r-q)f^2(s,q)\\
&\qquad\times\int\frac{\partial}{\partial q_l}\tilde{\Gamma}_\varepsilon(q-p)w_l(dp,ds)\,dr\,dq\\
&\quad+\sum_{l=1}^{2}\int\int G(u_1+u_2+u_3,r-q)(f(s,r)-f(s,q))f_\lambda(s,r)\\
&\qquad\times\int\frac{\partial}{\partial q_l}\tilde{\Gamma}_\varepsilon(q-p)w_l(dp,ds)\,dq\,dr\\
&\quad+\sum_{l=1}^{2}\int\int G(u_1+u_2+u_3,r-q)(f_\lambda(s,r)-f(s,r))f(s,q)\\
&\qquad\times\int\frac{\partial}{\partial q_l}\tilde{\Gamma}_\varepsilon(q-p)w_l(dp,ds)\,dq\,dr\\
&\quad+\sum_{l=1}^{2}\int\int G(u_1+u_2+u_3,r-q)(f(s,r)-f(s,q))f(s,q)\\
&\qquad\times\int\frac{\partial}{\partial q_l}\tilde{\Gamma}_\varepsilon(q-p)w_l(dp,ds)\,dq\,dr\Bigg\}\\
=\sum_{i=1}^{4}F_{i,\lambda}(ds).
\end{aligned}
$$

(ii.3) Clearly,

$$F_{1,\lambda}(ds) = \langle f^2(s), \nabla \cdot dW_\varepsilon(s)\rangle_0$$

(ii.4) Then

$$\begin{aligned}
&[II_\lambda(ds)] \\
&\quad \le \frac{32}{\varepsilon}\int_0^\infty\int_0^\infty\int_0^\infty \exp(-\lambda(u_1+\cdots+u_6))\lambda^6\,du_1\cdots du_6 \\
&\qquad \times\Big\{\int\int\int G(2(u_1+u_2+u_3), r-q)|f(s,q)-f(s,r)|\,|f_\lambda(s,r)| \\
&\qquad\quad \times G(2(u_4+u_5+u_6),\tilde r-\tilde q)|f(s,\tilde q)-f(s,\tilde r)| \\
&\qquad\qquad \times|f_\lambda(s,\tilde r)|\Big\}\,dq\,dr\,d\tilde q\,d\tilde r\,ds
\end{aligned}$$

(by the techniques used in the proof of Lemma 3.4) $\to 0$, as $\lambda\to\infty$ (by Lemma 3.8).

(ii.5) Similarly,

$$[F_{i,\lambda}(ds)] \to 0 \quad \text{as } \lambda\to\infty \text{ for } i=2,4.$$

(ii.6) Then

$$\begin{aligned}
[F_{3,\lambda}(ds)] &\le \sum_{l=1}^{2}\int\int\int|(R_\lambda-I)f(s,r)|\,|(R_\lambda-I)f(s,\tilde r)| \\
&\quad\times\int_0^\infty\int_0^\infty\int_0^\infty \lambda^6\exp(-\lambda(u_1+\cdots+u_6))\,du_1\cdots du_6 \\
&\quad\times G(u_1+u_2+u_3,r-q)G(u_4+u_5+u_6,\tilde r-\tilde q) \\
&\quad\times|f(s,q)|\,|f(s,\tilde q)|\left|\int\frac{\partial}{\partial q_l}\tilde\Gamma_\varepsilon(q-p)\frac{\partial}{\partial\tilde q_l}\tilde\Gamma_\varepsilon(\tilde q-p)\,dp\right| \\
&\quad\times dq\,d\tilde q\,dr\,d\tilde r\,ds \\
&\le \frac{1}{2\varepsilon}\|(R_\lambda-I)f\|_0^2\cdot\|R_\lambda|f|\,\|_0^2\,ds\to 0 \quad\text{as } \lambda\to\infty.
\end{aligned}$$

(ii.7) Since $f_\lambda^2(s,r)\to f^2(s,r)$, $dP\otimes ds\otimes dr$-a.e. and $f_\lambda^2(s)$ uniformly integrable,

$$\big[\langle f_\lambda^2(s),\nabla\cdot dW_\varepsilon(s)\rangle_0\big]\to\big[\langle f^2(s),\nabla\cdot dW_\varepsilon(s)\rangle_0\big],\qquad dP\otimes dt\text{-a.e. as } \lambda\to\infty,$$

whence we obtain uniform convergence on bounded intervals of the corresponding stochastic integrals. Using the identity

$$\langle f_\lambda,(\nabla f_\lambda)\cdot dW_\varepsilon(s)\rangle_0+\tfrac12\langle f_\lambda^2(s),\nabla\cdot dW_\varepsilon(s)\rangle_0=0$$

we obtain

$$(*)\qquad -\int_0^t\langle f_\lambda(s), \nabla R_\lambda(f(s)\,dW_\varepsilon(s))\rangle_0 \to -\tfrac{1}{2}\int_0^t\langle f^2(s), \nabla\cdot dW_\varepsilon(s)\rangle_0$$

in mean square, uniformly on bounded integrals. First of all, the preceding arguments show that the last quadratic variation integral in (3.2) tends a.s. to the corresponding integral in (1.33) uniformly on bounded intervals. Moreover, by choosing a subsequence $\lambda \to \infty$ in $(*)$, we obtain that the martingale in (3.2) tends to the martingale in the right-hand side of (1.33) a.s., uniformly on bounded intervals.

(iii) Next we will consider the first quadratic variation integral in (3.2) plus the first integral (containing $\langle X_{\varepsilon,\lambda}(s), \Delta X_{\varepsilon,\lambda}(s)\rangle_0$). Using the notation of the proof of Lemma 3.1, we obtain

$$\begin{aligned}
&B_1(s,\varepsilon,\lambda)\\
&\quad= \lambda^6\int_0^\infty\int_0^\infty\int_0^\infty \exp(-\lambda(u_1+\cdots+u_6))\,du_1\cdots du_6\\
&\qquad\qquad\times\int X_\varepsilon^2(s,r)\int(\Delta_q G(u_1+\cdots+u_6, r-q))\\
&\qquad\qquad\times(1-g_\varepsilon(r-q))\,dq\,dr\\
&\qquad+\lambda^6\int_0^\infty\int_0^\infty\int_0^\infty \exp(-\lambda(u_1+\cdots+u_6))\,du_1\cdots du_6\\
&\qquad\qquad\times\int\int(\Delta_q G(u_1+\cdots+u_6, r-q))(1-g_\varepsilon(r-q))\\
&\qquad\qquad\times(X_\varepsilon(s,q)-X_\varepsilon(s,r))X_\varepsilon(s,r)\,dq\,dr\\
&\quad= I_\lambda(s) + II_\lambda(s)\int(\Delta_q G(u,r-q))(1-g_\varepsilon(r-q))\,dq\\
&\quad= \sum_{l=1}^{2}\int\frac{(r_l-q_l)^2}{8\nu u\varepsilon}G(u,r-q)g_\varepsilon(r-q)\,dq\\
&\qquad\qquad\qquad\qquad\text{(after integration by parts)}\\
&\quad= \frac{1}{2\varepsilon}.
\end{aligned}$$

Thus, $I_\lambda(s) = 1/2\varepsilon\|X_\varepsilon(s)\|_0^2$. Also $II_\lambda(s) \to 0$, as $\lambda \to \infty$ by the same estimates as in the proof of Lemma 3.1 and by Lemma 3.8. By step (iv) in the proof of Lemma 3.1 and $\|R_\lambda\|_{\mathscr{L}(\mathbb{H}_0)} \le 1$, this together implies that a.s.,

$$2\nu n\int_0^t\|X_{\varepsilon,\lambda}(s)\|_0^{2(n-1)}B_1(s,\varepsilon,\lambda)\,ds \to \frac{\nu n}{\varepsilon}\int_0^t\|X_\varepsilon(s)\|_0^{2n}\,ds$$

uniformly on bounded intervals, as $\lambda \to \infty$.

(iv) Set $f(s) := X_\varepsilon(s)$, $f_\lambda(s) := X_{\varepsilon,\lambda}(s)$ and $F(s) = K_\delta * f(s)$.

(iv.1) Then

$$\begin{aligned}
&\langle f_\lambda(s), \nabla\cdot R_\lambda(f(s)F(s)) - (\nabla f_\lambda(s))\cdot F(s)\rangle_0 \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \exp(-\lambda(u_1+u_2+u_3))\lambda^3\, du_1 du_2 du_3 \\
&\quad\times\Bigg\{\sum_{l=1}^{2}\int\int f_\lambda(s,r)\frac{\partial}{\partial r_l}G(u_1+u_2+u_3, r-q)f(s,r) \\
&\qquad\times[F_l(s,q)-F_l(s,r)]\,dq\,dr \\
&\qquad+\sum_{l=1}^{2}\int\int f_\lambda(s,r)\frac{\partial}{\partial r_l}G(u_1+u_2+u_3,r-q)(f(s,q)-f(s,r)) \\
&\qquad\qquad\times[F_l(s,q)-F_l(s,r)]\,dq\,dr\Bigg\} \\
&= I_\lambda(s) + II_\lambda(s).
\end{aligned}$$

(iv.2) $I_\lambda(s) = \langle f_\lambda(s)f(s), R_\lambda\nabla\cdot F(s)\rangle_0 \to \langle f^2(s), \nabla\cdot F(s)\rangle_0$, as $\lambda\to\infty$, $dP\otimes dt$ a.e. by the continuity of $\nabla\cdot F(s)$.

(iv.3) $II_\lambda(s)\to 0$ as $\lambda\to\infty$ by the estimates in the proof of Lemma 3.2 and by Lemma 3.8. By Lemma 3.2 and similarly to step (ii.7), this implies a.s.

$$2n\int_0^t \|X_{\varepsilon,\lambda}(s)\|_0^{2(n-1)}\langle X_{\varepsilon,\lambda}(s), \nabla\cdot R_\lambda(X_\varepsilon(s)K_\delta * X_\varepsilon(s))\rangle_0\, ds$$

$$\to n\int_0^t \|X_\varepsilon(s)\|_0^{2(n-1)}\langle X_\varepsilon^2(s), (\nabla\cdot K_\delta)*X_\varepsilon(s)\rangle_0\, ds$$

uniformly on bounded intervals.

(v.1) By the previous steps, the convergence of (3.2) to (1.33) is uniform a.s., so we may assume that on the same measurable set Ω_0 with $P(\Omega_0)=1$: (1) $X_\varepsilon(t)\in\mathbf{H}_0$ uniformly in t; (2) $R_\lambda\mathscr{X}_\varepsilon = R_\lambda X_\varepsilon$ is continuous with values in $\mathbf{H}_0$; (3) $\|X_\varepsilon(\cdot)\|_0$ is continuous.

(v.2) If $\varphi\in\mathbf{H}_6$ and $\omega\in\Omega_0$, then

$$\begin{aligned}
|\langle X_\varepsilon(t)-X_\varepsilon(s),\varphi\rangle_0| &= |\langle X_{\varepsilon,\lambda}(t)-X_{\varepsilon,\lambda}(s), R_\lambda^{-1}\varphi\rangle_0| \\
&\le \|X_{\varepsilon,\lambda}(t)-X_{\varepsilon,\lambda}(s)\|_0\|R_\lambda^{-1}\varphi\|_0\to 0,
\end{aligned}$$

as $|t-s|\to 0$. That is, for $\omega\in\Omega_0$, $X_\varepsilon(\cdot)$ is weakly continuous if restricted to $\mathbf{H}_6$. Now it follows by a standard argument that $X_\varepsilon(\cdot)$ is weakly continuous on all of $\mathbf{H}_0$, if $\omega\in\Omega_0$, since $\sup_{0\le t\le T}\|X_\varepsilon(t)\|_0<\infty$ on Ω_0, for any $T>0$. The continuity of $\|X_\varepsilon(t)\|_0$ now implies (1.34).

(vi) Inequality (1.35) follows easily from (1.32), (3.15), (1.33) and the Burkholder–Davis–Gundy inequality. □

3.11. REMARK. Inequality (1.33) for $n=1$ shows that our stochastic Navier–Stokes equation cannot be treated by the usual variational methods on $\mathbf{H}_0$ (cf. Pardoux [28] and the generalization of Pardoux's variational approach by Krylov and Rozovskii [21]).

Finally, we prove uniqueness for (1.22) in the bilinear case.

3.12. PROOF OF THEOREM 1.10. (i) $Z_\varepsilon(t) := Y_\varepsilon(t) - X_\varepsilon(t) = -\sqrt{2\nu}\int_0^t T(t-s)\nabla(Z_\varepsilon(s)\,dW_\varepsilon(s))$.

(ii) Similarly to the derivation of (1.33), we obtain

$$E\|Z_\varepsilon(t)\|_0^2 = \frac{\nu}{\varepsilon}\int_0^t E\|Z_\varepsilon(s)\|_0^2\,ds,$$

which implies $(*)$ in Theorem 1.10 by the Gronwall lemma. □

3.13. REMARK. It is possible to derive uniqueness for (1.22) also for the case when $K_\delta \not\equiv 0$ since both the "smoothed" Euler equation and the diffusion equation ($K_\delta \equiv 0$) have unique weak solutions (cf. Kotelenez [19]).

4. The macroscopic limit. Let $\{r_{\varepsilon,N}\}_{\varepsilon>0} = \{r_{\varepsilon,N}(\cdot, r_N(0))\}_{\varepsilon>0}$ be the $\mathbf{R}^{2N}$-valued solution processes of (1.18) which, for any $\varepsilon > 0$, start in the same initial position $r_N(0)$.

4.1. LEMMA. *The family* $(r_{\varepsilon,N})_{\varepsilon>0}$ *is relatively compact on* $C([0,\infty);\mathbf{R}^{2N})$.

PROOF. (i) Since K_δ bounded and $E|\int_0^t\hat\Gamma_\varepsilon(r_\varepsilon^i(s)-p)w(dp,ds)|^2 = 2t$, we obtain, for any $t \ge 0$, a $c(N,t,a) < \infty$ such that

$$P\{|r_\varepsilon^i(t)| > M\} \le \frac{E|r_\varepsilon^i(t)|^2}{M^2} \le \frac{c(N,t)}{M^2},$$

which implies the compactness condition for the marginals $r_{\varepsilon,N}(t)$, $\forall t \ge 0$.

(ii) To obtain the "modulus of continuity" we compute a bound for the conditional expectation for $0 \le s \le t \le T$:

$$E\Big[|r_\varepsilon^i(t) - r_\varepsilon^i(s)|^2\Big|\mathscr{F}_s\Big] \le 2N^2c_\delta^2(t-s)^2 + 4(t-s).$$

(iii) Together (i) and (ii) imply relative compactness of $r_{\varepsilon,N}$ by Theorem 3.8.2 of Ethier and Kurtz [10] and the fact that the metric on the Skorohod space $D([0,\infty);\mathbf{R}^{2N})$ restricted to $C([0,\infty);\mathbf{R}^{2N})$ is equivalent to the metric of uniform convergence on bounded intervals. □

Let $\beta_N := (\beta^1,\dots,\beta^N)$ be a standard $\mathbf{R}^{2N}$-valued Brownian motion defined on $(\Omega,\mathscr{F},\mathscr{F}_{t\ge 0},P)$. Further, set $\tau_\varepsilon := \inf\{t \ge 0\colon\ r_{\varepsilon,N}(t, r_N(0)) \in \Lambda_N\}$, where Λ_N was defined by (1.36). Define continuous square-integrable martingales $M_{\varepsilon,N} := (M_\varepsilon^1,\dots,M_\varepsilon^N)$ by

$$M_\varepsilon^i(t) := \int_0^{t\wedge\tau_\varepsilon}\int\hat\Gamma_\varepsilon(r_\varepsilon^i(s)-p)w(dp,ds) + [\beta^i(t) - \beta^i(\tau_\varepsilon)]1_{\{t\ge\tau_\varepsilon\}},$$

$i = 1,\dots,N$, where we set $\beta^i(\tau_\varepsilon) = 0$, if $\tau_\varepsilon = \infty$. Denote by $\Rightarrow$ weak convergence.

4.2. LEMMA. *Suppose $r_N(0) \notin \Lambda_N$ a.s. Then*

$$M_{\varepsilon,N} \Rightarrow \beta_N \quad on\ C\big([0,\infty);\mathbf{R}^{2N}\big). \tag{4.1}$$

PROOF. (i) By the martingale central limit theorem, all we have to show is that the mutual quadratic variations of $M_{\varepsilon,N}$, $\langle M_\varepsilon^{i,l}, M_\varepsilon^{j,k}\rangle(t)$, tend for any $t \ge 0$ in probability to $t\cdot\delta_{i,j}\delta_{l,k}$ for $i,j = 1,\dots,N$ and $l,k = 1,2$, where l and k index the one-dimensional components of M_ε^i (resp., M_ε^j) (cf. Ethier and Kurtz [10], Theorem 7.1.4).

(ii) By Lemma 4.1, for any $\eta > 0$, there is a compact set $K_\eta \subset \mathbf{B} := C([0,\infty);\mathbf{R}^{2N})$ such that $\inf_\varepsilon P\{r_{\varepsilon,N}(\cdot, r_N(0)) \in K_\eta\} \ge 1-\eta$ (Ethier and Kurtz [10], Theorem 3.2.2).

(iii) Let $\mathbf{B}_\Lambda := \{q_N(\cdot) \in \mathbf{B}\colon q_N(0) \notin \Lambda_N\}$ and set $\tau := \inf\{t \ge 0\colon q_N(t) \in \Lambda_N\}$. Recalling that $g_\varepsilon(r) := \exp(-|r|^2/8\varepsilon)$, $r \in \mathbf{R}^2$, we define for $i,j \in \{1,\dots,N\}$ mappings $G^{i,j}\colon [0,1]\times\mathbf{B}_\Lambda \mapsto C([0,\infty);\mathbf{R})$ by

$$G^{i,j}(\varepsilon, q_N(\cdot))(t) := \begin{cases} \displaystyle\int_0^{t\wedge\tau} g_\varepsilon\big(q^i(s) - q^j(s)\big)\,ds \\ \quad +(t-\tau)1_{\{t\ge\tau\}}\cdot\delta_{i,j}, & \text{if } \varepsilon > 0,\\ t\cdot\delta_{i,j}, & \text{if } \varepsilon = 0.\end{cases}$$

Clearly, $G^{i,j}$ are continuous from $[0,1]\times\mathbf{B}_\Lambda$ into $C([0,\infty);\mathbf{R})$. By assumption, $K_\eta \subset \mathbf{B}_\Lambda$ and K_δ is compact [cf. step (ii)]. Therefore, the restriction of $G^{i,j}$ to $[0,1]\times K_\eta$ is uniformly continuous. In particular, for any $\rho > 0$ and $T > 0$ there is an $\varepsilon_{ij} > 0$ such that $\sup_{q_N(\cdot)\in K_\eta}\sup_{0\le t\le T} G^{i,j}(\varepsilon, q_N(\cdot))(t) \le \rho$, for all $\varepsilon \le \varepsilon_{ij}$ and $i \ne j$. The definition entails a.s. $G^{i,j}(\varepsilon, r_{\varepsilon,N}(\cdot)) = \langle M_\varepsilon^{i,l}, M_\varepsilon^{j,l}\rangle$, for $l = 1,2$. Hence, for $\varepsilon \le \varepsilon_{ij}$ and $i \ne j$, $P\{\omega\colon \langle M_\varepsilon^{i,l}, M_\varepsilon^{j,l}\rangle(t) > \rho\} \le \eta$. Since w_1 and w_2 are independent, (4.1) follows. □

Next we consider the more classical SODE for the positions of point vortices (cf. Marchioro and Pulvirenti [23]), where we assume the same initial condition $r_N(0) \notin \Lambda_N$ as for $r_{\varepsilon,N}(\cdot)$:

$$dr^i = \sum_{J=1}^{N} a_j K_\delta(r^i - r^j)\,dt + \sqrt{2\nu}\,d\beta^i, \qquad i = 1,\dots,N. \tag{4.2}$$

Clearly, (4.2) has a unique global continuous solution $r_N(\cdot, r_N(0))$.

If $q_N \in \mathbf{R}^{2N}$, we define $F(q_N) = (F^1(q_N),\dots,F^N(q_N))^\perp$ by $F^i(q_N) := \sum_{j=1}^N a_j K_\delta(q^i - q^j)$ and we define a continuous map $\mathbf{\Psi}\colon \mathbf{B}_\Lambda \to \mathbf{B}_\Lambda$ as the "pathwise" solution of the ODE:

$$\mathbf{\Psi}(q_N(\cdot))(t) = \int_0^t F\big(\mathbf{\Psi}(q_N(\cdot))(s)\big)\,ds + q_N(t).$$

Further, let $\mathbf{B}_{\Lambda,\tau} := \{q_N(\cdot) \in \mathbf{B}_\Lambda\colon q_N(t) \equiv q_N(t\wedge\tau)\}$, where τ is the first entrance time of q_N into Λ_N as defined in step (iii) of the proof of Lemma 4.2. $\Phi(q_N(\cdot))(t) := \mathbf{\Psi}(q_N(\cdot))(t\wedge\tau)$ defines a continuous map $\Phi\colon \mathbf{B}_\Lambda \to \mathbf{B}_{\Lambda,\tau}$ which satisfies

$$\Phi(q_N(\cdot))(t) = \int_0^{t\wedge\tau} F(\Phi(q_N(\cdot))(s))\,ds + q_N(t\wedge\tau). \tag{4.3}$$

For $q_N(t,\omega) = M_{\varepsilon,N}(t,\omega) + r_N(0,\omega)$, Φ is the solution of (1.18) if $t \le \tau_\varepsilon$ and, for $q_N(t,\omega) = \beta_N(t,\omega) + r_N(0,\omega)$, Φ is the solution of (4.2) if $t \le \tau_0$, where $\tau_0 := \inf\{t\colon r_N(t, r_N(0)) \in \Lambda_N\}$. Since Λ_N is nonattainable for (4.2) with $r_N(0) \notin \Lambda_N$, $\tau_0 = \infty$ a.s. (cf. Friedman [12], Corollary 11.4.3).

4.3. THEOREM. *Suppose $r_N(0) \notin \Lambda_N$ a.s. for all $N \in \mathbf{N}$. Then $\varepsilon \to 0$ implies*

$$r_{\varepsilon,N}(\cdot, r_N(0)) \Rightarrow r_N(\cdot, r_N(0)) \quad \text{on } C([0,\infty);\mathbf{R}^{2N}). \tag{4.4}$$

PROOF. The continuous mapping theorem (cf. Ethier and Kurtz [10], Corollary 3.1.9) implies $r_{\varepsilon,N}(\cdot\wedge\tau_\varepsilon, r_N(0)) \Rightarrow r_N(\cdot, r_N(0))$ by the preceding considerations. This in particular implies, for any $t \ge 0$,

$$\limsup_{\varepsilon\downarrow 0} P\{r_{\varepsilon,N}(t\wedge\tau_\varepsilon, r_N(0)) \in \Lambda_N\} \le P\{r_N(t, r_N(0)) \in \Lambda_N\} = 0,$$

whence $\lim_{\varepsilon\downarrow 0}\tau_\varepsilon = \infty$ a.s. □

We can now derive a macroscopic limit theorem as an easy consequence of our Theorem 4.3 and of Theorem 4.2 of Marchioro and Pulvirenti [23]. Recall that $X(\cdot, X(0))$ is the solution of (1.1) with initial condition $X(0)$, where we assume that $\int X^{\pm}(0,r)\,dr = a^{\pm}$. Further, denote by $\mathscr{Y}_{N,\delta}(t, \mathscr{X}_N(0))$ the empirical process associated with $r_N(\cdot, r_N(0))$, the solution of (4.2) [cf. (1.5)], where $\mathscr{X}_N(0) := \sum_{I=1}^N a_i \delta_r i_{(0)}$ and $\mathscr{X}_{\varepsilon,\delta}(t, \mathscr{X}_N(0))$ is the empirical process associated with (1.18), which is a solution of (1.22). Now δ indicates the dependence on the smoothing in K_δ.

4.4. PROOF OF THEOREM 1.11. By Marchioro and Pulvirenti [23] there is a sequence $\delta(N) \to 0$ such that, for any $\rho > 0$, $t > 0$ and $N \ge N(\rho, t)$,

$$\left|E\langle \mathscr{Y}_{N,\delta(N)}(t), \varphi\rangle - \langle X(t), \varphi\rangle\right| < \frac{\rho}{2}.$$

Moreover, by Theorem 4.3 for ρ and $N(\rho,t)$ there is an $\varepsilon(N(\rho,t))$ such that, for $\varepsilon \le \varepsilon(N(\rho,t))$,

$$\left|E\langle \mathscr{X}_{\varepsilon,\delta(N(\rho,t))}, \varphi\rangle - E\langle \mathscr{Y}_{N,\delta(N(\rho,t))}(t), \varphi\rangle\right| < \frac{\rho}{2}. \qquad \square$$

4.5. REMARK. Theorem 4.3 implies in particular that

$$\langle \mathscr{X}_{\varepsilon,\delta}(t), \varphi\rangle \Rightarrow \langle \mathscr{Y}_{N,\delta}(t), \varphi\rangle \quad \text{as } \varepsilon \to 0,$$

if $r_N(0) \notin \Lambda_N$ a.s. for all N. This means that we do not obtain a macroscopic distribution if $\varepsilon \to 0$, but N remains fixed, that is, if the vorticity keeps being concentrated in points.

However, we also expect macroscopic behavior for already smooth initial conditions, that is, where $X_\varepsilon(0) \in \mathbf{H}_0$, or smoother if $\varepsilon \to 0$.

5. Bilinear equations and generalizations.

5.1. REMARK. We may choose $g_\delta \equiv$ const. and thus $K_\delta \equiv 0$ and all results from Sections 2–4 will hold for the mild solution of

$$d\mathscr{X}(t) = \nu \Delta \mathscr{X} dt - \sqrt{2\nu}\, \nabla \cdot \left(\mathscr{X} \int \hat{\Gamma}_\varepsilon(\cdot, p) w(dp, ds) \right). \tag{5.1}$$

(In the macroscopic limit we just ignore K_δ.) Moreover, the results are not dimension dependent (even in the semilinear case working with some abstract smooth K_δ). An equation of type (5.1) was suggested by Molchanov [25] to describe the temperature field in a random flow.

The extension of our results to the vector-valued case is straightforward. In particular, by discretizing the momentum (see Kotelenez and Mann [20]), we can easily obtain by this approach a stochastic Navier–Stokes equation for the velocity field of a two-dimensional fluid. Again this equation would consist of the macroscopic NSE plus a state dependent fluctuation term of small order, similar to the stochastic term in (1.22). The salient feature in our approach is the derivation of the fluctuation term in (1.22) from the fluctuation "force" acting on the positions of the vortices, that is, its derivation from a microscopic model. This inevitably leads to a state dependent fluctuation term in (1.22) with the advantage that certain physical properties are conserved. Note that perturbation of a (parabolic) PDE by a state independent (Gaussian white noise) fluctuation term may deprive the resulting SPDE of the physical meaning attached to the PDE. This was shown for the reaction–diffusion equation (RDE) in Kotelenez [18] (a RDE perturbed by state independent white noise would no longer yield nonnegative solutions, so it could not be interpreted as a description of a mass distribution). A similar statement can be made concerning our NSE (1.1). Perturbations by state independent Gaussian white noise would yield solutions which do not preserve the orientation (which they should as a consequence of the conservation of the angular momentum; cf. Kotelenez and Mann [20]). Although adding Gaussian white noise to a PDE may render the resulting SPDE physically meaningless, it leads to a sometimes mathematically more accessible formalism and is widely used. For the NSE (for the velocity field), examples of this approach are Bensoussan and Temam [2], Albeverio and Cruzeiro [1] and the last chapters in Vishik and Fursikov [34]. It should be mentioned that a "justification" for adding a state independent noise is the argument that this presents an external random force acting on the system. Since it is hard to see how an external random "force" can make a distribution of particles negative, we believe that this kind of reasoning is not correct. One can, of course, let state independent fluctuation "forces" act on the positions of particles as done in (1.10), and one may therefore interpret the "forces" as external ones (in the spirit of the Ornstein–Uhlenbeck approximation to the Einstein–Smoluchowski theory of Brownian motion; cf. Nelson [26] and also van Kampen's comments on this interpretation [33], page 247). However, (1.10) leads to the (formal) SPDE (1.11) which also has a state dependent noise term. Moreover, in both (1.22) and (1.11) the resulting state dependent fluctuation term comes from the diffusion alone (if $\nu = 0$, then there are no fluctuations). On the other hand, if we include creation and annihilation, then there would be an additional (state dependent) fluctuation term (see Kotelenez [16] and also Dawson [6], where in [6] the resulting fluctuation term is due exclusively to the branching of particles). Let us briefly comment on correlations between the fluctuation "forces" acting on the positions of the vortices (resp., particles), which are more general than those given in (1.18). Such an assumption would yield an SPDE with state dependent diffusion coefficient, that is, a quasilinear SPDE (see Vaillancourt [32] for such a particle system as well as Kotelenez [19]). This generalization seems to be quite natural in areas like physical oceanography, where the molecular viscosity has to be replaced by the so-called eddy diffusion coefficient. The empirical determination of those coefficients yields spatially dependent diffusion coefficients (see Olson [27]), which will be distorted by noise (such as weather, etc.). Thus a more general task is to derive an SNSE (and its natural generalization) with state dependent diffusion coefficients, which have to be determined by some sort of smoothing procedure. This will be done in a forthcoming paper (cf. Kotelenez [19]).

5.2. REMARKS. (i) The generalization of the inviscid case to the viscous case can be interpreted as follows. In the inviscid case, the mezoscopic and macroscopic models coincide and are given by the Euler equation. In the viscous case, the mezoscopic and the macroscopic models are different. Furthermore, the mezoscopic model is described by a family of stochastic Navier–Stokes equations indexed by the small noise parameter $\varepsilon > 0$, and the macroscopic model, given by the Navier–Stokes equation, should be the limiting case of $\varepsilon = 0$ (and the limit for $\varepsilon \to 0$).

(ii) All results of this paper can be easily extended to higher dimensions for (abstract) particle systems as long as there is no creation or annihilation. The two dimensionality in this paper is only needed for the particular physical interpretation.

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