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GAUSSIAN LIMIT FIELDS FOR THE INTEGRATED PERIODOGRAM

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Functionals of a two-parameter integrated periodogram have been used for detecting a change in the spectral distribution of a stationary sequence. The bases for these results are functional central limit theorems for the integrated periodogram with a Gaussian limit field. We prove functional central limit theorems for a general linear sequence having a finite fourth moment which is shown to be the optimal moment condition. Our approach is via an approximation of the integrated periodogram by a finite linear combination of sample autocovariances. This gives special insight into the structure of the Gaussian limit field.

1. Introduction. The objective of this paper is to study the two-parameter process

$$\int_{-\pi}^{\lambda} I_{n,[nx],X}(y)\, f(y)\, dy, \qquad 0 \le x \le 1,\ -\pi \le \lambda \le \pi, \tag{1.1}$$

for a smooth function f and for the periodogram

$$I_{n,[nx],X}(\lambda) = \frac{1}{n}\left|\sum_{t=1}^{[nx]} X_t e^{-i\lambda t}\right|^2, \qquad 0 \le x \le 1,\ -\pi \le \lambda \le \pi,$$

of a sample $X_1, \ldots, X_n$ taken from a linear process

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \qquad t \in \mathbb{Z}. \tag{1.2}$$

The i.i.d. noise sequence $(Z_t)_{t\in\mathbb{Z}}$ is supposed to be mean zero with finite variance $\operatorname{var}(Z_0) = \sigma^2 > 0$. We will make the assumptions and notation precise in Section 2.

Two-parameter processes of type (1.1) have been used for detecting a change in the spectral distribution function of the sample $X_1, \ldots, X_n$. The integrated periodogram (1.1) serves here as an analogue to the empirical process for i.i.d. observations, and test statistics based on (1.1) have a structure which is similar to the corresponding Kolmogorov–Smirnov-type test statistics in nonparametric statistics. The basic idea is that the renormalized periodogram ordinates

$$\frac{1}{n}\left|\sum_{t=1}^{n} X_t e^{-i\lambda t}\right|^2 \Big/ |\psi(e^{-i\lambda})|^2 \quad \text{with } \psi(e^{-i\lambda}) = \sum_{j=-\infty}^{\infty} \psi_j e^{-i\lambda j}$$

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are asymptotically i.i.d. exponential at distinct frequencies $\lambda \in (0, \pi)$ and that the corresponding integrated periodogram has very much the same asymptotic behaviour as a sum of independent exponential random variables (r.v.'s). This fact has already been exploited by Grenander and Rosenblatt (1957) and Bartlett (1954) with their pioneering work on goodness-of-fit tests and also by Whittle (1953), who introduced a parameter estimator for autoregressive moving average (ARMA) processes based on the integrated periodogram.

Picard (1985) introduced tests for detecting a changepoint of the spectral distribution function F of a stationary Gaussian sequence (X_t). The test statistics are either

$$\sqrt{n} \sup_{\substack{0\le x\le 1\\ 0\le\lambda\le\pi}} \left| \frac{1}{2\pi n} \int_0^\lambda \left| \sum_{t=1}^{[nx]} X_t e^{-iyt} \right|^2 dy - \frac{[nx]}{n} F(\lambda) \right|$$

when F is known or

$$\sqrt{n} \sup_{\substack{0\le x\le 1\\ 0\le\lambda\le\pi}} \frac{1}{2\pi n} \left| \left(1 - \frac{[nx]}{n}\right) \int_0^\lambda \left| \sum_{t=1}^{[nx]} X_t e^{-iyt} \right|^2 dy \right.$$
$$\left. - \frac{[nx]}{n} \int_0^\lambda \left| \sum_{t=[nx]+1}^{n} X_t e^{-iyt} \right|^2 dy \right|$$

when F is unknown, or they are weighted versions of these statistics. Picard derived the asymptotic distributions of these test statistics from a two-parameter functional central limit theorem (FCLT) for the process (1.1). Giraitis and Leipus (1990, 1992) relaxed the condition of normality of (X_t) to a linear process with sufficiently high finite moments and derived a FCLT for (1.1) with a Gaussian limit field.

It is our intention to give another view of these results. Fundamental for our approach is the fact that the original problem for the periodogram of the stationary process (X_t) can be relaxed to the simpler problem for the periodogram of the i.i.d. noise sequence (Z_t). This is due to some specific continuity properties of the periodogram and of the sample autocovariances. Phillips and Solo (1992) have given a partial justification of this very useful fact. Furthermore, we observe that weak limits for linear combinations of sample autocovariances of the noise (Z_t) lead via a Slutsky argument to asymptotic results for the integrated periodogram. For example, the Whittle estimator for ARMA processes is based on the integrated periodogram of the observations $X_1, \dots, X_n$, and the derivation of its limit distribution relies mainly on the asymptotic behaviour of a finite number of such sample autocovariances [see Brockwell and Davis (1991), Chapter 10.8]. A similar approach allows us to derive goodness–of–fit test statistics based on the integrated periodogram [see Grenander and Rosenblatt (1957), Bartlett (1954), Priestley (1981), Dzhaparidze (1986) and Anderson (1993)].

These considerations are the starting point for our approach. In Section 3 we commence with a sequence of i.i.d. r.v.'s and derive a two-parameter FCLT for the integrated periodogram. This FCLT is a consequence of the limit theory for a finite vector of sample autocovariances.

FCLT's for empirical processes are standard and can be found in Shorack and Wellner (1986), for example. Under standard conditions, the resulting limit for the sequential empirical process is a *Kiefer process* $\widetilde{K}(x, t)$ which is a Gaussian two-parameter field for $x \geq 0$ and $t \in [0, 1]$ and satisfies

$$E\widetilde{K}(x, t) = 0$$

$$\operatorname{cov}\big(\widetilde{K}(x_1, t_1), \widetilde{K}(x_2, t_2)\big) = \min(x_1, x_2)(\min(t_1, t_2) - t_1 t_2).$$

The Gaussian limit field we obtain for the integrated periodogram is based on the process

$$K(x, \lambda) = 2\sum_{t=1}^{\infty} Y_t(x)\frac{\sin(\lambda t)}{t}, \qquad x \in [0, 1],\ \lambda \in [-\pi, \pi], \tag{1.3}$$

where $Y_1(x), Y_2(x), \ldots$ are i.i.d. Wiener processes with $\operatorname{var} Y_1(1) = \sigma^4$.

Using well known formulae for trigonometric functions [e.g., Gradshteyn and Ryzhik (1994), page 39] for $x_1, x_2 \in [0, 1]$ and $0 \leq \lambda_1 \leq \lambda_2 \leq \pi$, we obtain for the covariance function

$$\begin{aligned} &E(K(x_1, \lambda_1)K(x_2, \lambda_2)) \\ &\quad = 4\sigma^4 \min(x_1, x_2) \sum_{t=1}^{\infty} \frac{\sin(\lambda_1 t)\sin(\lambda_2 t)}{t^2} \\ &\quad = 2\pi^2\sigma^4 \min(x_1, x_2)\left(\min\left(\frac{\lambda_1}{\pi}, \frac{\lambda_2}{\pi}\right) - \frac{\lambda_1}{\pi}\frac{\lambda_2}{\pi}\right). \end{aligned} \tag{1.4}$$

Hence the process $K(x, \lambda)$ can be considered as a (suitably scaled) Kiefer process. This provides a further link between empirical process theory and the integrated periodogram. We still mention the well known property of a Kiefer process that for each fixed $x \neq 0$ the process $K(x, \lambda)$ defines a Brownian bridge; also for fixed $\lambda \neq 0$ we obtain a Wiener process. This is also immediate from (1.3) via the Lévy–Ciesielski or Paley–Wiener decomposition of a bridge process and by the scaling properties of independent Wiener processes.

In Section 4 we extend these results to the linear process (X_t) having a finite fourth moment. Our main result (Theorem 4.3) is a FCLT for the two-parameter process (1.1) with a two-parameter Gaussian field as limit. In the simplest case, Theorem 4.3 states that, for $x \in [0, 1]$ and $\lambda \in [-\pi, \pi]$,

$$\sqrt{n}\int_{-\pi}^{\lambda}\left(\frac{I_{n,[nx],X}(y)}{|\psi(e^{-i\lambda})|^2} - \frac{[nx]}{n}\,\sigma^2\right)dy \to_d (\lambda + \pi)Y_0(x) + K(x, \lambda)$$

in the Skorokhod space $\mathcal{D}([0, 1] \times [-\pi, \pi])$, where the Wiener process Y_0 is independent of $K(x, \lambda)$. This gives some insight into the structure of the Gaussian limit field. The course of the proof shows that the summand $Y_0(x)$ is due to the diagonal term in the quadratic form $\int_{-\pi}^{\lambda} I_{n,[nx],X}(y)/|\psi(e^{-i\lambda})|^2\,dy$, whereas the $Y_h(x)$ for $h \geq 1$ are the contributions from the sample autocovariances of the noise (Z_t) at lag h. Moreover, for a suitable random centering sequence, the Gaussian limit field is simply the Kiefer process $K(x, \lambda)$ defined in (1.3).

In Section 5 we prove the results from Sections 3 and 4, and in Section 6 we apply them to changepoint detection in some financial data sets.

2. Notation and assumptions. In this section we make the notation of the previous section precise. We also introduce basic assumptions.

Throughout we use the convention that $\sum_{i=a}^{b} a_i = 0$ for any sequence (a_i) provided $b < a$. The symbol c stands for positive constants which are possibly different from line to line or formula to formula and whose precise values are not of interest.

In order to state our results we need the definition of a p-stable r.v. We say that Y has a p-stable distribution ($Y =_d S_p(\sigma, \beta, \mu)$) if there are parameters $0 < p \leq 2$, $\sigma > 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$ such that its characteristic function has the form

$$E\, e^{i\theta Y} = \begin{cases} \exp\left\{-\sigma^p|\theta|^p\left(1 - i\beta(\operatorname{sign}\,\theta)\tan\dfrac{\pi p}{2}\right) + i\mu\theta\right\}, & \text{if } p \neq 1,\\[2ex] \exp\left\{-\sigma|\theta|\left(1 + \dfrac{2}{\pi}i\beta(\operatorname{sign}\,\theta)\ln|\theta|\right) + i\mu\theta\right\}, & \text{if } p = 1.\end{cases}$$

The case $p = 2$ corresponds to normal r.v.'s and we denote by $N(\mu, \sigma^2)$ a normal r.v. with mean μ and variance σ^2. In certain cases we will obtain stable limit processes. For their definition and properties we refer to the two recent monographs by Samorodnitsky and Taqqu (1994) and Janicki and Weron (1993).

We consider the linear process (1.2), where the innovations or the noise $(Z_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. r.v.'s with mean zero and finite variance σ^2. The assumption

$$\sum_{j=-\infty}^{\infty} |\psi_j|\, j < \infty \tag{2.1}$$

ensures that X_t is properly defined as an a.s. absolutely converging series. The function

$$|\psi(e^{-i\lambda})|^2 = \left|\sum_{j=-\infty}^{\infty} \psi_j\, e^{-i\lambda j}\right|^2, \qquad -\pi \leq \lambda \leq \pi,$$

is called the power transfer function (of the linear filter (ψ_j)), and

$$f(\lambda) = |\psi(e^{-i\lambda})|^2 \sigma^2/(2\pi)$$

is just the spectral density of the linear process (X_t). Condition (2.1) implies that f is bounded, hence belongs to any L^p.

Let now $(A_t)_{t\geq 1}$ be one of the sequences $(Z_t)_{t\in\mathbb{Z}}$ or $(X_t)_{t\in\mathbb{Z}}$. We define the periodogram for the sample $A_1, \dots, A_n$ as

$$I_{n,[nx],A}(\lambda) = n^{-1}\left|\sum_{t=1}^{[nx]} A_t e^{-i\lambda t}\right|^2, \qquad x \in [0,1],\ -\pi \leq \lambda \leq \pi,$$

and we also write for simplicity

$$I_{n,A}(\lambda) = n^{-1}\left|\sum_{t=1}^{n} A_t e^{-i\lambda t}\right|^2, \qquad -\pi \leq \lambda \leq \pi.$$

We will frequently make use of weak convergence in the Skorokhod space $\mathscr{D}$. In particular, we need the spaces $\mathscr{D}([a,b]\times[c,d],\mathbb{R}^m)$ of m-dimensional cadlag functions on $[a,b]\times[c,d]$ for finite $a < b$, $c < d$ and integer $m \geq 1$. We suppose that $\mathscr{D}([a,b]\times[c,d],\mathbb{R}^m)$ is equipped with the J_1-topology and the corresponding σ-algebra of the Borel sets [see Jacod and Shiryaev (1987), Bickel and Wichura (1971); also Billingsley (1968) and Pollard (1984) for special cases].

3. i.i.d. sequences. In this section we consider the case $(X_t) = (Z_t)$. We will derive a two-parameter FCLT for the integrated periodogram $(\int_{-\pi}^{\lambda} I_{n,[nx],Z}(y)\,dy)$ which is the basis for the corresponding results for general linear processes in Section 4.

THEOREM 3.1. (a) *Suppose that* $EZ^4 < \infty$. *Then*

$$\left(n^{-1/2}\sum_{t=1}^{[nx]}(Z_t^2-\sigma^2),\ n^{1/2}\int_{-\pi}^{\lambda}\left(I_{n,[nx],Z}(y) - n^{-1}\sum_{t=1}^{[nx]} Z_t^2\right)dy\right)$$
$$\to_d (Y_0(x),\ K(x,\lambda)), \qquad x\in[0,1],\ \lambda\in[-\pi,\pi],$$

in $\mathscr{D}([0,1]\times[-\pi,\pi],\mathbb{R}^2)$, *where* $Y_0(x)$ *and* $K(x,\lambda)$ *are independent stochastic processes,* $Y_0(x)$ *is a Wiener process with* $Y_0(0)\equiv 0$, $Y_0(1) =_d N(0,\mathrm{var}(Z^2))$ *and* $K(x,\lambda)$ *is a Kiefer process with covariance function* (1.4).

(b) *Suppose that* Z *is symmetric and*

$$P(Z^2 > x) = L(x)x^{-p}, \qquad x > 0,$$

for some $p \in (1,2)$ *and a slowly varying function* L. *Then for some slowly varying function* L_1,

$$\left(n^{-1/p}L_1(n)\sum_{t=1}^{[nx]}(Z_t^2-\sigma^2), n^{1/2}\int_{-\pi}^{\lambda}\left(I_{n,[nx],Z}(y) - n^{-1}\sum_{t=1}^{[nx]} Z_t^2\right)dy\right)$$
$$\to_d (Y_0(x),\ K(x,\lambda)), \qquad x\in[0,1],\ \lambda\in[-\pi,\pi],$$

in $\mathscr{D}([0,1]\times[-\pi,\pi],\mathbb{R}^2)$, *where* $Y_0(x)$ *and* $K(x,\lambda)$ *are independent stochastic processes,* $Y_0(x)$ *is a* p*-stable motion with* $Y_0(0)\equiv 0$, $Y_0(1) =_d S_p(1,\ 1,0)$ *and* $K(x,\lambda)$ *is a Kiefer process with covariance function* (1.4).

Now we obtain a FCLT for the integrated periodogram of the noise variables (Z_t) as an immediate consequence of Theorem 3.1.

COROLLARY 3.2. (a) *Suppose the assumptions of Theorem* 3.1(a) *hold. Then*

$$n^{1/2}\int_{-\pi}^{\lambda}\left(I_{n,[nx],Z}(y)-\frac{[nx]}{n}\sigma^2\right)dy \to_d S(x,\lambda)=(\lambda+\pi)Y_0(x)+K(x,\lambda),$$

$$x\in[0,1],\ \lambda\in[-\pi,\pi]$$

in $\mathscr{D}([0,1]\times[-\pi,\pi])$, *where* $Y_0(x)$ *and* $K(x,\lambda)$ *are as in Theorem* 3.1(a); *that is, the limit process is a two-parameter Gaussian field.*

(b) *Suppose the assumptions of Theorem* 3.1(b) *hold. Then for some slowly varying function* L_1,

$$n^{1-1/p}L_1(n)\int_{-\pi}^{\lambda}\left(I_{n,[nx],Z}(y)-\frac{[nx]}{n}\sigma^2\right)dy \to_d (\lambda+\pi)Y_0(x),$$

$$x\in[0,1],\ \lambda\in[-\pi,\pi],$$

in $\mathscr{D}([0,1]\times[-\pi,\pi])$, *where* $Y_0(x)$ *is a* p*-stable motion as in Theorem* 3.1(b); *that is, the limit process is a two-parameter stable process.*

Random centering yields a unifying result for variables Z satisfying the assumptions of either Theorem 3.1(a) or (b).

PROPOSITION 3.3. *Suppose* $\sigma^2<\infty$. *Then the two-parameter process*

$$n^{1/2}\int_{-\pi}^{\lambda}\left(I_{n,[nx],Z}(y)-n^{-1}\sum_{t=1}^{[nx]}Z_t^2\right)dy \to_d K(x,\lambda),$$

$$x\in[0,1],\ \lambda\in[-\pi,\pi],$$

in $\mathscr{D}([0,1]\times[-\pi,\pi])$, *where* $K(x,\lambda)$ *is a Kiefer process with covariance function* (1.4).

Consequently, random centering with $n^{-1}\sum_{t=1}^{[nx]}Z_t^2$ yields the same limit field independent of EZ^4 being finite or infinite provided $\sigma^2<\infty$. This changes dramatically if we use deterministic centering with σ^2. Then we get different limits according as $EZ^4<\infty$ or $EZ^4=\infty$. The random centering with $n^{-1}\sum_{t=1}^{[nx]}Z_t^2$ eliminates the leading (diagonal) term in the quadratic form given by the integrated periodogram. The remaining terms are of the same order, as becomes clear in Proposition 5.1. The random centering can be interpreted as a robustification of the limiting procedure.

4. Linear processes with finite fourth moment. In this section we extend the FCLT of Section 3 from the i.i.d. case to a general linear process (1.2). The key to our results is a standard decomposition which links the periodogram $I_{n,X}$ of the linear process (X_t) with the periodogram $I_{n,Z}$ of the i.i.d. sequence (Z_t) [e.g., Brockwell and Davis (1991), pages 346–347].

LEMMA 4.1. *We have*

$$I_{n,X}(\lambda) = |\psi(e^{-i\lambda})|^2 \, I_{n,Z}(\lambda) + n^{-1}R_n(\lambda), \qquad -\pi \le \lambda \le \pi,$$

where

$$R_n(\lambda) = \psi(e^{-i\lambda})J_n(\lambda)\, Y_n(-\lambda) + \psi(e^{i\lambda})J_n(-\lambda)\, Y_n(\lambda) + |Y_n(\lambda)|^2,$$

$$\psi(e^{-i\lambda}) = \sum_{j=-\infty}^{\infty} \psi_j e^{-i\lambda j}, \qquad J_n(\lambda) = \sum_{t=1}^{n} Z_t\, e^{-i\lambda t},$$

$$Y_n(\lambda) = \sum_{j=-\infty}^{\infty} \psi_j\, e^{-i\lambda j}\, U_{nj}(\lambda), \qquad U_{nj}(\lambda) = \sum_{t=1-j}^{n-j} Z_t\, e^{-i\lambda t} - \sum_{t=1}^{n} Z_t\, e^{-i\lambda t}.$$

This lemma enables us to write the integrated periodogram as

$$\begin{aligned} &n^{1/2}\int_{-\pi}^{\lambda} I_{n,[nx],X}(y)\,dy \\ &\quad = |\psi(e^{-i\lambda})|^2\, n^{1/2}\int_{-\pi}^{\lambda} I_{n,[nx],Z}(y)\,dy + n^{-1/2}\int_{-\pi}^{\lambda} R_{[nx]}(y)\,dy. \end{aligned} \tag{4.1}$$

Thus we can apply the FCLT for the integrated periodogram of i.i.d. (Z_t) (see Theorem 3.1) if the remainder in (4.1) is uniformly negligible (in λ and x). This is indeed the case if the fourth moment of Z exists.

LEMMA 4.2. *If* $EZ^4 < \infty$, *then*

$$n^{-1/2}\int_{-\pi}^{\lambda} R_{[nx]}(y)\,dy \to_P 0$$

uniformly in $\lambda \in [-\pi, \pi]$ *and* $x \in [0, 1]$.

The following result gives some insight into the structure of the Gaussian limit field of the integrated periodogram. Together with Corollary 3.2 this result also explains the influence of the sample autocovariances of the noise (Z_t) on the Gaussian limit field.

THEOREM 4.3. *Suppose the linear process* (X_t) *has i.i.d. mean-zero innovations* (Z_t) *and* $EZ^4 < \infty$. *Let* $f(\lambda)$ *and* $g(\lambda) = |\psi(e^{-i\lambda})|^2 f(\lambda)$ *be continuously differentiable functions on* $(-\pi, \pi)$. *Then*

$$\begin{aligned} &\sqrt{n}\int_{-\pi}^{\lambda}\left(I_{n,[nx],X}(y) - \sigma^2|\psi(e^{-iy})|^2\frac{[nx]}{n}\right)f(y)\,dy \\ &\quad \to_d g(\lambda)\,S(x,\lambda) - \int_{-\pi}^{\lambda} g'(y)S(x,y)\,dy, \qquad x \in [0,1],\ \lambda \in [-\pi,\pi], \end{aligned} \tag{4.2}$$

in $\mathscr{D}([0,1]\times[-\pi,\pi])$, *where* $S(x,\lambda)$ *is the two-parameter Gaussian field introduced in Corollary* 3.2.

If $\mathrm{var}(Z) = \sigma^2$ is unknown, it is reasonable to replace the centering sequence in Theorem 4.3 by a corresponding estimator based on the (Z_t). Revisiting the proof of Corollary 3.2 we see that the additional Wiener process $Y_0(x)$ in the limit is due to the centering with σ^2 instead of $n^{-1}\sum_{t=1}^n Z_t^2$. This makes the limit process $S(x, \lambda)$ more complicated. The following result suggests the use of a random centering sequence which can be calculated from the observations and which overcomes the additional term $Y_0(x)$ in the limit process.

THEOREM 4.4. *Suppose the assumptions of Theorem* 4.3 *hold and assume in addition that* $|\psi(e^{-i\lambda})|^2$ *is positive on* $[-\pi, \pi]$. *Then*

$$\sqrt{n}\int_{-\pi}^{\lambda}\left(I_{n,[nx],X}(y) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{I_{n,[nx],X}(z)}{|\psi(e^{-iz})|^2}\,dz\,|\psi(e^{-iy})|^2\right)f(y)\,dy$$

$$\to_d g(\lambda)\,K(x,\lambda) - \int_{-\pi}^{\lambda} g'(y)\,K(x,y)\,dy, \qquad x\in[0,1],\ \lambda\in[-\pi,\pi],$$

in $\mathscr{D}([0,1]\times[-\pi,\pi])$, *where* $K(x,\lambda)$ *is a Kiefer process with covariance function* (1.4).

The Gaussian limit field (4.2) coincides—although the representation is different—with the one in Picard (1985) and Giraitis and Leipus (1990, 1992) (except that we integrate from $-\pi$ to λ, whereas they integrate from $-\lambda$ to λ). An inspection of their proofs suggests that one needs at least a finite eighth moment of Z. Our conditions on the power transfer function $|\psi(e^{-i\lambda})|^2$ and on f are more restrictive than in Giraitis and Leipus (1990, 1992), who require that these functions belong to certain L^q spaces or that they are of bounded variation. Differentiability of f and g implies that these functions are bounded, hence belong to all L^q spaces. On the other hand, differentiability allows for the representation of the Gaussian field (4.2), which we think provides a more intuitive understanding of the limit.

REMARK. The approach of this section does not work for linear processes when the fourth moment of Z is infinite. For example, consider the MA(1)-process $X_t = Z_t + \theta Z_{t-1}$, $t\in\mathbb{Z}$, with symmetric Z and $P(Z^2 > x)\sim c_0x^{-p}$ as $x\to\infty$ for some $c_0>0$ and $p\in(1,2)$. Then $EZ^4=\infty$, but still $\sigma^2<\infty$. It is not difficult to see that

$$\begin{aligned}&\frac{n^{1-1/p}}{2\pi}\int_{-\pi}^{\pi}\left(I_{n,[nx],X}(\lambda) - \sigma^2|\psi(e^{-i\lambda})|^2\frac{[nx]}{n}\right)d\lambda\\ &\quad = n^{-1/p}(1+\theta^2)\sum_{t=1}^{[nx]}(Z_t^2-\sigma^2) - \theta^2 n^{-1/p}Z_{[nx]}^2 + o_P(1)\\ &\quad =: A_n(x) + B_n(x) + o_P(1)\end{aligned} \tag{4.3}$$

uniformly in $x\in[0,1]$, (A_n) converges in distribution to a p-stable process in $\mathscr{D}[0,1]$, but (B_n) is not tight and hence (A_n+B_n) is not tight. This situation remains the same if we replace σ^2 in (4.3) by its estimate $n^{-1}\sum_{t=1}^n Z_t^2$. The problems arise from the fact that the remainders $\int_{-\pi}^{\lambda} R_{[nx]}(y)\,dy$ are not negligible. That is, they are of the same order as $n^{1-1/p}\int_{-\pi}^{\lambda} I_{n,[nx],Z}(y)\,dy$ and do not converge weakly in $\mathscr{D}([0,1]\times[-\pi,\pi])$. Thus the FCLT for the two-parameter integrated periodogram of linear processes with infinite fourth moment does not hold. This means that these results are sensitive to large fluctuations in the innovations and therefore are not very robust. This failure came to us quite surprisingly since we showed in Klüppelberg and Mikosch (1993, 1994, 1996) and Mikosch, Gadrich, Klüppelberg and Adler (1995) that several statistical procedures in the frequency domain (such as estimation of the power transfer function, parameter estimation for ARMA processes and goodness-of-fit tests) remain valid with slight modifications for observations having even infinite variance. However, we mention that a one-parameter FCLT for the integrated periodogram (1.1) with $x = 1$ still holds true provided $\sigma^2 < \infty$. This can be used to derive goodness-of-fit test statistics in the spirit of Grenander and Rosenblatt (1957), Bartlett (1954), Dzhaparidze (1986) or Anderson (1993); see also Klüppelberg and Mikosch (1996) for the infinite variance case.

5. Proofs of the results. First we recall some results on sample autocovariances.

PROPOSITION 5.1. *Under the corresponding assumptions of Theorem* 3.1, *the following limit relations hold for every fixed* $m \geq 1$:

(a) $$\left(n^{-1/2}\sum_{t=1}^{n}(Z_t^2-\sigma^2),\, n^{-1/2}\sum_{t=1}^{n-1} Z_t Z_{t+1},\ldots,n^{-1/2}\sum_{t=1}^{n-m} Z_t Z_{t+m}\right) \to_d (Y_0, Y_1,\ldots,Y_m)$$

for independent mean-zero Gaussian r.v.'s $Y_0, Y_1,\ldots,Y_m$, *where* $\operatorname{var}(Y_0) = \operatorname{var}(Z^2) = EZ^4 - \sigma^4$ *and* $Y_1,\ldots,Y_m$ *are i.i.d. with* $\operatorname{var}(Y_1) = \sigma^4$;

(b) $$\left(n^{-1/p}L_1(n)\sum_{t=1}^{n}(Z_t^2-\sigma^2),\, n^{-1/2}\sum_{t=1}^{n-1} Z_t Z_{t+1},\ldots,n^{-1/2}\sum_{t=1}^{n-m} Z_t Z_{t+m}\right) \to_d (Y_0, Y_1,\ldots,Y_m)$$

for independent r.v.'s $Y_0, Y_1,\ldots,Y_m$ *and a slowly varying function* L_1. *Here* $Y_0 =_d S_p(1,1,0)$ *and* $Y_1,\ldots,Y_m$ *are i.i.d. with* $Y_1 =_d N(0,\sigma^4)$.

PROOF. Part (a) is standard and can be found in Brockwell and Davis [(1991), Chapter 7] for example. Part (b) can be proved by the point process techniques as developed in Davis and Resnick (1985, 1986). Here we prefer an elementary proof. We restrict ourselves to the case $m = 1$; the general case $m \geq 1$ can be handled analogously. Billingsley's CLT for mixing sequences [Billingsley (1968), Chapter 20] and the CLT for r.v.'s in the domain of attraction of a p-stable law G_p [e.g., Feller (1971) and Bingham, Goldie and Teugels (1987)] imply that

$$n^{-1/2}\sum_{t=1}^{n-1} Z_t Z_{t+1} \to_d N(0,\sigma^4), \qquad n^{-1/p} L_1(n)\sum_{t=1}^{n}(Z_t^2-\sigma^2) \to_d G_p. \tag{5.1}$$

Here L_1 is slowly varying. We show joint convergence of

$$\left(n^{-1/p} L_1(n)\sum_{t=1}^{n}(Z_t^2-\sigma^2),\, n^{-1/2}\sum_{t=1}^{n-1} Z_t Z_{t+1}\right)$$

via a characteristic function argument. Write $\varepsilon_t = \operatorname{sign} Z_t$. Then

$$\begin{aligned} &E\exp\left\{i\theta_1 n^{-1/p} L_1(n)\sum_{t=1}^{n}(Z_t^2-\sigma^2) + i\theta_2 n^{-1/2}\sum_{t=1}^{n-1} Z_t Z_{t+1}\right\} \\ &\quad = E\Bigg[\exp\left\{i\theta_1 n^{-1/p} L_1(n)\sum_{t=1}^{n}(Z_t^2-\sigma^2)\right\} \\ &\qquad \times E\left(\exp\left\{i\theta_2 n^{-1/2}\sum_{t=1}^{n-1}|Z_t Z_{t+1}|\varepsilon_t\varepsilon_{t+1}\right\}\Big|\, |Z_1|, |Z_2|, \ldots\right)\Bigg]. \end{aligned} \tag{5.2}$$

Notice that, given $(|Z_i|)$, the quadratic form in (ε_t),

$$A_n = \sum_{t=1}^{n-1}|Z_t Z_{t+1}|\varepsilon_t\varepsilon_{t+1} \Big/ \left(\sum_{t=1}^{n-1} Z_t^2 Z_{t+1}^2\right)^{1/2},$$

has mean zero and variance 1. From standard theory for random quadratic forms [e.g., Mikosch (1991)] it follows that the sequence (A_n) converges (conditionally) in distribution to the standard normal distribution provided

$$\sum_{t=1}^{n-1} Z_t^4 Z_{t+1}^4 \Big/ \left(\sum_{t=1}^{n-1} Z_t^2 Z_{t+1}^2\right)^2 \to 0$$

for almost all realizations of $(|Z_t|)$. This is easily seen since

$$\max_{t=1,\ldots,n-1} Z_t^2 Z_{t+1}^2 \Big/ \sum_{t=1}^{n-1} Z_t^2 Z_{t+1}^2 \to 0$$

for almost all realizations of $(|Z_t|)$. Thus we conclude that, for almost all realizations of $(|Z_t|)$,

$$E\left(\exp\left\{i\theta_2 n^{-1/2}\sum_{t=1}^{n-1}|Z_t Z_{t+1}|\varepsilon_t\varepsilon_{t+1}\right\}\Big|\, |Z_1|, |Z_2|, \ldots\right) \to \exp(-\theta_2^2\sigma^4/2).$$

This and relations (5.1) and (5.2) yield the statement. □

Proposition 5.1 can be extended to a FCLT.

PROPOSITION 5.2. *Under the corresponding assumptions of Theorem* 3.1 *the following limit relations hold for every fixed* $m \geq 1$:

$$\text{(a)}\quad \mathbf{Z}_n^{(A)} = \left(n^{-1/2}\sum_{t=1}^{[nx]}(Z_t^2-\sigma^2),\, n^{-1/2}\sum_{t=1}^{[nx]-1} Z_tZ_{t+1}, \dots,\, n^{-1/2}\sum_{t=1}^{[nx]-m} Z_tZ_{t+m}\right)$$
$$\to_d \big(Y_0(x), Y_1(x), \dots, Y_m(x)\big), \qquad x \in [0,1],$$

in $\mathscr{D}([0,1], \mathbb{R}^{m+1})$, *where* $Y_0(x), Y_1(x), \dots, Y_m(x)$ *are independent Wiener processes such that* $Y_i(0) \equiv 0$ *for all* i, $Y_0(1) =_d N(0, \operatorname{var}(Z^2))$ *and* $Y_i(1) =_d N(0, \sigma^4)$, $i = 1, \dots, m$.

$$\text{(b)}\quad \mathbf{Z}_n^{(B)} = \Bigg(n^{-1/p}L_1(n)\sum_{t=1}^{[nx]}(Z_t^2-\sigma^2),$$
$$n^{-1/2}\sum_{t=1}^{[nx]-1} Z_tZ_{t+1}, \dots,\, n^{-1/2}\sum_{t=1}^{[nx]-m} Z_tZ_{t+m}\Bigg)$$
$$\to_d \big(Y_0(x), Y_1(x), \dots, Y_m(x)\big), \qquad x \in [0,1],$$

in $\mathscr{D}([0,1], \mathbb{R}^{m+1})$, *where* $Y_0(x), Y_1(x), \dots, Y_m(x)$ *are independent processes,* $Y_0(x)$ *is a* p-*stable motion such that* $Y_0(0) \equiv 0$, $Y_0(1) =_d S_p(1,1,0)$ *and* $Y_1(x), \dots, Y_m(x)$ *are i.i.d. Wiener processes such that* $Y_1(0) \equiv 0$, $Y_1(1) =_d N(0, \sigma^4)$.

PROOF. (a) We have to show the convergence of the finite-dimensional distributions and the tightness in $\mathscr{D}([0,1], \mathbb{R}^{m+1})$ for $(\mathbf{Z}_n^{(A)})$.

The convergence of the finite-dimensional distributions follows from Proposition 5.1(a) by the Cramér–Wold device and the fact that, for every fixed $k \geq 1$, $(\sum_{t=1}^{n-k} Z_tZ_{t+k})_{n>k}$ is a sum process with k-dependent stationary increments and $\sum_{t=1}^{n} Z_t^2$ is a sum of i.i.d. r.v.'s.

For the tightness in $\mathscr{D}([0,1], \mathbb{R}^{m+1})$ we use the fact that each component of $(\mathbf{Z}_n^{(A)})$ converges weakly in $\mathscr{D}[0,1]$ to a Wiener process. This follows from Billingsley's FCLT's for mixing sequences [see Billingsley (1968), Chapter 20]. A straightforward generalisation of Lemma 4.4 in Resnick (1986) yields that the map from $(\mathscr{D}[0,1])^{m+1}$ into $\mathscr{D}([0,1], \mathbb{R}^{m+1})$ defined by

$$(x_0, \dots, x_m) \to (x_0(t), \dots, x_m(t))_{t\geq 0}$$

is continuous at $(x_0, \dots, x_m) \in \mathscr{D}[0,1] \times (\mathscr{C}[0,1])^m$. This and the sample path continuity of the limit processes assure that $(\mathbf{Z}_n^{(A)})$ is tight and converges weakly to the given limit.

(b) The convergence of the finite-dimensional distributions can be shown in the same way, by an application of Proposition 5.1(b).

To show tightness we have to modify our arguments slightly. Resnick's (1986) approach still works in this case if we can show that the components of

$(\mathbf{Z}_n^{(B)})$ are tight in $\mathscr{D}[0, 1]$. Applying a FCLT for processes with independent increments [e.g., Jacod and Shiryaev (1987) or Resnick (1986), Proposition 3.4] it follows that for $x \in [0, 1]$, $n^{-1/p} \sum_{t=1}^{[nx]} (Z_t^2 - \sigma^2)$ converges weakly to a p-stable process $Y_0(x)$ with independent stationary increments. To the remaining components of $(\mathbf{Z}_n^{(B)})$ we again apply Billingsley's FCLT for mixing sequences with a Wiener process as limit. □

PROOF OF THEOREM 3.1. (a) We write

$$\left(n^{-1/2} \sum_{t=1}^{[nx]} (Z_t^2 - \sigma^2),\ n^{1/2} \int_{-\pi}^{\lambda} \left(I_{n,[nx],Z}(y) - n^{-1} \sum_{t=1}^{[nx]} Z_t^2\right) dy\right)$$
$$= \left(n^{-1/2} \sum_{t=1}^{[nx]} (Z_t^2 - \sigma^2),\ 2\left(\sum_{t=1}^{[nx]} \frac{\sin(\lambda t)}{t}\left(n^{-1/2} \sum_{h=1}^{[nx]-t} Z_h Z_{h+t}\right)\right)\right).$$

For every fixed m, we conclude from Proposition 5.2(a) and from the continuous mapping theorem that

$$\left(n^{-1/2} \sum_{t=1}^{[nx]} (Z_t^2 - \sigma^2),\ 2 \sum_{t=1}^{m} \frac{\sin(\lambda t)}{t}\left(n^{-1/2} \sum_{h=1}^{[nx]-t} Z_h Z_{h+t}\right)\right)$$
$$\to_d \left(Y_0(x), 2 \sum_{t=1}^{m} \left(Z_t(x) \frac{\sin(\lambda t)}{t}\right)\right), \qquad x \in [0, 1],\ \lambda \in [-\pi, \pi],$$

in $\mathscr{D}([0, 1] \times [-\pi, \pi], \mathbb{R}^2)$. Thus it suffices to show that for all $\varepsilon > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left(\sup_{0\le x\le 1} \sup_{-\pi\le\lambda\le\pi} \left| \sum_{t=m+1}^{[nx]} \left(\frac{\sin(\lambda t)}{t} n^{-1/2} \sum_{h=1}^{[nx]-t} Z_h Z_{h+t}\right)\right| > \varepsilon\right) = 0; \tag{5.3}$$

see Billingsley [(1968), Theorem 4.2]. If Z has a finite fourth moment, we first apply a submartingale inequality in order to obtain that

$$P\left(\sup_{0\le x\le 1} \sup_{-\pi\le\lambda\le\pi} \left| \sum_{t=m+1}^{[nx]} \left(\frac{\sin(\lambda t)}{t} n^{-1/2} \sum_{h=1}^{[nx]-t} Z_h Z_{h+t}\right)\right| > \varepsilon\right)$$
$$\le \varepsilon^{-2}\, c\, E\left(\sup_{-\pi\le\lambda\le\pi} \left| \sum_{t=m+1}^{n} \frac{\sin(\lambda t)}{t} n^{-1/2} \sum_{h=1}^{n-t} Z_h Z_{h+t}\right|^2\right).$$

Then we follow the proof of Theorem 1 in Grenander and Rosenblatt [(1957), Chapter 6.4, pages 188–189]. We mention that Grenander and Rosenblatt refer to their Lemma 1 where an eighth moment of Z is required. A careful study of this proof and of the one on their page 189 shows that one only needs a finite fourth moment.

(b) We can apply the same arguments as under (a), but we still have to show (5.3) without a fourth moment condition. If Z has a symmetric distribution one can use the analogue of Lévy's inequality for quadratic forms [see

Theorem 6.2.1 in Kwapień and Woyczynski (1992)] to obtain that

$$P\left(\sup_{0\le x\le 1}\ \sup_{-\pi\le\lambda\le\pi}\left|\sum_{t=m+1}^{[nx]}\left(\frac{\sin(\lambda t)}{t}n^{-1/2}\sum_{h=1}^{[nx]-t}Z_hZ_{h+t}\right)\right|>\varepsilon\right)$$
$$\le cP\left(\sup_{-\pi\le\lambda\le\pi}\left|\sum_{t=m+1}^{n}\left(\frac{\sin(\lambda t)}{t}n^{-1/2}\sum_{h=1}^{n-t}Z_hZ_{h+t}\right)\right|>\varepsilon\right).$$

Now one can follow the lines of the proof of Theorem 3.2 in Klüppelberg and Mikosch (1996). Instead of the tail estimate for stable quadratic forms [Theorem 3.1 in Rosinski and Woyczynski (1987)], one applies Chebyshev's inequality; then all the inequalities remain valid with $\mu = 2$ (the parameter μ is defined in the paper mentioned). □

PROOF OF PROPOSITION 3.3. A careful study of the proof of Theorem 3.1 shows that the conditions $EZ^4 < \infty$ [part (a)] and regular variation for $P(Z^2 > x)$ [part (b)] are only needed to ensure the weak convergence of the properly normalised sequence $(\sum_{t=1}^n (Z_t^2 - \sigma^2))$. On the other hand, in Proposition 3.3 we are interested in the convergence of

$$n^{1/2}\int_{-\pi}^{\lambda}\left(I_{n,[nx],Z}(y) - n^{-1}\sum_{t=1}^{[nx]}Z_t^2\right)dy.$$

The latter quantity does not contain any terms with (Z_t^2) and depends only on the sample autocorrelations $n^{-1}\sum_{t=1}^{n-h}Z_tZ_{t+h}$, a finite vector of which converges to a multivariate Gaussian limit with mean zero and σ^4 times the identity as covariance matrix. For this convergence result, a finite variance is sufficient. Having this in mind, we can follow the lines of proof above which yield the statement of the proposition. □

PROOF OF THEOREM 4.3. Lemma 4.1 and integration by parts yield the following decomposition:

$$\begin{aligned}
&\sqrt{n}\int_{-\pi}^{\lambda}\left(I_{n,[nx],X}(y) - \sigma^2|\psi(e^{-iy})|^2\frac{[nx]}{n}\right)f(y)\,dy\\
&\quad = g(\lambda)\sqrt{n}\int_{-\pi}^{\lambda}\left(I_{n,[nx],Z}(y) - \sigma^2\frac{[nx]}{n}\right)dy\\
&\qquad - \int_{-\pi}^{\lambda}g'(y)\sqrt{n}\left(\int_{-\pi}^{y}\left(I_{n,[nx],Z}(z) - \sigma^2\frac{[nx]}{n}\right)dz\right)dy\\
&\qquad + f(\lambda)n^{-1/2}\int_{-\pi}^{\lambda}R_{[nx]}(y)\,dy\\
&\qquad - \int_{-\pi}^{\lambda}f'(y)n^{-1/2}\left(\int_{-\pi}^{y}R_{[nx]}(z)\,dz\right)dy.
\end{aligned}$$

An application of Corollary 3.2 and of Lemma 4.2 together with the continuous mapping theorem prove that the limit of this two-parameter field is given by (4.2). This concludes the proof. □

PROOF OF THEOREM 4.4. From Lemma 4.1 we see that

$$\sqrt{n}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{I_{[nx],X}(z)}{|\psi(e^{-iz})|^2}\,dz = n^{-1/2}\sum_{t=1}^{[nx]} Z_t^2 + n^{-1/2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R_{[nx]}(z)}{|\psi(e^{-iz})|^2}\,dz\,.$$

Moreover, Lemma 4.2, integration by parts and the continuous mapping theorem imply that

$$\begin{aligned} n^{-1/2}\int_{-\pi}^{\pi}\frac{R_{[nx]}(z)}{|\psi(e^{-iz})|^2}\,dz &= n^{-1/2}|\psi(e^{-i\lambda})|^{-2}\int_{-\pi}^{\pi}R_{[nx]}(z)\,dz \\ &\quad + n^{-1/2}\int_{-\pi}^{\pi}\frac{d}{d\lambda}\big(|\psi(e^{-i\lambda})|^{-2}\big)\left(\int_{-\pi}^{\lambda}R_{[nx]}(z)\,dz\right)d\lambda \\ &= o_P(1). \end{aligned}$$

Now the proof is similar to the one of Theorem 4.3 with σ^2 everywhere replaced by the random centering with $n^{-1}\sum_{t=1}^{n} Z_t^2$. □

PROOF OF LEMMA 4.2. The proof is an immediate consequence of the decomposition in Lemma 4.1 and of Lemmas 5.3 and 5.5. For ease of representation we restrict ourselves to a one-sided linear process $X_t = \sum_{j=0}^{\infty}\psi_j Z_{t-j}$, $t \in \mathbb{Z}$. The two-sided case does not cause any additional difficulties. □

LEMMA 5.3. *The relation*

$$n^{-1/2}\int_{-\pi}^{\pi}|Y_{[nx]}(y)|^2\,dy = o_P(1)$$

holds uniformly for $x \in [0,1]$.

PROOF. It suffices to show that the following terms are of the order $o_P(n^{-1/2})$ uniformly for $x \in [0,1]$:

$$I_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{j>[nx]}\psi_j e^{-i\lambda j}\sum_{t=1-j}^{[nx]-j} Z_t e^{-i\lambda t}\right|^2 d\lambda,$$

$$I_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{j>[nx]}\psi_j e^{-i\lambda j}\left(\sum_{t=1}^{[nx]} Z_t e^{-i\lambda t} + \sum_{t=1-j}^{0} Z_t e^{-i\lambda t}\right)\right|^2 d\lambda,$$

$$I_3 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{j=1}^{[nx]}\psi_j e^{-i\lambda j}\sum_{t=[nx]-j+1}^{[nx]} Z_t e^{-i\lambda t}\right|^2 d\lambda.$$

We have by (2.1) and since $EZ^4 < \infty$,

$$I_3 \le c\max_{0\le k\le n} Z_k^2\left(\sum_{j=1}^{[nx]}|\psi_j|\,j\right)^2 \le c\max_{0\le k\le n} Z_k^2 = o_p(\sqrt{n}).$$

A similar argument applies to I_2 and

$$I_{11} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{t=-[nx]+1}^{-1} Z_t \sum_{j=[nx]+1}^{[nx]-t} \psi_j \exp(-i\lambda(j+t))\right|^2 d\lambda$$

can be treated analogously. The lemma is proved for I_1 if we can show stochastic boundedness for

$$\begin{aligned} I_{12} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{t=-\infty}^{-[nx]} Z_t \sum_{j=1-t}^{[nx]-t} \psi_j \exp(-i\lambda(t+j))\right|^2 d\lambda \\ &= \sum_{t_1,\,t_2=-\infty}^{-[nx]} Z_{t_1} Z_{t_2} \sum_{j_1=1-t_1}^{[nx]-t_1} \sum_{\substack{j_2=1-t_2 \\ j_1+t_1=j_2+t_2}}^{[nx]-t_2} \psi_{j_1}\psi_{j_2}. \end{aligned}$$

Notice that

$$I_{12} \le \sum_{t=-\infty}^{0} Z_t^2 \sum_{j=1-t}^{\infty} \psi_j^2 + \sum_{-\infty<t_1\neq t_2\le 0}\left(|Z_{t_1}Z_{t_2}| \sum_{j=1}^{n} |\psi_{j-t_1}\psi_{j-t_2}|\right). \tag{5.4}$$

The series on the right-hand side of (5.4) does not depend on x and converges a.s. This concludes the proof. □

We will make use of the following lemma due to Bickel and Wichura (1971): let f_n be random elements assuming values in $\mathscr{D}([0,1]\times[-\pi,\pi])$. For a block $B = (x_1, x_2] \times (\lambda_1, \lambda_2] \subset [0,1]\times[-\pi,\pi]$ we define

$$f_n(B) = f_n(x_1, x_2) - f_n(x_1, \lambda_2) - f_n(x_2, \lambda_1) + f_n(\lambda_1, \lambda_2).$$

LEMMA 5.4. *Suppose that $f_n(0,\lambda) = f_n(x,0) = 0$ for all $0 < x < 1$, $-\pi < \lambda < \pi$, and that (μ_n) is a sequence of finite measures on $[0,1]\times[-\pi,\pi]$ converging weakly to a measure μ with continuous marginal distributions. If the relation*

$$E(\min(f_n(B), f_n(C)))^a \le c\,(\mu_n(B\cup C))^b, \qquad n \ge 1,$$

holds for some $a > 0$, $b > 1$ and for all disjoint blocks B and C in $[0,1]\times[-\pi,\pi]$ which have one edge in common, then (f_n) is tight in $\mathscr{D}([0,1]\times[-\pi,\pi])$.

LEMMA 5.5. *The relation*

$$n^{-1/2}\int_{-\pi}^{\lambda} J_{[nx]}(-y) Y_{[nx]}(y)\, dy = o_P(1)$$

holds uniformly for $x \in [0,1]$ and $\lambda \in [-\pi,\pi]$.

PROOF. By Lemma 5.3 it suffices to prove

$$
\begin{aligned}
& n^{-1/2} \int_{-\pi}^{\lambda} \Bigl(J_{[nx]}(-y) \sum_{s=1}^{[nx]} Z_s \sum_{j=[nx]-s+1}^{[nx]} \psi_j \exp(-iy(j+s)) \Bigr)\, dy \\
(5.5) \qquad & = n^{-1/2} \sum_{t=1}^{[nx]} \sum_{s=1}^{[nx]} Z_t Z_s \sum_{j=[nx]-s+1}^{[nx]} \psi_j \int_{-\pi}^{\lambda} \exp(-iy(-t+j+s))\, dy \\
& = o_P(1)
\end{aligned}
$$

uniformly for x and λ. Notice that, since $EZ^4 < \infty$,

$$
n^{-1/2} \Bigl| \sum_{t=1}^{[nx]} Z_t^2 \sum_{j=[nx]-t+1}^{[nx]} \psi_j \int_{-\pi}^{\lambda} e^{-iyj}\, dy \Bigr| \le c\, n^{-1/2} \max_{0 \le k \le n} Z_k^2 = o_P(1)
$$

uniformly for $x \in [0,1]$ and $\lambda \in [-\pi, \pi]$. Thus it suffices to prove the uniform convergence to zero for the right-hand side of (5.5) without the diagonal terms. We first show that the finite-dimensional distributions converge. This is immediate from the following calculations. Fix x and λ. Using one of the standard moment inequalities for random quadratic forms [e.g., Mikosch (1991), Lemma 1.3] and Hölder's inequality we obtain

$$
\begin{aligned}
& E\Bigl| n^{-1/2} \sum_{t=1}^{[nx]} \sum_{\substack{s=1 \\ s \ne t}}^{[nx]} Z_t Z_s \sum_{j=[nx]-s+1}^{[nx]} \psi_j \int_{-\pi}^{\lambda} \exp(-iy(t-j-s))\, dy \Bigr|^4 \\
& \le c n^{-2} \Bigl(\sum_{t=1}^{[nx]} \sum_{\substack{s=1 \\ s \ne t}}^{[nx]} \Bigl| \sum_{j=[nx]-s+1}^{[nx]} \psi_j \int_{-\pi}^{\lambda} \exp(-iy(t-j-s))\, dy \Bigr|^2 \Bigr)^2 \\
& \le c \Bigl(n^{-1} \sum_{t=1}^{[nx]} \sum_{\substack{s=1 \\ s \ne t}}^{[nx]} \Bigl(\sum_{j_1=[nx]-s+1}^{[nx]} \psi_{j_1}^2 \Bigr) \\
& \qquad \times \Bigl(\sum_{j_2=[nx]-s+1}^{[nx]} \Bigl| \int_{-\pi}^{\lambda} \exp(-iy(t-j_2-s))\, dy \Bigr|^2 \Bigr) \Bigr)^2 \\
& \le c \Bigl(n^{-1} \sum_{t=1}^{[nx]} \sum_{\substack{s=1 \\ s \ne t}}^{[nx]} \Bigl(\sum_{j_1=[nx]-s+1}^{[nx]} \psi_{j_1}^2 \Bigr) \Bigl(\sum_{j_2=[nx]-s+1}^{[nx]} \frac{1}{(t-(j_2+s))^2} \Bigr) \Bigr)^2 = o(1)
\end{aligned}
$$

uniformly for x and λ. Thus it remains to show the tightness via Lemma 5.4. Put

$$
A_n(x, \lambda) = n^{-1/2} \sum_{t=1}^{[nx]} \sum_{\substack{s=1 \\ s \ne t}}^{[nx]} Z_t Z_s \sum_{j=[nx]-s+1}^{[nx]} \psi_j \int_{-\pi}^{\lambda} \exp(-iy(t-j-s))\, dy.
$$

For $x \le x'$ and $\lambda \le \lambda'$ we introduce

$$\begin{aligned}
&A_n(x, x', \lambda, \lambda') \\
&\quad = A_n(x,\lambda) - A_n(x,\lambda') - A_n(x',\lambda) + A_n(x',\lambda') \\
&\quad = n^{-1/2} \sum_{t=1}^{[nx]} \sum_{s=1, s\ne t}^{[nx]} Z_s Z_t \Biggl(\sum_{j=[nx']-s+1}^{[nx']} \psi_j \int_\lambda^{\lambda'} \exp(-iy(t-j-s))\,dy \\
&\qquad\qquad - \sum_{j=[nx]-s+1}^{[nx]} \psi_j \int_\lambda^{\lambda'} \exp(-iy(t-j-s))\,dy \Biggr) \\
&\qquad + n^{-1/2} \sum_{\substack{1\le t\ne s\le [nx'] \\ t>[nx]\, or\, s>[nx]}} Z_t Z_s \sum_{j=[nx']-s+1}^{[nx']} \psi_j \int_\lambda^{\lambda'} \exp(-iy(t-j-s))\,dy \\
&\quad = A_1 + A_2.
\end{aligned}$$

Using a standard moment inequality for quadratic forms [e.g., Mikosch (1991), Lemma 1.3] and Hölder's inequality we see that

$$\begin{aligned}
EA_1^4 \le c \Biggl(n^{-1} \sum_{t=1}^{[nx]} \sum_{\substack{s=1\\ s\ne t}}^{[nx]} \Biggl| &\sum_{j=[nx']-s+1}^{[nx']} \psi_j \int_\lambda^{\lambda'} \exp(-iy(t-j-s))\,dy \\
&- \sum_{j=[nx]-s+1}^{[nx]} \psi_j \int_\lambda^{\lambda'} \exp(-iy(t-j-s))\,dy \Biggr|^2 \Biggr)^2 = J_1.
\end{aligned}$$

We split J_1 into different subsums and estimate those. Suppose first that $[nx'] - [nx] \ge [nx]$. Then

$$\begin{aligned}
J_1 &\le c\Biggl(n^{-1} \sum_{t=1}^{[nx]} \sum_{\substack{s=1\\ s\ne t}}^{[nx]} \Biggl| \sum_{j=[nx']-s+1}^{[nx']} \psi_j \int_\lambda^{\lambda'} \exp(-iy(t-j-s))\,dy \Biggr|^2 \Biggr)^2 \\
&\quad + c\Biggl(n^{-1} \sum_{t=1}^{[nx]} \sum_{\substack{s=1\\ s\ne t}}^{[nx]} \Biggl| \sum_{j=[nx]-s+1}^{[nx]} \psi_j \int_\lambda^{\lambda'} \exp(-iy(t-j-s))\,dy \Biggr|^2 \Biggr)^2 \\
&= J_{11} + J_{12}.
\end{aligned}$$

By Hölder's inequality,

$$J_{11} \le c\Biggl(n^{-1} \sum_{s=1}^{[nx]} \sum_{j_1=[nx']-s+1}^{[nx']} \psi_{j_1}^2 \sum_{j_2=[nx']-s+1}^{[nx']} \sum_{t=1}^{[nx]} \Biggl| \int_\lambda^{\lambda'} \exp(-iy(t-j_2-s))\,dy \Biggr|^2 \Biggr)^2.$$

Using the fact that

$$\sum_{r=-\infty}^{\infty} \Biggl| \int_\lambda^{\lambda'} \exp(-iyr)\,dy \Biggr|^2 \le c(\lambda' - \lambda),$$

we conclude

$$J_{11} \le c\left((\lambda' - \lambda)n^{-1} \sum_{s=1}^{[nx]} s \sum_{j_1=[nx']-s+1}^{[nx']} \psi_{j_1}^2\right)^2 \le c(\lambda' - \lambda)^2 (n^{-1}([nx'] - [nx]))^2.$$

Similarly we can estimate J_{12}:

$$J_{12} \le c(\lambda' - \lambda)^2 (n^{-1}([nx'] - [nx]))^2.$$

Similar calculations for $[nx'] - [nx] < [nx]$ yield the same upper estimates for EA_1^4, but also for EA_2^4. We omit details. In summary,

$$EA_n^4(x, x', \lambda, \lambda') \le c(\lambda' - \lambda)^2 (n^{-1}([nx'] - [nx]))^2$$

for all $n \ge 1$, $x \le x'$, $\lambda \le \lambda'$. Define

$$\mu_n((x, x'] \times (\lambda, \lambda']) = (\lambda' - \lambda)(n^{-1}([nx'] - [nx])).$$

The so-defined measure μ_n converges weakly to the measure μ defined by

$$\mu((x, x'] \times (\lambda, \lambda']) = (\lambda' - \lambda)(x' - x).$$

An application of Lemma 5.4 proves the tightness of the two-parameter process (5.5). This concludes the proof of the lemma. □

6. Some applications. Changepoint detection is an important question in many applied areas, for example, in meteorology, economics and engineering. One of the canonical assumptions in finance, for example, is that the price of a risky asset (e.g., exchange rate, interest rate, price of stock) can be modelled by geometric Brownian motion $G_t = \exp\{ct + \sigma B_t\}$, $t \ge 0$, where B is standard Brownian motion. The celebrated Black–Scholes pricing formula is based on the assumption of geometric Brownian motion. It is one of the backbones of modern portfolio theory. A vast amount of literature has recently appeared on the pricing of options, futures and other derivatives. We refer to Duffie (1992) and the literature cited therein for a mathematical treatment of the financial problems. Geometric Brownian motion is clearly a crude model for a price. However, because of its simplicity, it is widely applicable as a first approximation over a reasonable period of time. This can be several months or a shorter period of time for which geometric Brownian motion with constant volatility σ might be appropriate.

We consider the German stock index (DAX, closing values) over a period of 500 days (starting from July 1, 1988) and assume that this price corresponds to a geometric Brownian motion. Then the daily log returns $Z_t = \ln(G_t/G_{t-1})$ can be considered as realizations of i.i.d. $N(c, \sigma^2)$ r.v.'s. We work with centered data; that is, an estimated mean c has been subtracted. In Figure 1 the $(Z_t)_{t=1,\dots,500}$ are plotted. There is an obvious dramatic change of the model around day 330. In Figure 2 we plot the values

$$T_n = n^{1/2} \sup_{\substack{x\in[0,1]\\ \lambda\in[0,\pi]}} \left| \left(\int_{-\pi}^{\lambda} I_{n,[nx],Z}(y) - n^{-1} \sum_{t=1}^{[nx]} Z_t^2 \right) dy \right| \Big/ (2\sigma^2)$$

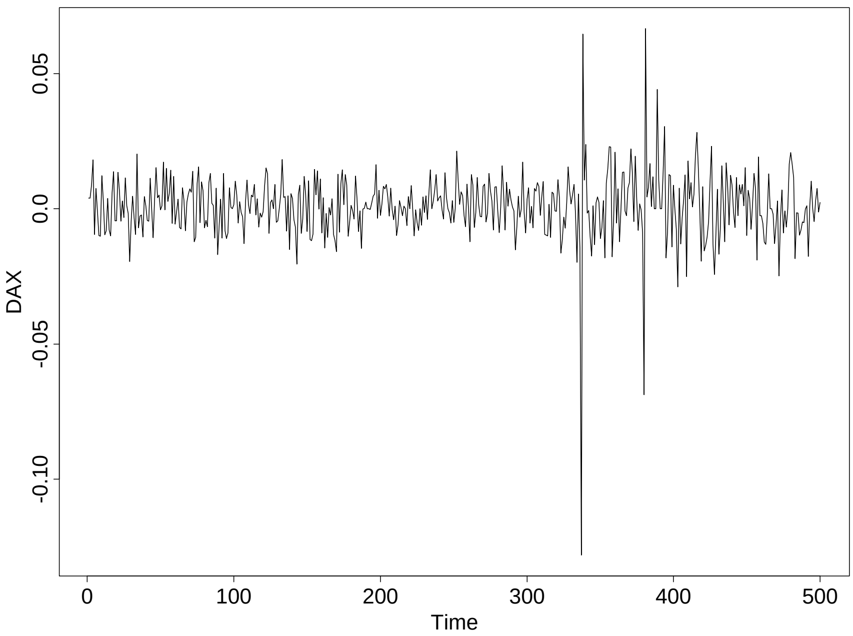

FIG. 1. *500 log returns of the DAX.*

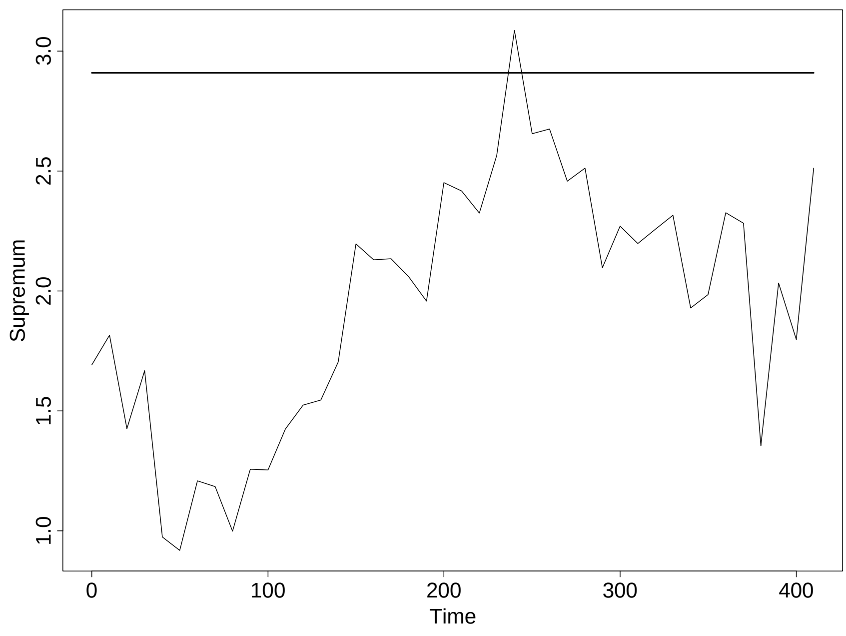

FIG. 2. *The changepoint statistic T_n with 95% confidence band for moving blocks of 90 days of the DAX index.*

calculated from the moving blocks $(Z_{l+1}, \ldots, Z_{l+n})$ for $n = 90$ and $l = 0, 10, 20, \ldots, 410$. The variance σ^2 is estimated for each block separately. Around day 240 we observe that the values of T_{90} are out of the asymptotic 95% confidence band (see Table 1) so that a change of the model in the last 10 days of the interval (241, 330) is very likely. We also see that around day 240 the values of T_{90} are quite high; that is, a change of the model happens in each moving block of length 90 with high probability. This fact makes the i.i.d. assumption on the log returns quite doubtful.

In a second example we consider 810 daily log returns (X_t) of the Japanese stock index (NIKKEI, closing data) starting from February 22, 1990. In a preliminary check we observed that in the interval $(1, 550)$ the sample autocorrelations (over moving blocks of lengths 200 and larger) do not change significantly. The sample autocorrelations over moving blocks in (550, 810) are not significantly different from zero. From the first 400 data we obtain the ARMA(1,2) model $X_t - 0.32X_{t-1} = Z_t - 0.26Z_{t-1} - 0.2Z_{t-2}$ via maximum likelihood estimation. Under the assumption that this model is correct we plot in Figure 3 the quantities

$$S_n = n^{1/2} \sup_{\substack{\lambda\in[-\pi,\,\pi]\\ x\in[0,\,1]}} \left| \int_{-\pi}^{\lambda} \left(\frac{I_{n,\,[nx],\,X}(y)}{|\psi(e^{-iy})|^2} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{I_{n,\,[nx],\,X}(z)}{|\psi(e^{-iz})|^2}\, dz \right) dy \right| \Big/ (2\sigma^2)$$

calculated from moving blocks $(X_{l+1}, \ldots, X_{l+n})$ of length $n = 100$ and $l = 0$, 10, $20, \ldots, 700$. The variance σ^2 is estimated for each block separately. We observe that the ARMA(1,2)–model is accepted in the first 570 days. Then the quantities S_{100} increase, indicating that there is a change in the model after day 570. The 95% confidence band is constructed from Table 1. We observe

TABLE 1
Quantiles of the supremum functional of $K(x, \lambda)/(2\sigma^2)$ on $[0, 1] \times [0, \pi]$

Quantile	T	**Quantile**	T
0.05	1.296149	0.75	2.267647
0.10	1.387182	0.80	2.383965
0.15	1.462891	0.85	2.484999
0.20	1.538464	0.90	2.660475
0.25	1.600525	0.91	2.720025
0.30	1.656383	0.92	2.773808
0.35	1.719021	0.93	2.814511
0.40	1.782333	0.94	2.862216
0.45	1.838113	0.95	2.909492
0.50	1.895355	0.96	3.039315
0.55	1.948501	0.97	3.094397
0.60	2.022626	0.98	3.246785
0.65	2.099422	0.99	3.477924
0.70	2.175465		

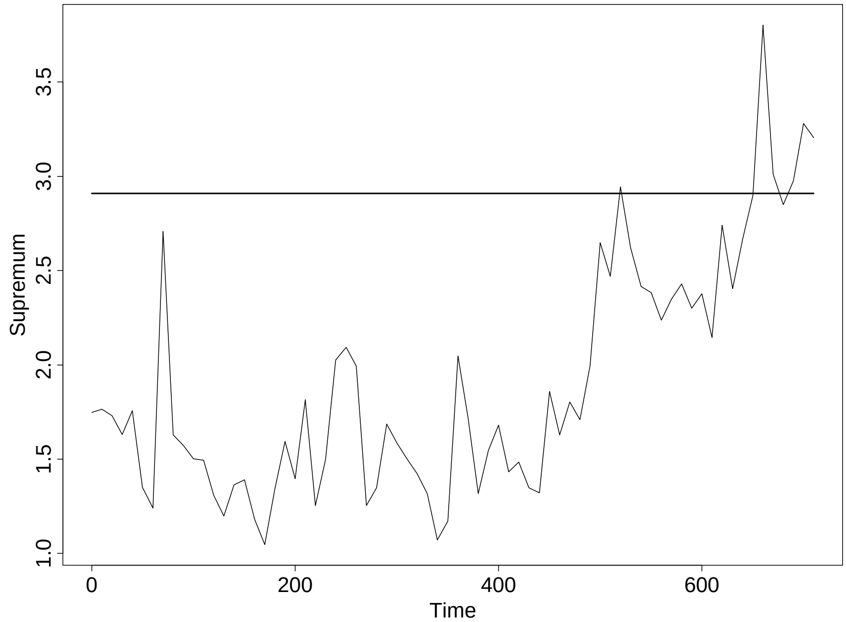

FIG. 3. *The changepoint statistic S_n with 95% confidence band for moving blocks of 100 days of the NIKKEI index.*

that the changepoint statistic S_n reacts quite sensitively to the change of the model. Indeed, the coefficients in the ARMA model are rather small: the dependence in the sequence is weak. Nevertheless, the change to white noise around day 570 is well illustrated.

An advantage of the statistics T_n and S_n is that they deliver reasonable results already for a medium sample size $n \approx 100$. The asymptotic quantiles of T_n were derived from Proposition 3.3

$$T_n \to_d T = \sup_{\substack{x\in[0,1]\\ \lambda\in[0,\pi]}} \frac{|K(x,\lambda)|}{2\sigma^2} =_d \sup_{\substack{x\in[0,1]\\ \lambda\in[0,\pi]}} \left|\sum_{t=1}^{\infty} \frac{\sin(\lambda t)}{t} Y_t(x)\right|,$$

where (Y_t) are i.i.d standard Brownian motions on $[0, 1]$. In view of Theorem 4.4, we may also conclude that $S_n \to_d T$ since in that case $g(x) \equiv 1$. The quantiles of T were obtained by simulation from the series representation (1.3) of the Kiefer process $K(x, \lambda)$. They are given in Table 1. We are not aware of such a table in the literature. Representation (1.3) offers a simple way to calculate these and other quantities related to the Kiefer process.

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