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COMPUTABLE EXPONENTIAL CONVERGENCE RATES FOR STOCHASTICALLY ORDERED MARKOV PROCESSES[1]

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Let $\{\Phi_t,\ t \geq 0\}$ be a Markov process on the state space $[0, \infty)$ that is stochastically ordered in its initial state. Examples of such processes include server workloads in queues, birth-and-death processes, storage and insurance risk processes and reflected diffusions. We consider the existence of a limiting probability measure π and an exponential "convergence rate" $\alpha > 0$ such that

$$\lim_{t \to \infty} e^{\alpha t} \sup_A \left| P_x[\Phi_t \in A] - \pi(A) \right| = 0$$

for every initial state $\Phi_0 \equiv x$.

The goal of this paper is to identify the largest exponential convergence rate α, or at least to find computationally reasonable bounds for such a "best" α. Coupling techniques are used to derive such results in terms of (i) the moment-generating function of the first passage time into state $\{0\}$ and (ii) solutions to drift inequalities involving the generator of the process. The results give explicit bounds for total variation convergence of the process; convergence rates for $E_x[f(\Phi_t)]$ to $\int f(y)\pi(dy)$ for an unbounded function f are also found. We prove that frequently the bounds obtained are the best possible. Applications are given to dam models and queues where first passage time distributions are tractable, and to one-dimensional reflected diffusions where the generator is the more appropriate tool. An extension of the results to a multivariate setting and an analysis of a tandem queue are also included.

1. Introduction. Suppose that $\{\Phi_t,\ t \geq 0\}$ is a time-homogeneous strong Markov process on the probability space $(\Omega, \mathscr{F}, P)$ that takes values in the state space $\mathbb{X} = [0, \infty)$. For regularity, we assume that the sample paths of $\{\Phi_t\}$ are right continuous, have left-hand limits and are nonexplosive. Frequently, as discussed in Down, Meyn and Tweedie (1995) and Meyn and Tweedie (1993b), it is known that an invariant distribution π exists and that Φ_t converges to π exponentially fast in the sense of total variation; that is,

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there is an $\alpha > 0$ such that

$$\lim_{t \to \infty} e^{\alpha t} \sup_{A \in \mathscr{B}(\mathbb{X})} |P_x[\Phi_t \in A] - \pi(A)| = 0 \tag{1.1}$$

for every $x \in \mathbb{X}$, where $\mathscr{B}(\mathbb{X})$ is the σ-algebra of Borel sets on $\mathbb{X}$ and the notation P_x indicates the initial condition $\Phi_0 \equiv x$. If $\alpha > 0$ satisfies (1.1), we call it an exponential "rate of convergence."

Our objective in this paper is to find values of α that satisfy (1.1) and, if possible, to identify the largest such α, for stochastically ordered (also called stochastically monotone) Markov processes. We will show that, for many Markov processes, the largest possible α in (1.1) is the radius of convergence of the moment-generating function of the first passage time of the chain into state {0}, and that this radius of convergence can frequently be bounded using "drift inequalities" based on the generator of the process if it is not computable explicitly. Hence, this paper extends to continuous time the chain results in Lund and Tweedie (1995).

We say that the random variable X_1 is stochastically larger than the random variable X_2 if $P[X_1 \le x] \le P[X_2 \le x]$ for all real x. Our primary assumption is that Φ_t is stochastically ordered in its initial state; that is, if $\{\Phi_t\}$ and $\{\Phi_t'\}$ are two copies of the process with the possibly random initial values Φ_0 and Φ_0' respectively, then Φ_t is stochastically larger than Φ_t' for all $t > 0$ whenever Φ_0 is stochastically larger than Φ_0'.

Many Markov processes are stochastically ordered in their initial state. For one example, we cite the server workload in an $M/G/1$ queue [Stoyan (1983)] where a higher initial workload produces a higher workload at all other times. Other examples of stochastically ordered Markov processes include birth-and-death processes [Van Doorn (1981)], storage processes [Brockwell, Resnick and Tweedie (1982)], insurance risk processes [Asmussen (1987), Meyn and Tweedie (1993b) and Prabhu (1980)] and reflected diffusions. Stochastic monotonicity has been seen to be crucial in the analysis of queueing networks, and single-class queueing networks and Petri nets are often monotone [Baccelli and Foss (1994), Meyn and Down (1994) and Shanthikumar and Yao (1989)].

Many of the above examples are pathwise ordered Markov processes; that is, a sample path of the process with a higher initial state is never below a sample path of the process with a lower initial state. For a general stochastically ordered Markov process, one can change the underlying probability space and construct a new process that is pathwise ordered and distributionally equivalent to the original process [see Kamae, Krengel and O'Brien (1977) for the arguments]. Without loss of generality, therefore, we henceforth assume that the process is pathwise ordered. Thus, if $\{\Phi_t(\omega)\}$ and $\{\Phi_t'(\omega)\}$ are two sample paths of the process for $\omega \in \Omega$ with $\Phi_0(\omega) \ge \Phi_0'(\omega)$, then $\Phi_t(\omega) \ge \Phi_t'(\omega)$ for all $t > 0$ as well.

2. Results. For notation, let $P^t(x, A) = P_x[\Phi_t \in A]$ for $t > 0$ and $A \in \mathscr{B}(\mathbb{X})$. The first passage (hitting) time into state {0} is defined as $\tau_0 = \inf\{t \ge 0:$

$\Phi_t = 0\}$. When $\Phi_0 = 0$, note that $\tau_0 = 0$. When $\Phi_0 \equiv x$, we denote the moment-generating function of τ_0 by

$$G_x(\alpha) = E_x[e^{\alpha\tau_0}];$$

for a general initial distribution $\mu(A) = P[\Phi_0 \in A]$, the notation $G_\mu(\alpha)$ and $E_\mu[\cdot]$ is employed.

The "taboo" probability of being in the set A at time t without passing through $\{0\}$ first is

$${}_0P^t(x, A) = P_x[\Phi_t \in A \cap \tau_0 \geq t]$$

for $x > 0$. We assume that $P^t(0, [\delta, \infty)) > 0$ for some $t > 0$ and $\delta > 0$. Thus, $\{0\}$ can be left and we further assume that, for each $y > x > 0$, the process can travel with positive probability from x to $[y, \infty)$ without passing through $\{0\}$; that is, there is a $t > 0$ such that

$$(2.1) \qquad {}_0P^t(x, [y, \infty)) > 0.$$

If (2.1) does not hold, then from the pathwise ordering of $\{\Phi_t\}$, one can show that the state space of the process can be reduced to a compact subset of $[0, \infty)$. In this case, most of our results can be easily modified (cf. Example 3.3).

Our first key convergence rate result is stated below and proven in Section 3.

THEOREM 2.1. *Suppose that* $\{\Phi_t\}$ *is a stochastically ordered Markov process satisfying* (2.1). *If* $G_x(\alpha) < \infty$ *for some* $\alpha > 0$ *and some* $x > 0$, *then there exists an invariant distribution* π *and a finite constant* M_x *such that*

$$(2.2) \qquad \sup_{A \in \mathscr{B}(\mathbb{X})} |P^t(x, A) - \pi(A)| \leq M_x e^{-\alpha t}$$

for every $x \geq 0$ *and* $t \geq 0$. *Furthermore,* $G_\pi(\alpha) < \infty$ *and* $M_x \leq G_x(\alpha) + G_\pi(\alpha)$.

Theorem 2.1 is a considerable improvement on known results for exponential convergence of general processes $\{\Phi_t\}$ [Thorisson (1983), Asmussen (1987), Meyn and Tweedie (1993b), Down, Meyn and Tweedie (1995) and Kalashnikov (1994)] which typically guarantee exponential convergence at some exponential rate $s > 0$ when $G_x(\alpha) < \infty$ for some $\alpha > 0$, but do not link the values of s and α.

Theorem 2.1 deals with the first passage time τ_0 rather than the process itself; frequently, as is illustrated in Sections 4 and 5, the probabilistic structure of τ_0 is readily available whereas the probabilistic structure of $\{\Phi_t\}$ is not.

Lemma 3.1 below shows that the radius of convergence of $G_x(\alpha)$ is the same for all $x > 0$; we denote this common radius of convergence by α^*. Thus, Theorem 2.1 shows that any $\alpha < \alpha^*$ satisfies (1.1) and (2.2). The proof of Theorem 2.1 will also show that (1.1) holds with $\alpha = \alpha^*$ when $G_x(\alpha^*) < \infty$ for some $x > 0$; however, we note that $G_x(\alpha^*)$ may not always be finite.

Our second key result, Theorem 2.2 below, shows that, essentially, one can obtain the same convergence rates for a stronger norm by examining the generator of $\{\Phi_t\}$. Theorem 2.2 also identifies a bound for the constant M_x. For this, we need the following concept of the *extended generator* of $\{\Phi_t\}$, which is a slightly restricted form of that in Davis (1993). Denote by $\mathscr{D}(\mathscr{A})$ the set of all functions $f\colon \mathbb{X} \to \mathbb{R}$ for which there is a measurable function $g\colon \mathbb{X} \to \mathbb{R}$ such that, for every $x \in \mathbb{X}$,

$$
\begin{aligned}
E_x[f(\Phi_t)] &= f(x) + E_x\left[\int_0^t g(\Phi_u)\,du\right], \\
&\int_0^t E_x[|g(\Phi_u)|]\,du < \infty.
\end{aligned} \tag{2.3}
$$

We write $\mathscr{A}f := g$ when (2.3) holds and call $\mathscr{A}$ the extended generator of $\{\Phi_t\}$. This defines an extension of the infinitesimal generator for Hunt processes. If the process is nonexplosive and (2.4) holds or $G_x(\alpha) < \infty$ for some $\alpha > 0$, then $\{\Phi_t\}$ is aperiodic, irreducible and positive Harris recurrent and hence has a unique invariant measure π. If V satisfies (2.4) below and $V(x) \to \infty$ as $x \to \infty$, then the process is automatically nonexplosive. If $\{\Phi_t\}$ is explosive, then (2.3) may be meaningless. We refer the reader to Meyn and Tweedie (1993a, 1993b) for general conditions for nonexplosivity based on the extended generator and for general discussions on the above issues.

Theorem 5.2 in Down, Meyn and Tweedie (1995) shows that if there exists a "drift function" $V\colon \mathbb{X} \to [1, \infty)$ and constants $c > 0$ and $b < \infty$ such that

$$
\mathscr{A}V(x) \le -cV(x) + b\mathbb{1}_{\{0\}}(x), \tag{2.4}
$$

then there exists some $\alpha > 0$ such that (2.2) holds. Theorem 2.2 below derives a much stronger result that links values of c in (2.4) to values of α in (2.2).

To examine the moment convergence of $\{\Phi_t\}$, we define the f-norm of a function $f\colon \mathbb{X} \to [0, \infty)$ as

$$
\|P^t(x, \cdot) - \pi\|_f = \sup_{g \le |f|} \left| E_x[g(\Phi_t)] - \pi(g) \right|, \tag{2.5}
$$

where, by stationarity,

$$
\pi(g) = \int_{[0,\infty)} g(x)\pi(dx) = E_\pi[g(\Phi_t)] \tag{2.6}
$$

for all $t \ge 0$.

THEOREM 2.2. *Suppose that* $\{\Phi_t\}$ *is a stochastically ordered Markov process satisfying* (2.1).

(i) *If* (2.4) *holds, then* $G_x(c) < \infty$ *for all* $x > 0$; *hence*, (2.2) *and* (1.1) *also hold for* $\alpha \le c$.

(ii) *If* $V(x)$ *satisfies* (2.4) *and is nondecreasing in* x, *then*

$$
\|P^t(x, \cdot) - \pi\|_V \le 2e^{-ct}\left[V(x)\left[1 - \mathbb{1}_{\{0\}}(x)\right] + b/c\right].
$$

Part (i) of Theorem 2.2 identifies total variation exponential convergence at rate c from (2.4); part (ii) establishes convergence of all sub V-moments up to the exponential rate c. In Section 7, we show how a multivariate version of Theorem 2.2 can be used to analyze a tandem queue.

We next address the sharpness of Theorems 2.1 and 2.2. As in the chain case [Lund and Tweedie (1995)], it is not true that α^* will be the best exponential convergence rate for all stochastically ordered processes; that is, it is not generally true that (2.2) fails for some $x \geq 0$ when $\alpha > \alpha^*$. This is because our convergence rates are derived from couplings at the "minimal" state $\{0\}$. It is possible for the process to "couple more quickly" in a state other than $\{0\}$; we illustrate this possibility in Example 3.3. Furthermore, there exist extreme cases such as storage models that never empty ($\tau_0 = \infty$ when $x > 0$), but where $\alpha > 0$ satisfying (2.2) exist [see Lund (1995)]. Hence, in general, our rates are only bounds.

However, many stochastically ordered Markov processes are ordered in an additional manner where α^* can indeed be shown to be the best possible exponential convergence rate. Suppose that $\pi_0 = \pi(\{0\}) > 0$. Now let $\{\Phi_t(\omega)\}$ and $\{\Phi'_t(\omega)\}$ denote sample paths of the process with $\Phi_0(\omega) = x$ and $\Phi'_0(\omega) = x'$ and suppose that $0 \leq x < x'$. Our stronger ordering supposes the existence of $\kappa > 0$ and $\Delta > 0$ such that

$$\Phi_t(\omega) < \Phi'_t(\omega), \tag{2.7}$$

whenever $t \leq \inf\{u > 0\colon \Phi_u(\omega) = 0\} + \Delta$ and $x' > x + \kappa$; that is, the ordering is strict until Δ units of time after the lower starting sample path first returns to $\{0\}$ whenever the sample paths begin at least κ units apart. The ordering in (2.7) is a variant of that used for chains in Lund and Tweedie (1995).

One example of a process satisfying (2.7) is the server workload in an $M/G/1$ queue; we elaborate on this in Section 4. Another class of stochastically ordered Markov processes satisfying (2.7) is the class of storage models considered in Brockwell, Resnick and Tweedie (1982). For a stochastically ordered Markov process satisfying (2.1) and (2.7), we prove the following result in Section 3.

THEOREM 2.3. *Suppose that* $\{\Phi_t\}$ *is a stochastically ordered Markov process satisfying* (2.1). *Suppose that* $\pi_0 > 0$ *and that* (2.7) *holds. Let* α^* *be the common (for all* x*) radius of convergence of* $G_x(\alpha)$. *Then, if* $\alpha > \alpha^*$,

$$\limsup_{t \to \infty} e^{\alpha t} \sup_{A \in \mathscr{B}(\mathbb{X})} \left| P^t(0, A) - \pi(A) \right| = \infty.$$

Theorem 2.3 shows that, when $x = 0$, $\pi_0 > 0$ and (2.7) holds, the process cannot converge at an exponential rate that is larger than α^*. Thus, when (2.7) holds, α^* is the best possible exponential convergence rate for a stochastically ordered Markov process. While Theorem 2.1 establishes exponential convergence at rate α^* when $G_x(\alpha^*) < \infty$, we have as yet been unable to prove "divergence" at rate α^* when $G_x(\alpha^*) = \infty$.

One can also obtain convergence rates for "unordered" Markov processes that are pathwise dominated by stochastically ordered processes. Suppose that $\{\Phi_t\}$ and $\{\tilde{\Phi}_t\}$ are Markov processes on $(\Omega, \mathscr{F}, P)$ with $\{\Phi_t\}$ pathwise dominated by the stochastically ordered process $\{\tilde{\Phi}_t\}$: $\Phi_t(\omega) \le \tilde{\Phi}_t(\omega)$ for all $t \ge 0$ and $\omega \in \Omega$ whenever $\Phi_0(\omega) \le \tilde{\Phi}_0(\omega)$. For notation, we use $\tilde{\pi}$ for the limiting distribution of $\{\tilde{\Phi}_t\}$ if it exists, $\tilde{\tau}_0 = \inf\{t \ge 0: \tilde{\Phi}_t = 0\}$ and $\tilde{G}_x(\alpha) = E_x[e^{\alpha \tilde{\tau}_0}]$. The following result, whose proof is identical to that in Lund and Tweedie (1995) for discrete-time chains, now follows.

THEOREM 2.4. *Suppose that $\{\Phi_t\}$ is a possibly nonstochastically ordered Markov process that is pathwise bounded by a stochastically ordered Markov process $\{\tilde{\Phi}_t\}$ that satisfies* (2.1). *Further suppose that $\{\Phi_t\}$ has the limiting distribution π and that $\tilde{G}_x(\alpha) < \infty$ for some $x > 0$. Then*

$$\sup_{A \in \mathscr{B}(\mathbb{X})} |P_x[\Phi_t \in A] - \pi(A)| \le \tilde{M}_x e^{-\alpha t}$$

for all $x \ge 0$ and $t > 0$ where $\tilde{M}_x \le \tilde{G}_x(\alpha) + \tilde{G}_{\tilde{\pi}}(\alpha) < \infty$.

3. Proofs. Our first lemma establishes three properties of stochastically ordered Markov processes. We say that $\{\Phi_t\}$ is stochastically increasing in t if $\Phi_{t+t'}$ is stochastically larger than Φ_t for $t, t' \ge 0$.

LEMMA 3.1. *Suppose that $\{\Phi_t\}$ is a stochastically ordered Markov process satisfying* (2.1). *Then*:

(i) *Φ_t is stochastically increasing in t when $x = 0$.*
(ii) *$G_x(\alpha) < \infty$ for some $x > 0$ if and only if $G_x(\alpha) < \infty$ for every $x > 0$.*
(iii) *$G_\pi(\alpha) < \infty$ if and only if $G_x(\alpha) < \infty$ for some $x > 0$.*

PROOF. Part (i) is well known [cf. Theorem 4.9.3 of Lindvall (1992)]. For (ii), the pathwise ordering of $\{\Phi_t\}$ shows that $G_x(\alpha)$ is nondecreasing in x for fixed α, so if $G_x(\alpha) < \infty$ for some $x > 0$, then $G_y(\alpha) \le G_x(\alpha) < \infty$ for all $y < x$. Let $y > x$ and choose $t > 0$ such that ${}_0P^t(x, [y, \infty)) > 0$. The inequality $G_x(\alpha) \ge {}_0P^t(x, [y, \infty)) e^{\alpha t} G_y(\alpha)$ now gives $G_y(\alpha) < \infty$ as well.

For (iii), use the pathwise ordering of $\{\Phi_t\}$ to get $G_\pi(\alpha) \ge G_x(\alpha)\pi([x, \infty))$ for all $x > 0$; hence, if $G_\pi(\alpha) < \infty$, then $G_x(\alpha) < \infty$ for some $x > 0$. For the other direction, we apply Theorem 5.1 of Down, Meyn and Tweedie (1995) to the function $V(x) = G_x(\alpha)$ for $x > 0$ with $V(0) = 1$ to establish the existence of a $\lambda < 1$, $d < \infty$ and $h > 0$ such that

$$\int_{[0,\infty)} V(y) P^h(x, dy) \le \lambda V(x) + d. \tag{3.1}$$

Since π is also invariant for the skeleton chain $\{\Phi_{nh}, n \ge 0\}$ for every fixed h, we apply Theorem 14.3.7 of Meyn and Tweedie (1993c) with $f(x) = (1 -$

$\lambda)V(x)$ and $s(x) \equiv d$ to obtain

$$G_\pi(\alpha) = \int_{[0,\infty)} V(x)\pi(dx) \le d[1-\lambda]^{-1} < \infty,$$

as required. □

Proof of Theorem 2.1. We follow Lund and Tweedie (1995) and couple a stationary trajectory of the process with a trajectory of the process that starts from the initial level x. Let $\{\Phi_t^1\}$ and $\{\Phi_t^2\}$ be two copies of the process with the initial conditions $\Phi_0^1 \equiv x$ and $\Phi_0^2 = X$, where X is a random variable on $(\Omega, \mathscr{F}, P)$ with the invariant distribution π. For a fixed $t > 0$, Φ_t^1 and Φ_t^2 may be statistically dependent; however, because π is invariant, $\{\Phi_t^2\}$ is stationary: $P_\pi[\Phi_t^2 \in A] = \pi(A)$ for all $A \in \mathscr{B}(\mathbb{X})$ and $t > 0$. Define $T = \inf\{t \ge 0: \Phi_t^1 = \Phi_t^2\}$ and use the coupling inequality [Lindvall (1992)] to get

$$\sup_{A \in \mathscr{B}(\mathbb{X})} \left|P^t(x,A) - \pi(A)\right| \le P_{x,\pi}[T > t]. \tag{3.2}$$

We remark that the strong Markov property and the right-continuous sample paths of $\{\Phi_t\}$ are needed for (3.2). The subscripts on P in (3.2) indicate the dependence of T on the initial distributions.

The crucial observation is that, because of the pathwise ordering of $\{\Phi_t\}$, once the process with the larger initial starting value has reached state $\{0\}$, the process with the smaller initial starting value must also be in state $\{0\}$. Thus, $P_{x,\pi}[T > t] \le P_\nu[\tau_0 > t]$ where $\nu(A) = P[\max(X, x) \in A]$ for $A \in \mathscr{B}(\mathbb{X})$. Hence, from (3.2) we get

$$\sup_{A \in \mathscr{B}(\mathbb{X})} \left|P^t(x,A) - \pi(A)\right| \le P_\nu[\tau_0 > t]. \tag{3.3}$$

Now use the Markov inequality to get

$$\begin{aligned} P_\nu[\tau_0 > t] &= P[X \le x]P_x[\tau_0 > t] + \int_{(x,\infty)} P_u[\tau_0 > t]\pi(du) \\ &\le P_x[\tau_0 > t] + P_\pi[\tau_0 > t] \\ &\le e^{-\alpha t}[G_x(\alpha) + G_\pi(\alpha)]. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) establishes (1.1) and (2.2) for any $\alpha < \alpha^*$. We note that $M_x = G_x(\alpha) + G_\pi(\alpha) < \infty$ follows from the assumption $G_x(\alpha) < \infty$ and part (iii) of Lemma 3.1. From (3.3), (3.4) and part (iii) of Lemma 3.1, we see that (2.2) is also valid for $\alpha = \alpha^*$ when $G_x(\alpha^*) < \infty$.

To establish (1.1) for $\alpha = \alpha^*$ when $G_x(\alpha^*) < \infty$, multiply (3.3) by $e^{\alpha^* t}$ and use the first inequality in (3.4) to get

$$e^{\alpha^* t} \sup_{A \in \mathscr{B}(\mathbb{X})} \left|P^t(x,A) - \pi(A)\right| \le e^{\alpha^* t}P_x[\tau_0 > t] + e^{\alpha^* t}P_\pi[\tau_0 > t]. \tag{3.5}$$

The finiteness of $G_x(\alpha^*)$ and $G_\pi(\alpha^*)$ yield

$$\lim_{t\to\infty} e^{\alpha^* t}P_x[\tau_0 > t] = \lim_{t\to\infty} e^{\alpha^* t}P_\pi[\tau_0 > t] = 0,$$

which establishes (1.1) for $\alpha = \alpha^*$ when used in (3.5). □

We now move to the proof of Theorem 2.2, which uses a similar but slightly more subtle coupling argument. The following lemma, which links the hitting times of $\{0\}$ to solutions of the generator drift inequality (2.4), is a special case of Theorem 6.1 of Down, Meyn and Tweedie (1995).

LEMMA 3.2. *Suppose that V is a drift function that satisfies* (2.4). *Then, for any $s \le c$ and $x > 0$,*

$$V(x) \ge G_x(s) + (c-s)E_x\left[\int_0^{\tau_0} e^{st}V(\Phi_t)\,dt\right].$$

The proof of Theorem 6.1 in Down, Meyn and Tweedie (1995) shows that Lemma 3.2 still holds if one weakens (2.4) to $\mathscr{A}V(x) \le -cV(x)$ for $x > 0$.

PROOF OF THEOREM 2.2. For (i), choose $s = c$ in Lemma 3.2 to get $G_x(c) \le V(x) < \infty$ as required.

For (ii), we need to refine the coupling argument used in the proof of total variation convergence in Theorem 2.1. Let us consider process copies $\{\Phi_t^1\}, \{\Phi_t^2\}$ starting from x and 0 and copies $\{\Phi_t^3\}, \{\Phi_t^4\}$ starting from X and 0 where X has distribution π. Define the state $\{0\}$ coupling times by $T_1 = \inf\{t \ge 0: \Phi_t^1 = \Phi_t^2 = 0\}$ and $T_2 = \inf\{t \ge 0: \Phi_t^3 = \Phi_t^4 = 0\}$ for the respective process pairs. Using (2.6), the triangle inequality and the same coupling arguments that produced (3.2)–(3.4), one obtains

$$\begin{aligned}
|E_x[g(\Phi_t)] - \pi(g)| &\le E_{x,0}\left[|g(\Phi_t^1)|\mathbb{1}_{[T_1>t]}\right] + E_{x,0}\left[|g(\Phi_t^2)|\mathbb{1}_{[T_1>t]}\right] \\
&\quad + E_{\pi,0}\left[|g(\Phi_t^3)|\mathbb{1}_{[T_2>t]}\right] + E_{\pi,0}\left[|g(\Phi_t^4)|\mathbb{1}_{[T_2>t]}\right] \\
&\le E_{x,0}\left[V(\Phi_t^1)\mathbb{1}_{[T_>t]}\right] + E_{x,0}\left[V(\Phi_t^2)\mathbb{1}_{[T_1>t]}\right] \\
&\quad + E_{\pi,0}\left[V(\Phi_t^3)\mathbb{1}_{[T_2>t]}\right] + E_{\pi,0}\left[V(\Phi_t^4)\mathbb{1}_{[T_2>t]}\right] \\
&\le 2E_x\left[V(\Phi_t)\mathbb{1}_{[\tau_0>t]}\right] + 2E_\pi\left[V(\Phi_t)\mathbb{1}_{[\tau_0>t]}\right]
\end{aligned} \tag{3.6}$$

for any function g such that $|g| \le V$. The last inequality in (3.6) follows from the monotonicity of V and the inequalities $\Phi_t^2 \le \Phi_t^1$ and $\Phi_t^4 \le \Phi_t^3$ for all $t \ge 0$.

Noting that the right-hand side of (3.6) is free of g, we take a supremum over $|g| \le V$ and use (2.5) to get

$$\|P^t(x,\cdot) - \pi\|_V \le 2E_x\left[V(\Phi_t)\mathbb{1}_{[\tau_0>t]}\right] + 2E_\pi\left[V(\Phi_t)\mathbb{1}_{[\tau_0>t]}\right]. \tag{3.7}$$

To bound $E_x[V(\Phi_t)\mathbb{1}_{[\tau_0>t]}]$, we note from Meyn and Tweedie (1993b) that $M_t = e^{ct}V(\Phi_t)\mathbb{1}_{[\tau_0>t]}$ is a supermartingale. To establish finiteness of $E_x[M_t] < \infty$ for all $t > 0$ and $x \ge 0$, note that, from the nondecreasing V, it is sufficient to show that $\pi(V) = E_\pi[V(\Phi_t)] < \infty$. Part (ii) of Theorem 4.3 in Meyn and Tweedie (1993b) with $f(x) = V(x)$ gives $\pi(V) \le b/c$ as required.

Now let $\{S_n\}$ be a sequence of stopping times with $S_n \to \infty$ as $n \to \infty$. Optional stopping gives

$$E_x\left[e^{c\min(t,S_n)}V(\Phi_{\min(t,S_n)})\mathbb{1}_{[\tau_0>\min(t,S_n)]}\right] \le E_x[M_0] = V(x)\left[1-\mathbb{1}_{\{0\}}(x)\right]$$

for each $t > 0$ and $x \ge 0$. Letting $n \to \infty$ and applying Fatou's lemma gives

$$E_x\left[V(\Phi_t)\mathbb{1}_{[\tau_0>t]}\right] \le e^{-ct}V(x)\left[1-\mathbb{1}_{\{0\}}(x)\right]. \tag{3.8}$$

Combining (3.7) with (3.8) and $\pi(V) \le b/c$ gives

$$\|P^t(x,\cdot)-\pi\|_V \le 2e^{-ct}\left[V(x)\left[1-\mathbb{1}_{\{0\}}(x)\right]+b/c\right] \tag{3.9}$$

and finishes the proof of (ii). □

PROOF OF THEOREM 2.3. This proof requires consideration of three copies of the process: $\{\Phi_t^1\}$, $\{\Phi_t^2\}$ and $\{\Phi_t^3\}$. For initial conditions, we take $\Phi_0^1 \equiv 0$, $\Phi_0^2 = X$, where X is random with distribution π, and $\Phi_0^3 \equiv \Delta$. Arguing with the coupling time $T = \inf\{t \ge 0: \Phi_t^1 = \Phi_t^2\}$, we have

$$\begin{aligned}\sup_{A\in\mathscr{B}(\mathbb{X})} &\left|P^t(0,A)-\pi(A)\right| \\ &\ge \left|P^t(0,\{0\})-\pi(\{0\})\right| \\ &= \left|P_{0,\pi,\Delta}\left[\Phi_t^1=0\cap T>t\right]-P_{0,\pi,\Delta}\left[\Phi_t^2=0\cap T>t\right]\right|;\end{aligned} \tag{3.10}$$

the three subscripts on P in (3.10) denote the three initial distributions. From the pathwise ordering of $\{\Phi_t\}$, if $\Phi_t^2 = 0$, then $\Phi_t^1 = 0$ and $T \le t$; thus, $P_{0,\pi,\Delta}[\Phi_t^2 = 0 \cap T > t] = 0$ and (3.10) gives

$$e^{\alpha t}\sup_{A\in\mathscr{B}(\mathbb{X})}\left|P^t(0,A)-\pi(A)\right| \ge e^{\alpha t}P_{0,\pi,\Delta}\left[\Phi_t^1=0\cap T>t\right]. \tag{3.11}$$

Now suppose that $\{\Phi_t^3\}$ first hits state $\{0\}$ sometime during $(t-\Delta, t]$ and returns to state $\{0\}$ at time t. Since $\Phi_t^1 \le \Phi_t^3$ for all $t \ge 0$, $\Phi_t^1 = 0$; furthermore, if $X > \kappa + \Delta$, then $T > t$ by the ordering assumption in (2.7). Thus,

$$\begin{aligned}&P_{0,\pi,\Delta}\left[\Phi_t^1=0\cap T>t\right] \\ &\qquad\ge P_{0,\pi,\Delta}\left[\tau_0^3\in(t-\Delta,t]\cap\Phi_t^3=0\cap X>\kappa+\Delta\right],\end{aligned} \tag{3.12}$$

where $\tau_0^3 = \inf\{t > 0: \Phi_t^3 = 0\}$. Investigating the right-hand side of (3.12), we first notice that the evolution of $\{\Phi_t^3\}$ in t is independent of X; hence,

$$\begin{aligned}&P_{0,\pi,\Delta}\left[\tau_0^3\in(t-\Delta,t]\cap\Phi_t^3=0\cap X>\kappa+\Delta\right] \\ &\qquad= P_{\Delta}\left[\tau_0^3\in(t-\Delta,t]\cap\Phi_t^3=0\right]\pi((\kappa+\Delta,\infty)).\end{aligned} \tag{3.13}$$

Since (2.1) holds, $\pi((x,\infty)) > 0$ for all $x \ge 0$; hence, $\pi((\kappa+\Delta,\infty)) > 0$. Part (i) of Lemma 3.1 shows that $P_0[\Phi_t = 0]$ decreases in t to π_0; using this with the strong Markov property proves

$$P_{\Delta}\left[\tau_0^3\in(t-\Delta,t]\cap\Phi_t^3=0\right] \ge \pi_0 P_{\Delta}\left[\tau_0^3\in(t-\Delta,t]\right]. \tag{3.14}$$

Combining (3.11)–(3.14) gives

$$e^{\alpha t} \sup_{A \in \mathscr{B}(\mathbb{X})} \left| P^t(0, A) - \pi(A) \right| \geq e^{\alpha t} \pi_0 \pi((\kappa + \Delta, \infty)) P_\Delta\left[\tau_0^3 \in (t - \Delta, t]\right];$$

hence, the theorem is proven if we show that

$$\limsup_{t \to \infty} e^{\alpha t} P_\Delta\left[\tau_0^3 \in (t - \Delta, t]\right] = \infty \tag{3.15}$$

when $\alpha > \alpha^*$.

To establish (3.15), we examine τ_0^3 rounded up to the nearest multiple of Δ. Set $\tilde{\tau}_0^3 = \Delta\lceil \tau_0/\Delta \rceil$, where $\lceil u \rceil$ denotes the smallest integer larger than or equal to u. First notice that α^* is also the radius of convergence of $E_x[e^{\alpha \tilde{\tau}_0^3}]$ for any $x > 0$. Equation (6.2) in Lund and Tweedie (1995) shows that

$$\limsup_{n \to \infty} e^{\alpha n} P_\Delta\left[\tilde{\tau}_0^3 = n\Delta\right] = \infty$$

when $\alpha > \alpha^*$; thus, $\limsup_{n \to \infty} e^{\alpha n} P_\Delta[(n-1)\Delta < \tau_0^3 \leq n\Delta] = \infty$ and (3.15) follows. □

The following example shows that α^* is not always the best exponential convergence rate for a stochastically ordered Markov process.

THEOREM 3.3. *Consider a finite capacity store* $\{\Phi_t\}$ *where inputs replenishing the store's content arrive according to a Poisson process with arrival rate* λ. *The store's capacity is K units and each arrival completely fills the store; input exceeding the store's capacity is discarded. The store releases content at a unit rate when nonempty.*

It is clear that $\{\Phi_t\}$ is a pathwise ordered strong Markov process with an invariant measure π. Notice that (2.1) holds for all $x, y \in (0, K]$ with $y > x$; hence, Lemma 3.1 and Theorem 2.1 remain valid. Let $\{\Phi_t^1\}$ and $\{\Phi_t^2\}$ be initial level x and stationary copies of this process, respectively. Define $T^* = \inf\{t \geq 0: \Phi_t^1 = \Phi_t^2 = K\}$ as the first time when both stores are filled and note that T^* has an exponential distribution with parameter λ regardless of the value of x. Since $\Phi_t^1 = \Phi_t^2$ for $t \geq T^*$, T^* is a coupling time of $\{\Phi_t^1\}$ and $\{\Phi_t^2\}$ and (2.2) holds for any $\alpha < \lambda$.

To compare this rate to the one obtained by coupling in $\{0\}$, we first note that $G_x(\alpha) < \infty$ for some $\alpha > 0$ by geometric trials. Now choose $x = K$ and condition on the first arrival time to get

$$G_K(\alpha) = \int_0^K e^{\alpha u} G_K(\alpha) \lambda e^{-\lambda u}\, du + \int_K^\infty e^{\alpha K} \lambda e^{-\lambda u}\, du,$$

which can be solved for $G_K(\alpha)$:

$$G_K(\alpha) = \frac{(\lambda - \alpha) e^{(\alpha - \lambda)K}}{\lambda e^{(\alpha - \lambda)K} - \alpha}. \tag{3.16}$$

Thus, we can find the radius of convergence of $G_K(\alpha)$, which may well be less than λ. For a specific comparison, take $K = 1$ and $\lambda = 2$. The numerator in (3.16) is positive whenever $\alpha < 2$; however, the denominator of (3.16) is 0 whenever α is a solution to $2e^{-2} = \alpha e^{-\alpha}$, the smallest of which is 0.41 to two decimal places. Thus, α^* is approximately 0.41 and $G_K(\alpha) = \infty$ when $\alpha > 0.41$. This rate is much smaller than the rate of 2^- obtained with the coupling at $\{K\}$. Finally, we note that (2.7) is clearly violated.

4. Application: dam processes. This section applies the results of Section 2 to the dam processes of Prabhu (1980). The results extend the $M/G/1$ server workload convergence rates of Cohen (1982) to dam processes.

Consider a dam process $\{\Phi_t\}$ driven by a time-homogeneous Lévy input process $\{A_t\}$. Water is released from the dam at a unit rate when present; that is, the release rule is $r(u) = \mathbb{1}_{(0,\infty)}(u)$. As noted in Brockwell, Resnick and Tweedie (1982) and Prabhu (1980), $\{\Phi_t\}$ is a pathwise ordered strong Markov process satisfying (2.1). When $\Phi_0 = x$, the sample paths of $\{\Phi_t\}$ satisfy the storage equation

$$\Phi_t = x + A_t - \int_0^t r(\Phi_u)\,du = x + A_t - t + \int_0^t \mathbb{1}_{\{0\}}(\Phi_u)\,du. \tag{4.1}$$

Now consider two sample paths of the process, say $\{\Phi_t\}$ and $\{\Phi'_t\}$, driven by the same sample path of $\{A_t\}$, but starting from the initial levels x and x', respectively. We take $x < x'$ and note that if $\Phi_u > 0$ for all $u \le t$, then $\Phi'_u > 0$ for all $u \le t$, and, by (4.1), $\Phi'_u - \Phi_u = x' - x$ for all $u \le t$. It now follows that (2.7) holds with $\kappa = \Delta = 1$.

The input process $\{A_t\}$ has stationary and independent increments. When finite, the moment-generating function of A_t has the form $E[e^{sA_t}] = e^{t\phi(s)}$, where

$$\phi(s) = \int_0^\infty (e^{su} - 1)\,\nu(du). \tag{4.2}$$

In (4.2), ν is a σ-finite measure supported on $(0, \infty)$ that satisfies $\int_0^\infty \min(u, 1)\nu(du) < \infty$. Define the "traffic intensity" by $\rho = E[A_1]$; it is well known [Prabhu (1980)] that a proper limiting measure π exists if and only if $\rho < 1$, in which case $\pi_0 = 1 - \rho > 0$. It is also known that (2.2) holds for some $\alpha > 0$ [Tuominen and Tweedie (1979) and Lund (1995)] when $E[e^{sA_1}]$ is finite for some $s > 0$. It follows from Theorems 2.1 and 2.3 that the best possible exponential rate of convergence is α^*, the radius of convergence of $G_x(\alpha) = E_x[e^{\alpha\tau_0}]$ for any $x > 0$.

Thus, we assume that $\rho < 1$ and that $E[e^{sA_1}]$ is finite for some $s > 0$. Arguing as in Prabhu (1980), or taking a martingale approach as in Rosenkrantz (1983) or Kella and Whitt (1992), one obtains $G_x(\alpha) = e^{x\eta(s)}$, where $\eta(s)$ is a solution to the functional equation $\eta(s) = s + \phi(\eta(s))$. Hence, α^* is the largest s where $\eta(s) < \infty$.

To identify this largest s, define $f(s) = s - \phi(s)$ and note that $\eta(s)$ solves $s = f(\eta(s))$. Observe that $|f(s)| < \infty$ whenever $E[e^{sA_1}] < \infty$. Other properties of f include: $f(0) = 0$, $f'(0) = 1 - \rho > 0$, $f''(s) \le 0$ for all $s > 0$ and $f(s) \to -\infty$ as $s \to \infty$. Hence, f is a continuous concave function and the supremum $\beta^* = \sup\{f(s)\colon s \ge 0\}$ is finite. For each $s \in (0, \beta^*)$, there are at most two positive values of $\eta(s)$ such that $f(\eta(s)) = s$, but one value of $\eta(s)$, from the concavity of f, decreases as s increases. Hence, for $s < \beta^*$, $\eta(s)$ is the unique "nondecreasing inverse" of f and is finite. Now let $s > \beta^*$ and suppose that $\eta(s) < \infty$. Then $f(x) \le \beta^* < s$ for every $x \ge 0$. Choosing $x = \eta(s)$ and applying $\eta(s) = s + \phi(\eta(s))$ produces the contradiction $s = f(\eta(s)) < s$; hence, $\eta(s) = \infty$ and $\alpha^* = \beta^*$.

Hence, we have proven the following result.

THEOREM 4.1. *Suppose that* $\{\Phi_t\}$ *is a dam process with the Lévy input process* $\{A_t\}$ *and the unit release rule* $r(u) = \mathbb{1}_{(0,\infty)}(u)$. *If* $E[A_1] < 1$ *and* $E[e^{sA_1}] < \infty$ *for some* $s > 0$, *then* (1.1) *and* (2.2) *hold for* $\alpha < \alpha^* = \sup\{s - \phi(s)\colon s > 0\}$ *and* (1.1) *and* (2.2) *fail when* $x = 0$ *and* $\alpha > \alpha^*$.

To attain exponential convergence at rate α^*, one need only check that $E[e^{sA_1}] < \infty$ for $s = \arg\sup[f(s)]$. We have been unable to establish whether this finiteness holds in generality.

Convergence rates for some nonunit release storage models can be obtained from a comparison and Theorem 2.4. Let $\{\Phi_t^*\}$ be a storage process with Lévy input $\{A_t\}$ and release rate $r^*(u)$ when the storage level is u [see Brockwell, Resnick and Tweedie (1982)]. Suppose that $E[A_1] < 1$ and that $r^*(u) > 1$ for all $u > 0$. In this case, Brockwell, Resnick and Tweedie (1982) show that an invariant measure π^* exists for $\{\Phi_t^*\}$ and that the ordering $\Phi_t^* \le \Phi_t$ for all $t \ge 0$ holds when the two processes are defined from the same sample path of $\{A_t\}$. Hence, by Theorem 2.4, (2.2) also holds for $\{\Phi_t^*\}$ when $\alpha < \sup\{s - \phi(s)\colon s > 0\}$.

Theorem 4.1 reproduces the explicit convergence rate for the server workload in the $M/M/1$ queue obtained by Morse (1958). Here, $\{A_t\}$ is a compound Poisson process

$$A_t = \sum_{i=1}^{N_t} Y_i,$$

where $\{N_t\}$ is a Poisson process with arrival rate λ and $\{Y_i\}$ is an i.i.d. sequence of random variables with density function $\mu e^{-\mu x}$ for $x > 0$. We assume that $\rho = \lambda/\mu < 1$ and notice that $E[e^{sY_1}] < \infty$ whenever $s < \mu$. Computations show that $f(s) = s - \lambda s[\mu - s]^{-1}$ for $s < u$ and that $\alpha^* = (\sqrt{\mu} - \sqrt{\lambda})^2$. In this case, exponential convergence at rate α^* is indeed achieved: $\arg\sup[f(s)] = \mu - \sqrt{\mu\lambda}$ and $E[e^{sA_1}] < \infty$ for all $s < \mu$. This rate has also appeared in a birth-and-death process setting [Van Doorn (1985) and Zeifman (1991)].

5. Application: periodic queues. In this section, we derive the best exponential convergence rate of the server workload process $\{\Phi_t\}$ in the periodic single-server queueing model of Afanas'eva (1985). Here, customers arrive at the queue according to a Poisson process with bounded intensity function $\lambda(\cdot)$. We take the period of the system to be 1: $\lambda(n+\gamma)=\lambda(\gamma)$ for each natural number n and $\gamma \in [0,1)$. If the ith customer arrives at the queue at time $n+\gamma$, then the workload induced to the server has the distribution function $H_\gamma(\cdot)$ which only depends on γ. The server works at a unit rate when work is available for processing.

Let A_t denote the total workload submitted to the server during $[0,t]$. When finite, the moment-generating function of A_t takes on a periodic form of (4.2):

$$E[e^{sA_t}] = \exp\left[\int_0^t \int_0^\infty (e^{su}-1) H_\gamma(du)\lambda(\gamma)\, d\gamma\right], \tag{5.1}$$

where $H_\gamma(\cdot)$ and $\lambda(\gamma)$ are extended periodically to all $\gamma \in [0,\infty)$. The mean workload added to the queue over one seasonal cycle is

$$\rho = E[A_1] = \int_0^1 \int_0^\infty x H_\gamma(dx)\lambda(\gamma)\, d\gamma;$$

Lund (1994) shows that $\{\Phi_t\}$ converges to a proper periodic limiting distribution, denoted by $\pi_\gamma(\cdot)$, $0 \le \gamma < 1$, uniformly in the season in the sense that

$$\lim_{n\to\infty} \sup_{\gamma\in[0,1)} \sup_B \left|\mathbb{P}_x[\Phi_{n+\gamma} \in B] - \pi_\gamma(B)\right| = 0$$

for all $x \ge 0$ if and only if $\rho < 1$.

To get a convergence rate for this model, we first note that $\{\Phi_t\}$ is a pathwise ordered process. When the distribution of Φ_0 is $\pi_0(\cdot)$, Lund (1994) shows that $\{\Phi_t\}$ is rendered periodically stationary in the sense that $\Phi_{n+\gamma}$ has distribution $\pi_\gamma(\cdot)$. Hence, coupling arguments with a periodically stationary version of the process can be used:

$$\sup_B \left|\mathbb{P}_x[\Phi_{n+\gamma} \in B] - \pi_\gamma(B)\right| \le P_{x,\pi_0}[T > n+\gamma] \le P_{x,\pi_0}[T > n], \tag{5.2}$$

where T is the state $\{0\}$ coupling time between an initial level x and a periodically stationary copy of the process. Taking a supremum in (5.2) over γ and multiplying by $e^{\alpha(n+\gamma)}$ gives

$$e^{\alpha(n+\gamma)} \sup_{\gamma\in[0,1)} \sup_B \left|\mathbb{P}_x[\Phi_{n+\gamma} \in B] - \pi_\gamma(B)\right| \le e^{\alpha(n+1)} P_{x,\pi_0}[T > n]. \tag{5.3}$$

From the Markov inequality and the arguments that led to (3.3) and (3.4), we see that the right-hand side of (5.3) tends to 0 as $n \to \infty$ if $E_{\pi_0}[e^{\alpha\tau_0}]$ is finite; hence, total variation convergence at the exponential rate α is achieved uniformly in the season γ whenever $E_{\pi_0}[e^{\alpha\tau_0}] < \infty$.

To identify values of α where $E_{\pi_0}[e^{\alpha\tau_0}] < \infty$, we make a comparison to a Lindley random walk. Set $\Phi_0^* = X_0$ where X_0 has distribution $\pi_0(\cdot)$ and

$$\Phi_n^* = \max[\Phi_{n-1}^* + A(n) - A(n-1) - 1, 0] \tag{5.4}$$

for $n \geq 1$. Sample path comparisons show that $\tau_0 = \inf\{t \geq 0: \Phi_t = 0\} \leq \tau_0^* = \inf\{n \geq 0: \Phi_n^* = 0\}$ for each $x \geq 0$. Hence, $E_{\pi_0}[e^{\alpha\tau_0}] < \infty$ whenever $E_{\pi_0}[e^{\alpha\tau_0^*}] < \infty$. Now use Example 7.3 of Lund and Tweedie (1995) to get $E_{\pi_0}[e^{\alpha\tau_0^*}] < \infty$ whenever $\alpha < -\ln(\psi^*)$, where $\psi^* = \inf\{e^{-\alpha}E[e^{\alpha A_1}]: \alpha > 0\}$. When $\rho < 1$ and $E[e^{sA_1}] < \infty$ for some $s > 0$, we note that $\psi^* < 1$.

To see that $-\ln(\psi^*)$ is the best possible exponential convergence rate, note that, as in Section 4, (2.7) holds with $\kappa = \Delta = 1$. Following the proof of Theorem 2.3 and specializing to $\gamma = 0$ gives

$$e^{\alpha n} \sup_B \left| P[\Phi_n \in B] - \pi_\gamma(B) \right| \geq M e^{\alpha n} P_1[\tau_0 \in (n-1, n]], \tag{5.5}$$

where $M = \pi_0((2, \infty)) \inf_{\gamma \in [0,1)} \pi_\gamma(\{0\})$. Here, we have used a seasonal strong Markov property along with the fact that $P_0[\Phi_{n+\gamma} = 0] \downarrow \pi_\gamma(\{0\})$ for every $\gamma \in [0, 1)$ [see Lund (1994)]. From the boundedness of $\lambda(\cdot)$, one can show that $\pi_0((2, \infty)) > 0$ and $\inf_{\gamma \in [0,1)} \pi_\gamma(\{0\}) > 0$; hence, $M > 0$ and (5.5) shows that exponential convergence does not happen if $\limsup_{n \to \infty} e^{\alpha n} P_1[\tau_0 \in (n-1, n]] > 0$.

Now observe that $P_1[\tau_0 \in (n-1, n]] \geq P_1[\tau_0^* = n]$ where $\{\Phi_n^*\}$ is governed by (5.4) except for $\Phi_0^* \equiv 1$. Hence, $\limsup_{n\to\infty} e^{\alpha n} P_1[\tau_0^* = n] = \infty$ whenever α exceeds the radius of convergence of $E_1[e^{s\tau_0^*}]$. But the convergence radii of $E_1[e^{s\tau_0^*}]$ and $E_0[e^{s\tau_0^{**}}]$ are identical where $\tau_0^{**} = \inf\{n > 0: \Phi_n^* = 0\}$ (note that τ_0^* and τ_0^{**} differ only when $x = 0$) and the latter is known to be $-\ln(\psi^*)$ [see Heathcote (1967)]. Hence, $-\ln(\psi^*)$ is indeed the best exponential convergence rate possible. For other results on the convergence rates of Lindley random walks, see Veraverbeke and Teugels (1975, 1976).

We summarize our work in the following theorem.

THEOREM 5.1. *Suppose that $\{\Phi_t\}$ is the server workload process in a periodic queue with input $\{A_t\}$ as described in* (5.1). *If $\rho < 1$, $\lambda(\gamma)$ is bounded over $\gamma \in [0, 1)$ and $E[e^{sA_1}] < \infty$ for some $s > 0$, then*

$$\lim_{n \to \infty} e^{\alpha n} \sup_{\gamma \in [0,1)} \sup_B \left| \mathbb{P}_x[\Phi_{n+\gamma} \in B] - \pi_\gamma(B) \right| = 0 \tag{5.6}$$

for all $x \geq 0$ and $\alpha < -\ln(\psi^*)$ where $\psi^* = \inf\{e^{-\alpha}E[e^{\alpha A_1}]: \alpha > 0\}$. Furthermore, (5.6) *fails for $x = 0$ when $\alpha > -\ln(\psi^*)$.*

6. Application: diffusion models on $[0, \infty)$. This section considers reflected diffusions where Theorem 2.2 is applicable. The process $\{\Phi_t\}$ is governed on $(0, \infty)$ by the stochastic differential equation

$$d\Phi_t = a(\Phi_t)\, dt + \sigma(\Phi_t)\, dB_t, \tag{6.1}$$

where σ is a C^∞ function with $|\sigma(x)| \leq \gamma$ for all x and $\{B_t\}$ is standard Brownian motion. We will not describe the behavior at $\{0\}$ in further detail as it is irrelevant to our future arguments, but we assume that the reflection at $\{0\}$ is done in such a manner that $\{\Phi_t\}$ has continuous sample paths. It is well known [see Chapter 6 of Lindvall (1992) for results and references] that $\{\Phi_t\}$ is a pathwise ordered Markov process that satisfies (2.1).

Working directly with the generator of the reflected process presents unnecessary complications. Instead, let us consider an unreflected process $\{\Phi_t^*\}$ that takes values on the whole of $(-\infty, \infty)$ governed by

$$d\Phi_t^* = a(\Phi_t^*)\,dt + \sigma(\Phi_t^*)\,dB_t.$$

Here, the domain of a and σ are extended to $(-\infty, \infty)$ in a smooth manner. We denote all quantities related to $\{\Phi_t^*\}$ with the superscript $*$. As Equation (5.5) in Chapter 6 of Lindvall (1992) shows, when $\{\Phi_t\}$ and $\{\Phi_t^*\}$ are driven with the same Brownian motion and $\Phi_0 = \Phi_0^* > 0$, $\Phi_t = \Phi_t^*$ up to the first hitting time of $\{0\}$. Hence, $\tau_0 = \tau_0^*$ and convergence rates for $\{\Phi_t\}$ can be obtained by studying $\{\Phi_t^*\}$.

Ichihara and Kunita (1974) show that, for any V with continuous first and second derivatives, the unreflected generator $\mathscr{A}^*$ satisfies

$$\mathscr{A}^*V(x) = a(x)V'(x) + \tfrac{1}{2}[\sigma(x)]^2V''(x). \tag{6.2}$$

We analyze two separate versions of this model with different $a(\cdot)$.

6.1. *Affine drift.* Assume that $a(\cdot)$ satisfies

$$a(x) \le -a(1+x) \tag{6.3}$$

for all $x \ge 0$ for some $a > 0$. If we choose $V(x) = 1 + x$, we have that, for $x > 0$,

$$\mathscr{A}^*V(x) = a(x) \le -aV(x). \tag{6.4}$$

From the remark following Lemma 3.2, choose $s = c = a$ to get $G_x^*(a) \le 1 + x$ for all $x > 0$. Hence, $G_x(a) < 1 + x$ also and, by Theorem 2.1 or 2.2, a is an exponential rate of convergence for $\{\Phi_t\}$.

The same approach can be used to investigate the exponential convergence of higher-order moments of the process. Let $n \ge 1$ and consider polynomial solutions, $V_n(x)$, to the drift equation (6.4) which can be rewritten as

$$(1+x)V_n'(x) \ge V_n(x) = [\gamma^2/2a]V_n''(x) \tag{6.5}$$

when (6.3) and $|\sigma(x)| \le \gamma$ are used. Solutions to (6.5) may be constructed as $V_1(x) = 1 + x$, $V_2(x) = x^2 + (\gamma^2/a + 1)x + 1$ and, for general n, by

$$V_n(x) = x^n + \sum_{j=1}^{n-2} \left(\frac{\gamma^2}{2a}\right)^j \frac{n!}{(n-j)!} x^{n-j} + \left[n!\left(\frac{\gamma^2}{2a}\right)^{n-1} + 1\right]x + 1.$$

We note that such solutions are nondecreasing C^∞ functions and that $V_n(x) \geq 1$ for all $n \geq 1$ and $x \geq 0$. Thus, for $s < a$, we have from the remark following Lemma 3.2 that

$$V_n(x) \geq (a - s) E_x\left[\int_0^{\tau_0^*} e^{st} V_n(\Phi_t^*)\, dt\right]$$

for $x > 0$. But the same inequality holds for $\{\Phi_t\}$ since the starred and unstarred processes are identical up to time τ_0. Equation (3.9) now gives

$$\|P^t(x, \cdot) - \pi\|_{V_n} \leq 2e^{-at}[V_n(x) + b/a] < \infty.$$

Since $V_n(x) \geq \alpha_n + \beta_n x^n$ for some $\alpha_n, \beta_n > 0$, we obtain

$$|E_x[\Phi_t^n] - \pi(x^n)| \leq \frac{2e^{-at}}{\beta_n}\left[V_n(x) + \frac{b}{a}\right]; \tag{6.6}$$

hence, exponential convergence of the polynomial moments of $\{\Phi_t\}$ also occurs up to the exponential rate a. Bounds for b in (6.6) will clearly depend on the behavior of the reflection of $\{\Phi_t\}$ at 0.

Lastly, note that, when $a \geq 2\gamma^2$, $V(x) = \exp(ax/\gamma^2)$ is also a solution to (6.4). Arguing as before, we obtain

$$\left|E_x\left[\exp\left(a\Phi_t/\gamma^2\right)\right] - \pi\left(\exp(ax/\gamma^2)\right)\right| \leq 2e^{-at}\left[\exp(ax/\gamma^2) + b/a\right]. \tag{6.7}$$

We summarize our work in the following theorem.

THEOREM 6.1. *Suppose that* $\{\Phi_t\}$ *is the reflected diffusion governed by* (6.1) *on* $(0, \infty)$ *with* $a(x) \leq -a(1 + x)$ *for all* $x \geq 0$ *and* $|\sigma(x)| \leq \gamma$ *for all* $x \geq 0$. *Then*:

(i) *There exists an invariant measure* π *with* $G_\pi(a) < \infty$.
(ii) *Convergence to* π *occurs exponentially at rate* a *in the sense of* (1.1) *and* (2.2).
(iii) *The polynomial moments of* $\{\Phi_t\}$ *converge in the sense of* (6.6), *and if* $a \geq 2\gamma^2$ *the exponential moments of* $\{\Phi_t\}$ *converge in the sense of* (6.7); *in both, the behavior of the reflection at* $\{0\}$ *is relevant only through the constant* b.

We have given this result only as an example of the methodology, although it appears new and of considerable interest in its own right. Clearly, we could be much more delicate with our assumptions, and the result should remain the same: if the drift is more strongly negative than $-a(1 + x)$, then a is an exponential rate of convergence for the process.

6.2. *Constant drift*. Consider the regulated Brownian motion model studied by Abate and Whitt (1987). Here, $a(x) \equiv -\mu < 0$ and $\sigma(x) \equiv 1$ and (6.2) gives

$$\mathscr{A}^* V(x) = -\mu V'(x) + \tfrac{1}{2} V''(x) \tag{6.8}$$

for $x > 0$. Using $V_m(x) = e^{mx}$ in (6.8) gives $\mathscr{A}^* V_m(x) = (m^2/2 - m\mu)V_m(x)$. Choosing $m = \mu$ minimizes $(m^2/2 - m\mu)$ and gives $\mathscr{A}^* V_\mu(x) = -(\mu^2/2)V_\mu(x)$. Hence, exponential convergence with rate at least $\mu^2/2$ is achieved for this process. Specializing to moments, we have from (3.9) that

$$\|P^t(x, \cdot) - \pi\|_{V_\mu} \le 2e^{-\mu^2 t/2}\left[e^{\mu x} + 2b/\mu^2\right]. \tag{6.9}$$

Applying the crude bound $V_\mu(x) \ge (\mu x)^n/n!$ for $x \ge 0$ in (6.9) gives convergence of all polynomial moments of $\{\Phi_t\}$:

$$|E_x[\Phi_t^n] - \pi(x^n)| \le \frac{2e^{-\mu^2 t/2} n!}{\mu^n}\left[e^{\mu^2 x/2} + \frac{2b}{\mu^2}\right]. \tag{6.10}$$

We will not explore the value of b here. This is the same convergence rate for the moments of $\{\Phi_t\}$ obtained in Corollary 1.1.2 of Abate and Whitt (1987). While our V-norm methods give moment convergence of all subexponential functions, we note that the constants obtained in (6.10) are considerably worse than those computed (with rather more effort) by Abate and Whitt (1987).

7. Multivariate monotonicities: a tandem queue application. Finally, we consider a d-variate Markov process $\{\Phi_t\}$ on $\mathbb{R}_+^d = [0, \infty)^d$ and a function f: $\mathbb{R}_+^d \to [0, \infty)$. In queueing network applications, f might typically be the total customer population or workload in the network. In this section, we derive a convergence rate for $f(\Phi_t)$ and apply the results to a tandem queue.

Suppose that $\{\Phi_t\}$ is pathwise ordered in each of its components (much less stringent assumptions are possible) and suppose there is a function V: $\mathbb{R}_+^d \to [1, \infty)$, nondecreasing in each component, such that

$$\mathscr{A}V(x) \le -cV(x) + b\mathbb{1}_{\{0^d\}}(x),$$

where $0^d = (0, \dots, 0)'$ and c and b are positive real numbers. Let $\{\Phi_t\}$ and $\{\Phi_t'\}$ denote initial level x and stationary versions of the process, respectively; since the process is pathwise ordered, $T = \inf\{t \ge 0: \ \Phi_t = \Phi_t' = 0^d\}$ is a coupling time of $\{\Phi_t\}$ and $\{\Phi_t'\}$. The methods used to prove Theorem 2.2 also apply to general spaces; hence, one obtains $G_x(\alpha) = E_x[e^{\alpha\tau_0}] < \infty$ for all $x \in \mathbb{R}_+^d$ where, as before, $\tau_0 = \inf\{t \ge 0: \Phi_t = 0^d\}$; as in (3.9),

$$\|P^t(x, \cdot) - \pi\|_V \le 2e^{-ct}[V(x) + b/c] < \infty. \tag{7.1}$$

With $G_x(\alpha) < \infty$, the proof of Theorem 2.1 can be easily adapted to the current setting. Thus, $\{\Phi_t\}$ converges to stationarity in a d-variate total variation sense:

$$\lim_{t \to \infty} e^{\alpha t} \sup_{A \in \mathscr{B}(\mathbb{R}_+^d)} |P_x[\Phi_t \in A] - \pi(A)| = 0. \tag{7.2}$$

In many cases, the quantity of interest will be a scalar function of the multidimensional process. For example, in the tandem queue below, the total queue size at time t is formed from the coordinates of $\{\Phi_t\}$ with an f defined by $f(x_1, x_2) = x_1 + x_2$; here, the x_i represent the queue lengths at each node. The process $\{f(\Phi_t)\}$ can be analyzed with the coupling mapping inequality [see page 13 of Lindvall (1992)]. This inequality shows that if (7.2) holds and f is a measurable function, then $f(\Phi_t)$ converges in total variation at the exponential rate α to a random variable distributed as $f(X)$, where X has distribution π.

We now briefly apply these results by considering a pair of $M/M/1$ queues in tandem: customers arrive as a Poisson stream with unit rate to the first queue, where they are serviced with mean service time μ_1^{-1}. After service is completed at the first queue, each customer immediately departs and joins the second queue where the mean service time is μ_2^{-1}. After service is completed at the second queue, the customers leave the system. This is a Jackson network and it is known that the process is ergodic whenever the load condition $\rho_i = \mu_i^{-1} < 1$ is satisfied at each queue.

Geometric considerations from Meyn and Down (1994) suggest

$$V(x_1, x_2) = A^{x_1-1} + B^{x_1+x_2-1} + \gamma A^{-\alpha(x_1-1)} B^{-\beta(x_1+x_2-1)}$$

as a candidate drift function where $A, B > 1$ and $\gamma, \alpha, \beta > 0$. Applying the generator to V [see Meyn and Down (1994)] gives

$$\begin{aligned}\mathscr{A}V(x_1, x_2) = {}& (A-1)A^{x_1-1} + (B-1)B^{x_1+x_2-1} \\ & + \gamma(A^{-\alpha}B^{-\beta} - 1)A^{-\alpha(x_1-1)}B^{-\beta(x_1+x_2-1)} \\ & + \mu_1 \mathbb{1}_{(x_1 \ge 1)}\big[(A^{-1}-1)A^{x_1-1} \\ & \qquad + \gamma(A^{\alpha}-1)A^{-\alpha(x_1-1)}B^{-\beta(x_1+x_2-1)}\big] \\ & + \mu_2 \mathbb{1}_{(x_2 \ge 1)}\big[(B^{-1}-1)B^{x_1+x_2-1} \\ & \qquad + \gamma(B^{\beta}-1)A^{-\alpha(x_1-1)}B^{-\beta(x_1+x_2-1)}\big].\end{aligned}$$

The parameters in V can be determined from the network parameters numerically. For instance, when $\mu_1 = 3$ and $\mu_2 = 2$, using the computer program Mathematica, it is found that, with $\alpha = 3/2$, $\beta = 3/10$, $\gamma = 4/10$, $A = 1.06$ and $B = 1.03$, V satisfies $\mathscr{A}V(x_1, x_2) \le -c(x_1, x_2)V(x_1, x_2)$, where, for $(x_1, x_2) \ne (0,0)$, $c(x_1, x_2)$ is lower bounded by 0.002 (approximately). Furthermore, for these parameter values, one can check that V is nondecreasing in each coordinate. For large (x_1, x_2), $c(x_1, x_2)$ is lower bounded by approximately 0.02; we believe that a more sophisticated argument would yield drift of this order of magnitude for all $(x_1, x_2) \ne (0,0)$.

Hence, this network converges exponentially with rate at least 0.002. The total queue population is obtained by adding the components of $\{\Phi_t\}$: $f(x_1, x_2) = x_1 + x_2$. By the above arguments, $f(\Phi_t)$ converges at the exponential rate 0.002 as well. Finally, for convergence of moments, notice that V satisfies

$V(x_1, x_2) \geq \kappa f(x_1, x_2)$, where $\kappa = \ln(B)/B > 0$. Hence, by (7.1),

$$\lim_{t \to \infty} e^{st} \left| E_x[f(\Phi_t)] - \pi(f) \right| = 0$$

for each x and $s < 0.002$ and the total queue population moments converge up to rate 0.002 as well.

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