The Annals of Applied Probability
1996, Vol. 6, No. 1, 76–91

A MARKOVIAN STORAGE MODEL

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We investigate a storage model where the input and the demand are additive functionals on a Markov chain J. The storage policy is to meet the largest possible portion of the demand. We first derive results for the net input process embedded at the epochs of transitions of J, which is a Markov random walk. Our analysis is based on a Wiener–Hopf factorization for this random walk; this also gives results for the busy period of the storage process. The properties of the storage level and the unsatisfied demand are then derived.

1. Introduction. In this paper we investigate the storage model in which the storage level $Z(t)$ at time t satisfies almost surely (a.s.) the integral equation

$$Z(t) = Z(0) + \int_0^t a(J(s))\,ds - \int_0^t r(Z(s), J(s))\,ds, \tag{1}$$

where

$$r(x,j) = \begin{cases} d(j), & x > 0, \\ \min(a(j), d(j)), & x \le 0, \end{cases} \tag{2}$$

with the condition $Z(0) \ge 0$. Here $J = \{J(t),\ t \ge 0\}$ is a nonexplosive Markov chain on a countable state space E, and a and d are nonnegative functions on E. Equation (1) states that when the Markov chain J is in state j at time t, input into the storage (buffer) occurs at rate $a(j)$, while the demand occurs at rate $d(j)$ and the storage policy is to meet the largest possible portion of this demand. Let us denote by

$$A(t) = \int_0^t a(J(s))\,ds, \qquad D(t) = \int_0^t d(J(s))\,ds \tag{3}$$

the input and the (actual) demand during a time interval $(0, t]$. It can be proved that $(A, J) = \{(A, J)(t),\ t \ge 0\}$ and $(D, J) = \{(D, J)(t),\ t \ge 0\}$ are Markov additive processes (MAPs) on the state space $\mathbb{R}_+ \times E$. A storage model with a more general input process (X, J) of the Markov additive type has been investigated in [11]. Here $X(t)$ consists of jumps of positive size as well as a (cumulative) drift $A(t)$. However, the analysis of that paper cannot be applied in the present model because in [11] it is assumed that

$$a(j) < d(j) \quad \text{for } j \in E. \tag{4}$$

Received November 1993; revised November 1995.

AMS 1991 *subject classifications.* 60K330, 60J15, 60J25.

Key words and phrases. Additive functional, busy period, communication systems, integral equation, Markov-additive processes, Markov random walk, storage models, Wiener–Hopf factorization.

This assumption will make (1) trivial since then the storage level is decreasing so that it will eventually reach 0 after a random length of time and remain at 0 after that. Therefore in this paper we do not assume (4) and use a completely different approach to analyze our model. We denote

$$E_1 = \{j \in E: a(j) \le d(j)\}, \qquad E_0 = \{j \in E: a(j) > d(j)\}. \tag{5}$$

The model represented by (1) occurs in data communication systems. Virtamo and Norros [17] have investigated a model in which a buffer receives input of data from an $M/M/1$ queueing system at a constant rate c_0 so long as the system is busy, and transmits these data at a maximum rate c_1 $(< c_0)$. Denoting by $J(t)$ the queue length, we can represent the buffer content $Z(t)$ at time t by

$$Z(t) = Z(0) + \int_0^t c_0 1_{\{J(s) > 0\}}\, ds - \int_0^t c_1 1_{\{[J(s) > 0] \vee [Z(s) > 0]\}}\, ds. \tag{6}$$

This equation is of type (1) with

$$a(j) = \begin{cases} c_0, & j > 0, \\ 0, & j = 0, \end{cases} \qquad d(j) = c_1, \qquad j \ge 0.$$

Clearly, $a(0) < d(0)$, but $a(j) > d(j)$ for $j > 0$. Thus $E_0 = \{1, 2, \dots\}$ and $E_1 = \{0\}$. The input during $(0, t]$ is given by $c_0 B(t)$, where

$$B(t) = \int_0^t 1_{\{J(s) > 0\}}\, ds,$$

this being the part of the time interval $(0, t]$ during which the server is busy. For a survey of earlier storage models of data communication systems satisfying (1), see [11]. A brief description of two of these models follows.

Anick, Mitra and Sondhi [2] study a model for a data-handling system with N sources and a single transmission channel. The input rate is $a(j) = j$, where j is the number of sources that are "on," and the maximum output rate is a constant c, so that $d(j) = c$. We see in this case $E_0 = \{[c] + 1, [c] + 2, \dots\}$ and $E_1 = \{0, 1, \dots, [c]\}$.

Gaver and Lehoczky [5] investigate a model for an integrated circuit and packet switching multiplexer, with input of data and voice calls. There are $s + u$ output channels, of which s are for data transmission, while the remaining u are shared by data and voice calls (with calls having preemptive priority over data). Here calls arrive in a Poisson process and have exponentially distributed holding times. The model gives rise to (1) for the data buffer content $Z(t)$ with

$$a(j) = c_0, \qquad d(j) = c_2(s + j), \qquad j = 0, 1, \dots, u,$$

where j is the number of channels out of u not occupied by calls, c_0 is the (constant) data arrival rate and c_2 is the output rate capacity per channel.

The net input of our model $X(t)$ is given by

$$X(t) = A(t) - D(t) = \int_0^t x(J(s))\, ds, \tag{7}$$

with $x = a - d$. The net input process is thus an MAP which is nonincreasing during periods in which the environment J is in E_1 and is increasing when J is in E_0. Its sample functions are continuous a.s. and differentiable everywhere except at the transition epochs T_n, $n \geq 0$, of the Markov chain J. Since

$$\int_0^t r(Z(s), J(s))\, ds$$
$$= \int_0^t d(J(s))\, ds + \int_0^t \min\{a(J(s)) - d(J(s)), 0\} 1_{\{Z(s) \leq 0\}}\, ds,$$

we can rewrite (1) in the form

$$Z(t) = Z(0) + X(t) + \int_0^t x(J(s))^- 1_{\{Z(s) \leq 0\}}\, ds, \tag{8}$$

where $y^- = \max\{-y, 0\}$. Here the integral

$$I(t) = \int_0^t x(J(s))^- 1_{\{Z(s) \leq 0\}}\, ds \tag{9}$$

represents the amount of unsatisfied demand during $(0, t]$. If we let $J_n = J(T_n)$, then from (8) we obtain, for $T_n \leq t \leq T_{n+1}$,

$$Z(t) = Z(T_n) + x(J_n)(t - T_n) + x(J_n)^- \int_{T_n}^t 1_{\{Z(s) \leq 0\}}\, ds \tag{10}$$

and from (9)

$$I(t) = I(T_n) + x(J_n)^- \int_{T_n}^t 1_{\{Z(s) \leq 0\}}\, ds. \tag{11}$$

This shows that in order to study the process (Z, I, J) it may be of interest to first study the properties of the embedded process $Z(T_n), I(T_n), J(T_n))$.

The following is a brief summary of the results of this paper. In Section 2 we give the solution of the integral equation (8); a particular consequence of the solution is that $Z(T_n)$ and $I(T_n)$ may be identified as functionals on the process $(T_n, X(T_n), J(T_n))$, which is a Markov random walk (MRW—the discrete-time analog of an MAP). So the properties of this MRW are investigated in Section 3, the key result being a Wiener–Hopf factorization due to Presman [14]; see [12] and [13]. In Section 4 the results of Section 3 are used to study the properties of the storage level and the unsatisfied demand.

Rogers [15] investigates the model we consider, with the Markov chain having finite state space. His analysis is based on the Wiener–Hopf factorization of finite Markov chains, from which the invariant distribution of the storage level is derived. Methods for computing the invariant law of the storage level are discussed by Rogers and Shi [16]. Asmussen [3] and Karandikar and Kulkarni [7] investigate a storage process identified as the reflected Brownian motion (BM) modulated by a finite state Markov chain. In the case where the variance components of this BM are all 0 their storage process reduces to that of our paper.

Our analysis allows for the Markov chain to have infinite state space, but in [15], [3] and [7] this space is finite. We believe that some results in the paper by Asmussen [3] and especially in the paper by Rogers [15] could be generalized, without much effort, to the infinite state space case. The same is not true for numerical methods to compute the storage quantities of interest. In fact, the development of efficient numerical methods when the Markov chain has infinite state space is likely to be the subject of future research in the area of communication systems. In [15], [3] and [7], only the steady-state behavior of the storage level is studied, whereas we derive the time-dependent, as well as the steady-state, behavior of both the storage level and the unsatisfied demand (see Examples 1 and 2). It should also be noted that the specificity of our net input (namely, piecewise linearity) is not too relevant for our analysis, so the techniques of the paper can be applied to other net inputs. This makes our approach potentially more powerful.

We shall denote by $N = (\nu_{jk})$ the generator matrix of J and assume that J has a stationary distribution $(\pi_j,\ j \in E)$. For analytical convenience we assume that $x(j) \neq 0$ for $j \in E$.

2. Preliminary results. We start by solving the integral equation (8). Proceeding as in the proof of Theorem 1 in [11], we have the following result.

LEMMA 1. *We are given a stochastic process J, as defined above, on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$, and additive functionals A and D on J as given in* (3). *The integral equation* (8) *with $Z(0) \geq 0$ a.s. has $\mathscr{P}$-a.s. the unique solution*

$$Z(t) = Z(0) + X(t) + I(t), \tag{12}$$

where

$$I(t) = [Z(0) + m(t)]^- = \Big[Z(0) + \inf_{0 \leq \tau \leq t} X(\tau)\Big]^- . \tag{13}$$

One of the consequences of the solution (12) is (10) since

$$Z(t) = \max\{Z(0) + X(t), X(t) - m(t)\} \geq [Z(0) + X(t)]^+ \geq 0. \tag{14}$$

We next prove some preliminary results concerning the embedded process $Z(T_n)$, $I(T_n)$, $J(T_n)$. We let

$$Z_n = Z(T_n), \qquad I_n = I(T_n), \qquad S_n = X(T_n), X_{n+1} = S_{n+1} - S_n, \tag{15}$$

so that $X_{n+1} = x(J_n)(T_{n+1} - T_n)$.

LEMMA 2. *For $T_n \leq t \leq T_{n+1}$ we have*

$$\begin{aligned} Z(t) &= [Z_n + x(J_n)(t - T_n)]^+, \\ I(t) &= I_n + [Z_n + x(J_n)(t - T_n)]^- , \end{aligned} \tag{16}$$

where

$$Z_n = Z_0 + S_n + I_n, \qquad I_n = [Z_0 + m_n]^- = \Big[Z_0 + \min_{0 \leq r \leq n} S_r\Big]^- . \tag{17}$$

PROOF. Using (10), we may conclude that

$$Z(t) = \max\{0, Z_n + x(J_n)(t - T_n)\}, \qquad n \geq 0, t \in [T_n, T_{n+1}],$$

which proves (16). As a consequence, $Z_{n+1} = \max\{0, Z_n + X_{n+1}\}$ for $n \geq 0$, which implies (17) in view of a result familiar in queueing systems (e.g., Theorem 8 in [10], Chapter 2). □

Lemma 2 shows that in order to study the process (Z_n, I_n, J_n) it suffices to investigate the MRW (T_n, S_n, J_n), which we do in the next section. We note that, if $T_n \leq t \leq T_{n+1}$, then $X(t) = S_n + x(J_n)(t - T_n)$. Thus $\min(S_n, S_{n+1}) \leq X(t) \leq \max(S_n, S_{n+1})$, which, in turn, implies that a.s. $\liminf X(t) = \liminf S_n$ and $\limsup X(t) = \limsup S_n$. Similarly, if we denote, for $t \geq 0$ and $n = 0, 1, \dots,$

$$M(t) = \sup_{0 \leq \tau \leq t} X(\tau), \qquad M_n = \max_{0 \leq r \leq n} S_r, \tag{18}$$

we may conclude that

$$\lim_{t \to \infty} M(t) = \lim_{n \to \infty} M_n = M \leq +\infty, \tag{19}$$

$$\lim_{t \to \infty} m(t) = \lim_{n \to \infty} m_n = m \geq -\infty. \tag{20}$$

These statements show that some conclusions about the fluctuation behavior of the net input process may be drawn from the associated MRW (S_n, J_n). This, in turn, has implications for the storage level and unsatisfied demand since these processes depend on the net input. We denote by $(\pi_j^*, j \in E)$ the stationary distribution of (J_n), so that

$$\pi_j^* = \frac{(-\nu_{jj})\pi_j}{\sum_{k \in E}(-\nu_{kk})\pi_k}. \tag{21}$$

Also, define the *net input rate* $\bar{x} = \sum_{j \in E} \pi_j x(j)$, where we assume the sum exists, but may be infinite. We then have the following.

THEOREM 1 [Fluctuation behavior of $X(t)$]. *We have a.s.*:

(i) $X(t)/t \to \bar{x}$.

(ii) *If* $\bar{x} > 0$, *then* $\lim X(t) = +\infty$, $m > -\infty$ *and* $M = +\infty$.

(iii) *If* $\bar{x} = 0$, *then* $\liminf X(t) = -\infty$, $\limsup X(t) = +\infty$, $m = -\infty$ *and* $M = +\infty$.

(iv) *If* $\bar{x} < 0$, *then* $\lim X(t) = -\infty$, $m = -\infty$ *and* $M < +\infty$.

PROOF. The proof of (i) is standard, but is given here for completeness. We have

$$\frac{X(t)}{t} = \frac{1}{t}\int_0^t x(J(s))\, ds = \frac{1}{t}\int_0^t \left\{[x(J(s))]^+ - [x(J(s))]^-\right\} ds. \tag{22}$$

Now since J is ergodic,

$$\frac{1}{t}\int_0^t [x(J(s))]^+ \, ds = \sum_{j \in E} \frac{[x(j)]^+}{t} \int_0^t 1_{\{J(s)=j\}} \, ds \to \sum_{j \in E} [x(j)]^+ \pi_j \quad \text{as } t \to \infty, \text{ a.s.,} \tag{23}$$

and similarly $(1/t)\int_0^t [x(J(s))]^- \, ds \to \sum_{j \in E} [x(j)]^- \pi_j$ a.s. as $t \to \infty$, so that, using (22), we conclude that $\lim_{t \to \infty} X(t)/t = \bar{x}$ a.s. Define the mean increment in the MRW (S_n, J_n):

$$\mu^* = \sum_{j \in E} \pi_j^* E[X_1 \mid J_0 = j]. \tag{24}$$

Since $E[X_1 \mid J_0 = j] = x(j)/(-\nu_{jj})$, it follows that $\bar{x} = \mu^* \sum_{k \in E} (-\nu_{kk}) \pi_k$. Statements (ii)–(iv) follow from this and (19) and (20), by using Proposition 2 of Prabhu and Tang [12] and Theorem 8 of Prabhu, Tang and Zhu [13]. These last two results describe the fluctuation behavior of the MRW (S_n, J_n). □

3. The MRW (T_n, S_n, J_n). In this section we investigate the properties of the MRW (T_n, S_n, J_n). We note that the conditional distribution of the increments $(T_n - T_{n-1}, S_n - S_{n-1})$, given J_{n-1}, is singular, since $X_n = S_n - S_{n-1} = x(J_n)(T_n - T_{n-1})$ a.s. The distribution of (T_1, X_1, J_1) is best described by the transform matrix

$$\Phi(\theta, \omega) = \big(\phi_{jk}(\theta, \omega)\big) = \big(E[\exp(-\theta T_1 + i\omega X_1); J_1 = k \mid J_0 = j]\big) \tag{25}$$

for $\theta > 0$, ω real and $i = \sqrt{-1}$. We find that

$$\begin{aligned} \Phi(\theta, \omega) &= \big(\phi_{jk}(\theta, \omega)\big) = \big(\alpha_j(\theta, \omega) p_{jk}\big) \\ &= \big(\alpha_j(\theta, \omega)\delta_{jk}\big)\big(p_{jk}\big) = \alpha(\theta, \omega) P, \end{aligned} \tag{26}$$

where

$$\alpha_j(\theta, \omega) = \frac{-\nu_{jj}}{-\nu_{jj} + \theta - i\omega x(j)}, \qquad p_{jk} = \frac{\nu_{jk}}{(-\nu_{jj})}, \quad k \neq j, \qquad p_{jj} = 0. \tag{27}$$

For the time-reversed MRW $(\hat{T}_n, \hat{S}_n, \hat{J}_n)$ corresponding to the given MRW, we have

$$\begin{aligned} \hat{\Phi}(\theta, \omega) &= \big(\hat{\phi}_{jk}(\theta, \omega)\big) = \Big(E\Big[\exp\big(-\theta \hat{T}_1 + i\omega \hat{X}_1\big); \hat{J}_1 = k \mid \hat{J}_0 = j\Big]\Big) \\ &= \left(\frac{\pi_k^*}{\pi_j^*} E[\exp(-\theta T_1 + i\omega X_1); J_1 = j \mid J_0 = k]\right) = \hat{P}\alpha(\theta, \omega), \end{aligned} \tag{28}$$

where $\hat{P}$ is the transition probability matrix of the time-reversed chain $\hat{J}$, namely,

$$\hat{P} = \big(\hat{p}_{jk}\big) = \left(\frac{\pi_k^*}{\pi_j^*} p_{kj}\right). \tag{29}$$

Since the T_n are nondecreasing a.s., the fluctuating theory of the MRW (T_n, S_n, J_n) is adequately described by (S_n). We now define the descending ladder epoch $\overline{N}$ of the MRW (T_n, S_n, J_n) and the ascending ladder epoch N of the time-reversed MRW $(\hat{T}_n, \hat{S}_n, \hat{J}_n)$:

$$\overline{N} = \min\{n\colon S_n < 0\}, \qquad N = \min\{n\colon \hat{S}_n > 0\}. \tag{30}$$

(Here we adopt the convention that the minimum of an empty set is $+\infty$.) It should be noted that both N and $\overline{N}$ are strong ladder epochs, which is reasonable since the increments of S_n and $\hat{S}_n$ in each case have an absolutely continuous distribution. The random variables $S_{\overline{N}}$ and $\hat{S}_N$ are the ladder heights corresponding to $\overline{N}$ and N. We also denote the transforms (in matrix form)

$$\bar{\chi} = \left(\bar{\chi}_{jk}(z, \theta, \omega)\right) = \left(E\left[z^{\overline{N}} exp\left(-\theta T_{\overline{N}} + i\omega S_{\overline{N}}\right); J_{\overline{N}} = k \mid J_0 = j\right]\right), \tag{31}$$

$$\chi = \left(\chi_{jk}(z, \theta, \omega)\right) = \left(\frac{\pi_k^*}{\pi_j^*} E\left[z^N exp\left(-\theta\hat{T}_N + i\omega\hat{S}_N\right); \hat{J}_N = j \mid \hat{J}_0 = k\right]\right), \tag{32}$$

where $0 < z < 1$, $\theta > 0$, $i = \sqrt{-1}$ and ω is real. Connecting these two transforms is the Wiener–Hopf factorization, first established by Presman [14] analytically and interpreted in terms of the ladder variables defined above by Prabhu, Tang and Zhu [13]. The result is the following:

LEMMA 3 (Wiener–Hopf factorization). *For the MRW* (T_n, S_n, J_n) *with* $0 < z < 1$, $\theta > 0$ *and* ω *real,*

$$I - z\Phi(\theta, \omega) = [I - \chi(z, \theta, \omega)][I - \bar{\chi}(z, \theta, \omega)]. \tag{33}$$

We shall use this factorization and the special structure of our MRW to indicate how the transforms χ and $\bar{\chi}$ can be computed in the general case. It turns out that our results contain information concerning the descending ladder epoch $\overline{T}$ of the net input process (X, J) and the ascending ladder epoch T of the time-reversed process $(\hat{X}, \hat{J})$, which is defined as follows:

$$\hat{J}(t) = \hat{J}_n, \qquad \hat{T}_{n-1} < t \le \hat{T}_n, \qquad \hat{X}(t) = \int_0^t x\left(\hat{J}(s)\right) ds. \tag{34}$$

Thus

$$\overline{T} = \inf\{t > 0\colon X(t) \le 0\}, \qquad T = \inf\{t > 0\colon \hat{X}(t) \ge 0\}. \tag{35}$$

We note that $X(\overline{T}) = 0$ and $\hat{X}(T) = 0$ a.s. For $0 < z < 1$, $\theta > 0$ we define the transforms

$$\zeta = \left(\zeta_{jk}(z, \theta)\right) = \left(E\left[z^{\overline{N}} e^{-\theta\overline{T}}; J(\overline{T}) = k \mid J(0) = j\right]\right), \tag{36}$$

$$\eta = \left(\eta_{jk}(z, \theta)\right) = \left(\frac{\pi_k^*}{\pi_j^*} E\left[z^N e^{-\theta T}; \hat{J}(T) = j \mid \hat{J}(0) = k\right]\right). \tag{37}$$

THEOREM 2. *For* $0 < z < 1$, $\theta > 0$ *and* ω *real, we have*

$$\bar{\chi}(z, \theta, \omega) = \zeta(z, \theta)\Phi(\theta, \omega), \qquad \chi(z, \theta, \omega) = \alpha(\theta, \omega)\eta(z, \theta). \tag{38}$$

PROOF. An inspection of the sample paths of (X, J) will show that $J(\overline{T}) = J_{\overline{N}-1}$ and

$$T_{\overline{N}} - \overline{T} = \frac{S_{\overline{N}}}{x(J(\overline{T}))} \quad \text{a.s.}$$

Since $\overline{T}$ is a stopping time for (X, J), we see that, given $J(\overline{T}) = l$, $S_{\overline{N}}/x(J(\overline{T}))$ is independent of $\overline{T}$ and has the same distribution as T_1, given $J_0 = l$. Therefore

$$\begin{aligned}
&\bar{\chi}_{jk}(z, \theta, \omega) \\
&\quad = \sum_{l \in E} E\left[z^{\overline{N}} \exp\left(-\theta\left[\overline{T} + \frac{S_{\overline{N}}}{x(J_{\overline{N}-1})}\right] + i\omega S_{\overline{N}}\right); J_{\overline{N}-1} = l, J_{\overline{N}} = k \mid J_0 = j\right] \\
&\quad = \sum_{l \in E} E\left[z^{\overline{N}} e^{-\theta \overline{T}}; J(\overline{T}) = l \mid J_0 = j\right] \\
&\qquad \times E\left[\exp\left(-\theta \frac{S_{\overline{N}}}{x(J_{\overline{N}-1})} + i\omega S_{\overline{N}}\right); J_{\overline{N}} = k \mid J_{\overline{N}} - 1 = l\right] \\
&\quad = \sum_{l \in E} \zeta_{jl}(z, \theta) E\left[\exp(-\theta T_1 + i\omega X_1); J_1 = k \mid J_0 = l\right] \\
&\quad = \sum_{l \in E} \zeta_{jl}(z, \theta)\phi_{lk}(\theta, \omega).
\end{aligned}$$

Thus $\bar{\chi}(z, \theta, \omega) = \zeta(z, \theta)\Phi(\theta, \omega)$. The proof of $\chi(z, \theta, \omega) = \alpha(\theta, \omega)\eta(z, \theta)$ is similar. □

In general, for an $(|E| \times |E|)$ matrix A we block-partition A in the form

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix},$$

with the rows and columns of A_{00} corresponding to the states in E_0. We now have

$$\zeta = \begin{pmatrix} \mathbf{0} & \zeta_{01} \\ \mathbf{0} & \zeta_{11} \end{pmatrix}, \qquad \eta = \begin{pmatrix} \eta_{00} & \eta_{01} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad I = \begin{pmatrix} I_{00} & \mathbf{0} \\ \mathbf{0} & I_{11} \end{pmatrix}, \tag{39}$$

where I is the identity matrix. From (33) and Theorem 2 we have the following.

THEOREM 3. *We have, for* $0 < z < 1$, $\theta > 0$ *and* ω *real,*

$$\chi = \begin{pmatrix} \alpha_{00}\eta_{00} & \alpha_{00}\eta_{01} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \bar{\chi} = \begin{pmatrix} \zeta_{01}\Phi_{10} & \zeta_{01}\Phi_{11} \\ z\Phi_{10} & z\Phi_{11} \end{pmatrix} \tag{40}$$

and

$$(41)\qquad I_{00} - \bar{\chi}_{00} = (I_{00} - \chi_{00})^{-1}[(I_{00} - z\Phi_{00}) - z\chi_{01}\Phi_{10}],$$

$$(42)\qquad \bar{\chi}_{01} = (I_{00} - \chi_{00})^{-1}[z\Phi_{01} - \chi_{01}(I_{11} - z\Phi_{11})],$$

where the inverse exists in the specified domain.

We note that if we let

$$(43)\qquad \bar{\gamma} = \left(\bar{\gamma}_{jk}(z,\theta)\right) = \left(E\left[z^{\bar{N}}\exp(-\theta T_{\bar{N}});\ J_{\bar{N}-1} = k \mid J_0 = j\right]\right),$$

$$(44)\qquad \gamma = \left(\gamma_{jk}(z,\theta)\right) = \left(\frac{\pi_k^*}{\pi_j^*}E\left[z^{N}\exp\left(-\theta\hat{T}_N\right);\ \hat{J}_N = j \mid \hat{J}_0 = k\right]\right),$$

then, with $I^\theta = (-\delta_{jk}\nu_{jj}/[-\nu_{jj} + \theta])$,

$$(45)\qquad \bar{\gamma} = \zeta I^\theta, \qquad \gamma = I^\theta\eta,$$

with, due to (45), the results for (ζ, η) being equivalent to those for $(\bar{\gamma}, \gamma)$. As a matter of convenience we express some of the remaining results of this section in terms of $(\bar{\gamma}, \gamma)$.

COROLLARY 1. *For* $0 < z < 1$, $\theta > 0$, *with* $R = (r_{jk}(\theta)) = (\delta_{jk} \mid x(j)|/[-\nu_{jj} + \theta])$, *we have*

$$(46)\qquad \gamma_{00} + (I_{00} - \gamma_{00})\bar{\gamma}_{01}P_{10} = z\left[I_{00}^\theta P_{00} + \gamma_{01}I_{11}^\theta P_{10}\right],$$

$$(47)\qquad \gamma_{01} + (I_{00} - \gamma_{00})\bar{\gamma}_{01}P_{11} = z\left[I_{00}^\theta P_{01} + \gamma_{01}I_{11}^\theta P_{11}\right],$$

$$(48)\qquad R_{00}\bar{\gamma}_{01}P_{10} + (I_{00} - \gamma_{00})\bar{\gamma}_{01}R_{11}P_{10} = z\gamma_{01}R_{11}I_{11}^\theta P_{10},$$

$$(49)\qquad R_{00}\bar{\gamma}_{01}P_{11} + (I_{00} - \gamma_{00})\bar{\gamma}_{01}R_{11}P_{11} = z\gamma_{01}R_{11}I_{11}^\theta P_{11}.$$

PROOF. We equate the real parts of the identity (41) and put $\omega = 0$. This yields (46). We also equate the imaginary parts of (41), divide by ω and let $\omega \to 0$. This yields (48), in view of (46). The proof of (47) and (49) is similar, starting with the identity (42). □

Theorem 3 shows that the submatrices $\bar{\chi}_{00}$ and $\bar{\chi}_{01}$ are determined by χ_{00} and χ_{01}. Corollary 1 can be used in some important cases to reduce the computation to a single (matrix) equation for $\bar{\gamma}_{01}$, as we will show in the following. Case (i) arises in models with $|E_1| > 1$, while case (ii) covers the situation with $|E_1| = 1$. Details of the computations are omitted.

Case (i). If the submatrix P_{11} has an inverse, then

$$(50)\qquad \gamma_{00} = zI_{00}^\theta P_{00} + R_{00}\bar{\gamma}_{01}R_{11}^{-1}P_{10},$$

$$(51)\qquad \gamma_{01} = zI_{00}^\theta P_{01} + R_{00}\bar{\gamma}_{01}R_{11}^{-1}P_{11},$$

where $\bar{\gamma}_{\mathbf{01}}$ satisfies the equation

$$\begin{aligned}\bar{\gamma}_{\mathbf{01}}\left[R_{\mathbf{11}}^{-1}P_{\mathbf{10}}\right]\bar{\gamma}_{\mathbf{01}} - \left[R_{\mathbf{00}}^{-1}\left(I_{\mathbf{00}} - zI_{\mathbf{00}}^{\theta}P_{\mathbf{00}}\right)\bar{\gamma}_{\mathbf{01}} + \bar{\gamma}_{\mathbf{01}}R_{\mathbf{11}}^{-1}\left(I_{\mathbf{11}} - zP_{\mathbf{11}}I_{\mathbf{11}}^{\theta}\right)\right]& \\ +z^2R_{\mathbf{00}}^{-1}I_{\mathbf{00}}^{\theta}P_{\mathbf{01}}I_{\mathbf{11}}^{\theta} = 0.&\end{aligned} \tag{52}$$

Case (ii). If $P_{\mathbf{11}} = \mathbf{0}$ and $r_{jj}(\theta) = r_1^{\theta}$, $j \in E_1$, then

$$\gamma_{\mathbf{00}} = zI_{\mathbf{00}}^{\theta}P_{\mathbf{00}} + \frac{1}{r_1^{\theta}}R_{\mathbf{00}}\bar{\gamma}_{\mathbf{01}}P_{\mathbf{10}}, \qquad \gamma_{\mathbf{01}} = zI_{\mathbf{00}}^{\theta}P_{\mathbf{01}}, \tag{53}$$

$$\begin{aligned}\left(\bar{\gamma}_{\mathbf{01}}P_{\mathbf{10}}\right)^2 - \left[r_1^{\theta}R_{\mathbf{00}}^{-1}\left(I_{\mathbf{00}} - zI_{\mathbf{00}}^{\theta}P_{\mathbf{00}}\right) + I_{\mathbf{00}}\right]\left(\bar{\gamma}_{\mathbf{01}}P_{\mathbf{10}}\right)& \\ + z^2r_1^{\theta}R_{\mathbf{00}}^{-1}I_{\mathbf{00}}^{\theta}P_{\mathbf{01}}I_{\mathbf{11}}^{\theta}P_{\mathbf{10}} = 0.&\end{aligned} \tag{54}$$

EXAMPLE 1. Consider the Gaver–Lehoczky [5] model with a single output channel, in which the channel is shared by data and voice calls (with calls having preemptive priority over data). Here $J(t) = 0$ if a call is in progress at time t (i.e., the channel is not available for data transmission), and $J(t) = 1$ otherwise. Thus J has a two-state space $\{0, 1\}$ and

$$a(0) = a(1) = c_0, \qquad d(0) = 0, \qquad d(1) = c_2, \qquad c_0 < c_2,$$

so that $E_{\mathbf{0}} = \{0\}$, $E_{\mathbf{1}} = \{1\}$, $x(0) = c_0$ and $x(1) = c_0 - c_2 = -c_1$. Let the arrival rate and service rate of calls be denoted by λ and μ, respectively. Then

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad N = (\nu_{jk}) = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}.$$

We may now use (45), (53) and (54) to conclude that

$$\eta_{01} = z, \qquad \eta_{00} = \frac{\sigma_1}{\sigma_0}\zeta_{01},$$

$$\sigma_1\zeta_{01}^2 - \left[(\sigma_0 + \sigma_1) + \left(\frac{1}{c_0} + \frac{1}{c_1}\right)\theta\right]\zeta_{01} + z^2\sigma_0 = 0,$$

where $\sigma_0 = \mu/c_0$ and $\sigma_1 = \lambda/c_1$. This implies that

$$\begin{aligned}\zeta_{01}(z,\theta) = (2\sigma_1)^{-1}\Big[&(\sigma_0 + \sigma_1) + (1/c_0 + 1/c_1)\theta \\ &- \sqrt{\left[(\sigma_0 + \sigma_1) + (1/c_0 + 1/c_1)\theta\right]^2 - 4z^2\sigma_0\sigma_1}\Big].\end{aligned} \tag{55}$$

For an $M/M/1$ queue with arrival and service rates σ_1 and σ_0, respectively, we denote the busy period by $\overline{T}^*$ and the number of customers served during the busy period by $\overline{N}^*$. From (55) we have the following (see Section 2.8 of [9]):

$$\begin{aligned}\zeta_{01}(z,\theta) &= E\left[z^{\overline{N}}\exp(-\theta\overline{T});\, J(\overline{T}) = 1 \mid J(0) = 0\right] \\ &= E\left[z^{2\overline{N}^*}\exp\left(-\theta\left(\frac{1}{c_0} + \frac{1}{c_1}\right)\overline{T}^*\right)\right].\end{aligned}$$

EXAMPLE 2. Consider the Virtamo–Norros [17] model, given by (6). Denote $c = c_1/c_0$ and $\rho = \lambda/\mu$, in which case $0 < c < 1$ and $\rho > 0$. Equation (54) holds for this model and may be proved to be equivalent to

$$\zeta_{n0}\left[\zeta_{10} - \frac{c\mu + \lambda + \mu}{\lambda(1-c)}\right] - \frac{zc}{\rho(1-c)}\left[\zeta_{n-1,0} + \rho\zeta_{n+1,0}\right] = 0, \qquad n \geq 1. \tag{56}$$

If $J(0) = 1$, it is known from Aalto [1] that $\overline{T}$ is equal to the busy period of an $M/M/1$ queue with arrival rate λ and service rate $c\mu$. Thus

$$\zeta_{10}(1,\theta) = \frac{c\mu + \lambda + \theta}{2\lambda} - \sqrt{\left(\frac{c\mu + \lambda + \theta}{2\lambda}\right)^2 - \frac{c\mu}{\lambda}}\,. \tag{57}$$

Using (56), the transforms

$$\zeta_{n0}(1,\theta) = \left(E\left[e^{-\theta\overline{T}} \mid J(0) = n\right]\right), \qquad n \geq 1,$$

may be computed recursively, starting with $\zeta_{10}(1,\theta)$ as given in (57).

4. The main results. With the properties for the MRW (T_n, S_n, J_n) established in Section 3, we are now in a position to derive the main results of the paper. We first state the following results for the embedded process (Z_n, I_n, J_n), which follow easily from Theorems 3 and 4 of Prabhu and Tang [12].

LEMMA 4. *If* $Z_0 = 0$ *a.s., then, for* $\theta > 0$ *and* ω_1, ω_2 *real, we have*

$$\left(\sum_{n=0}^{\infty} E\left[\exp(-\theta T_n + i\omega_1 Z_n + i\omega_2 I_n);\, J_n = k \mid J_0 = j\right]\right)^{-1} = \left[I - \chi(z,\theta,\omega_1)\right]\left[I - \overline{\chi}(z,\theta,-\omega_2)\right]. \tag{58}$$

LEMMA 5. *Suppose that* $\mu^* < 0$. *Then* $(Z_n, J_n) \to_{\mathscr{D}} (Z_\infty^*, J_\infty^*)$ *as* $n \to \infty$, *for all initial distributions, where* J_∞^* *is the stationary version of* (J_n) *and* $(E[\exp(i\omega Z_\infty^*);\, J_\infty^* = k])$ *is the kth element of the row vector*

$$\pi^*\left[I - \chi(1,0,0)\right]\left[I - \chi(1,0,\omega)\right],$$

where $\pi^* = (\pi_j^*, j \in E)$.

For finite t, the distribution of $(Z(t), I(t), J(t))$ can be found from (16) using Lemma 4. We note that since $\overline{T} = \inf\{t > 0\colon X(t) \leq 0\}$ we have

$$\overline{T} = \inf\{t > 0\colon Z(t) = 0 \mid Z(0) = 0\}. \tag{59}$$

Thus $\overline{T}$ is the busy period of the storage. The transform of $(\overline{T}, J(\overline{T}))$ is $\zeta(1,\theta)$ given by

$$\zeta(1,\theta) = \left(\zeta_{jk}(1,\theta)\right) = \left(E\left[e^{-\theta\overline{T}};\, J(\overline{T}) = k \mid J(0) = j\right]\right). \tag{60}$$

In Section 3 it was shown how this transform can be computed. We recall that $\bar{x}$ is the *net input rate*.

THEOREM 4. *The busy period* $\overline{T}$ *defined by* (59) *has the following properties*:

(i) *Given* $J(0) \in E_1$, $\overline{T} = 0$ *a.s.*
(ii) *If* $\bar{x} < 0$, *then* $\overline{T} < \infty$ *a.s.*

PROOF. (i) The statement follows immediately from the fact that $x(j) < 0$ for $j \in E_1$.

(ii) Since $\bar{x} < 0$ we have $\mu^* < 0$ [see the proof of Theorem 1(ii)]. This implies that the descending ladder epoch of the MRW (T_n, S_n, J_n) is finite ($\overline{N} < \infty$ a.s.) by virtue of Proposition 2 of Prabhu and Tang [12]. This, in turn, implies that $T_{\overline{N}} < \infty$ a.s. The statement now follows since $\overline{T} \le T_{\overline{N}}$. □

The limit behavior of the process $(Z(t), I(t), J(t))$ as $t \to \infty$ can also be obtained from that of the embedded process (Z_n, I_n, J_n) as $n \to \infty$, by using Lemma 5. The following theorems characterize this limit behavior.

THEOREM 5. *The process* $Z(t), I(t), J(t))$ *has the following properties*:

(i) *If* $\bar{x} > 0$, *then* $I(t) \to (Z(0) + m)^- < +\infty$ *and* $Z(t)/t \to \bar{x}$ *a.s.*; *in particular*, $Z(t) \to +\infty$ *a.s.*
(ii) *If* $\bar{x} = 0$, *then* $I(t) \to +\infty$ *and* $\limsup Z(t) = +\infty$ *a.s.*
(iii) *If* $\bar{x} < 0$, *then* $I(t)/t \to -\bar{x}$ *and* $Z(t)/t \to 0$ *a.s.*; *in particular*, $I(t) \to +\infty$ *a.s.*

PROOF. (i) We first note that $I(t)$ converges as indicated by Theorem 1(ii) and (13). The rest of the statement follows directly from Theorem 1(i) and (1).

(ii) From Theorem 1(iii) and (13), $I(t) \to (Z(0) + m)^- = +\infty$. Also, from (12), since $I(t)$ is nonnegative, $Z(t) \ge Z(0) + X(t)$. Using Theorem 1(iii), we obtain

$$\limsup Z(t) \ge \limsup [Z(0) + X(t)] = +\infty.$$

(iii) Since $X(t)$ has continuous sample functions and $X(t)/t \to \bar{x} < 0$, by Theorem 1(i), standard analytical arguments show that

$$\lim \frac{m(t)}{t} = \lim \frac{X(t)}{t} = \bar{x} < 0.$$

The desired results now follow from (13) and (14). □

THEOREM 6. *If* $\bar{x} < 0$, *then*, *for* $z_0, z \ge 0$ *and* $j, k \in E$ *and with* (Z_∞^*, J_∞^*) *being the limit distribution of* (Z_n, J_n) *as given in Lemma* 5,

$$\begin{aligned} &\lim_{t \to \infty} P\{Z(t) \le z;\ J(t) = k \mid Z(0) = z_0,\ J(0) = j\} \\ (61)\qquad &= \pi_k \int_{0-}^{\infty} P\{Z_\infty^* \in dv \mid J_\infty^* = k\} P\{X_1 \le z - v \mid J(0) = k\}. \end{aligned}$$

PROOF. Let $N(t) = \sup\{n\colon T_n \le t\}$. We have

$$
\begin{aligned}
&P\{Z(t) \le z;\, J(t) = k \mid Z(0) = z_0,\, J(0) = j\}\\
&\quad = \int_{0-}^{\infty} P\{Z_{N(t)} \in dv;\, J_{N(t)} = k \mid Z_0 = z_0,\, J_0 = j\}\\
&\qquad\qquad \times P\{Z(t) \le z \mid Z(T_{N(t)}) = v,\, J(T_{N(t)}) = k,\, Z(0) = z_0,\, J(0) = j\}\\
&\quad = P\{J_{N(t)} = k \mid J_0 = j\} \int_{0-}^{\infty} P\{Z_{N(t)} \in dv \mid Z_0 = z_0,\, J_{N(t)} = k,\, J_0 = j\}\\
&\qquad\qquad \times P\{Z(t) \le z \mid Z(T_{N(t)}) = v,\, J(T_{N(t)}) = k\}\\
&\quad = P\{J(t) = k \mid J(0) = j\} \int_{0-}^{\infty} P\{Z_{N(t)} \in dv \mid Z_0 = z_0,\, J_{N(t)} = k,\, J_0 = j\}\\
&\qquad\qquad \times P\Big\{\big[v + x(k)(t - T_{N(t)})\big]^{+} \le z \mid J(T_{N(t)}) = k\Big\}.
\end{aligned}
$$

Since $P\{J(t) = k \mid J(0) = j\} \to \pi_k$ a.s. as $t \to \infty$, the statement follows from the fact that as $t \to \infty$ the following two results hold. Given $J(T_{N(t)}) = k$, $(t - T_{N(t)})$ has the distribution of T_1 in the limit, given $J(0) = k$, so that

$$
\begin{aligned}
&P\Big\{\big[v + x(k)(t - T_{N(t)})\big]^{+} \le z \mid J(T_{N(t)}) = k\Big\}\\
&\quad \to P\big\{[v + X_1]^{+} \le z \mid J(0) = k\big\}\\
&\quad = P\{X_1 \le z - v \mid J(0) = k\}.
\end{aligned}
$$

Since $\bar{x} < 0$ we have $\mu^* < 0$ and $N(t) \to \infty$; thus, using Lemma 5, we conclude that

$$
P\{Z_{N(t)} \in dv \mid Z_0 = z_0,\, J_{N(t)} = k,\, J_0 = j\} \to P\{Z_\infty^* \in dv \mid J_\infty^* = k\}. \qquad \square
$$

In case $\bar{x} < 0$, we denote by (Z_∞, J_∞) the limit random variable of $(Z, J)(t)$, which, in view of Theorem 6, is independent of the initial distribution.

COROLLARY 2. *If* $\bar{x} < 0$, *we have the following*:

(i) *For* $z \ge 0$ *we have*

$$
\begin{aligned}
&P\{Z_\infty \le z \mid J_\infty = k\}\\
&\quad = P\{Z_\infty^* \le z \mid J_\infty^* = k\}\\
&\qquad \times \left(1 - E\left[\exp\left(-\frac{\nu_{kk}}{x(k)}(Z_\infty^* - z)\right)\middle|\, Z_\infty^* \le z,\, J_\infty^* = k\right]\right), \qquad k \in E_0,
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
&P\{Z_\infty > z \mid J_\infty = k\}\\
&\quad = P\{Z_\infty^* > z \mid J_\infty^* = k\}\\
&\qquad \times \left(1 - E\left[\exp\left(-\frac{\nu_{kk}}{x(k)}(Z_\infty^* - z)\right)\middle|\, Z_\infty^* > z,\, J_\infty^* = k\right]\right), \qquad k \in E_1.
\end{aligned}
\tag{63}
$$

(ii) *We have*

$$(64)\quad P\{Z_\infty = 0 \mid J_\infty = k\} = \begin{cases} 0, & k \in E_0, \\ E\left[\exp\left(-\dfrac{\nu_{kk}}{x(k)} Z_\infty^*\right) \middle| J_\infty^* = k\right], & k \in E_1. \end{cases}$$

PROOF. (i) Let $z \geq 0$ and $k \in E_0$. From (61) we have

$$\begin{aligned} &P\{Z_\infty \leq z \mid J_\infty = k\} \\ &\quad = \int_{0-}^{z} P\{Z_\infty^* \in dv \mid J_\infty^* = k\} P\{X_1 \leq z - v \mid J(0) = k\} \\ &\quad = \int_{0-}^{z} P\{Z_\infty^* \in dv \mid J_\infty^* = k\}\left(1 - \exp\left(\frac{\nu_{kk}}{x(k)}(z - v)\right)\right) \\ &\quad = P\{Z_\infty^* \leq z \mid J_\infty^* = k\} \\ &\qquad \times \left(1 - \int_{0-}^{z} P\{Z_\infty^* \in dv \mid Z_\infty^* \leq z,\, J_\infty^* = k\} \exp\left(-\frac{\nu_{kk}}{x(k)}(v - z)\right)\right) \\ &\quad = P\{Z_\infty^* \leq z \mid J_\infty^* = k\}\left(1 - E\left[\exp\left(-\frac{\nu_{kk}}{x(k)}(Z_\infty^* - z)\right) \middle| Z_\infty^* \leq z,\, J_\infty^* = k\right]\right). \end{aligned}$$

This gives (62). The proof of (63) is similar.

(ii) The statement follows from the fact that, using (61), we have

$$P\{Z_\infty = 0 \mid J_\infty = k\} = \int_{0-}^{\infty} P\{Z_\infty^* \in dv \mid J_\infty^* = k\} P\{X_1 \leq -v \mid J(0) = k\}. \quad \square$$

We denote by $I_k(t)$ the unsatisfied demand in state k during $(0, t]$, so that

$$(65)\quad I_k(t) = \int_0^t x(J(s))^- 1_{\{Z(s)=0,\ J(s)=k\}}\, ds = x(k)^- \int_0^t 1_{\{Z(s)=0,\ J(s)=k\}}\, ds.$$

If $k \in E_0$, then $x(k)^- = 0$ and $I_k(t) = 0$. If $k \in E_1$, we have the following important result for the performance analysis of the system.

COROLLARY 3. *If* $\bar{x} < 0$ *and* $k \in E_1$, *then*

$$(66)\quad \lim_{t\to\infty} \frac{I_k(t)}{t} = -x(k)\pi_k E\left[\exp\left(-\frac{\nu_{kk}}{x(k)} Z_\infty^*\right) \middle| J_\infty^* = k\right]$$

and

$$(67)\quad \lim_{t\to\infty} \frac{I_k(t)}{I(t)} = \frac{x(k)\pi_k E\left[\exp\left(-(\nu_{kk}/x(k))Z_\infty^*\right) \mid J_\infty^* = k\right]}{\sum_{j\in E_1} x(j)\pi_j E\left[\exp\left(-(\nu_{jj}/x(j))Z_\infty^*\right) \mid J_\infty^* = j\right]}.$$

PROOF. Using Theorem 6, we conclude that

$$\frac{1}{t}\int_0^t 1_{\{Z(s)=0,\ J(s)=k\}}\, ds \to P\{Z_\infty = 0,\ J_\infty = k\}.$$

This implies (66) in view of (64) and (65). Also, since $I(t) = \sum_{j \in E_1} I_j(t)$, (67) follows by using (66). □

EXAMPLE 1 (Continuation). We note that $\pi_0^* = \pi_1^* = 1/2$, $\zeta_{01}(1,0) = 1$, $\eta_{01}(1,0) = 1$ and $\eta_{00}(1,0) = \rho$, with $\rho = \sigma_1/\sigma_0$. We now assume $\rho < 1$. Since

$$\begin{aligned}&\pi^*[1 - \chi(1,0,0)][I - \chi(1,0,\omega)]^{-1}\\&\quad= \frac{1}{2}\left[(1-\rho) + \rho\frac{\sigma_0 - \sigma_1}{(\sigma_0 - \sigma_1) - i\omega}\frac{\sigma_0 - \sigma_1}{(\sigma_0 - \sigma_1) - i\omega}\right],\end{aligned}$$

we conclude from Lemma 5 that $P\{Z_\infty^* > z \mid J_\infty^* = 0\} = \rho \exp(-(\sigma_0 - \sigma_1)z)$ for $z \geq 0$, and similarly $P\{Z_\infty^* > z \mid J_\infty^* = 1\} = \exp(-(\sigma_0 - \sigma_1)z)$. With $\pi_0 = \lambda/(\lambda + \mu)$ and $\pi_1 = \mu/(\lambda + \mu)$ we conclude, using Theorem 6, that, for $z \geq 0$,

$$\begin{aligned}P\{Z_\infty > z;\, J_\infty = 0\} &= \pi_0 \exp\left(-(\sigma_0 - \sigma_1)z\right),\\P\{Z_\infty > z;\, J_\infty = 1\} &= \pi_1 \rho \exp\left(-(\sigma_0 - \sigma_1)z\right).\end{aligned}$$

Finally, using Corollary 3, we conclude that a.s.

$$I_0(t) = 0\ \forall\, t \quad \text{and} \quad \lim_{t \to \infty} \frac{I_1(t)}{t} = c_1 \pi_1 (1 - \rho).$$

We note that this example has been considered also by Chen and Yao [4], Gaver and Miller [6] and Kella and Whitt [8] (in the context of storage models for which the net input is alternatingly nonincreasing and nondecreasing) and by Karandikar and Kulkarni [7] (Case 1 of Example 1, Section 6) with the storage level being a particular case of a Markov-modulated reflected Brownian motion.

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