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ABSTRACT NONLINEAR FILTERING THEORY IN THE PRESENCE OF FRACTIONAL BROWNIAN MOTION

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We develop the filtering theory in the case where both the signal and the observation are solutions of some stochastic differential equation driven by a multidimensional fractional Brownian motion. We show that the classical approach fails to give a closed equation for the filter and we develop another approach using an auxiliary process-valued semimartingale which solves this problem theoretically.

1. Introduction. We pursue the study, initiated in Coutin and Decreusefond (1997), Decreusefond and Üstünel (1998), of the fractional Brownian motion, in particular of the stochastic differential equations driven by such a process. We here address the following filtering problem [see Kleptsyna, Kloeden and Anh (1996a, b) for a related problem]. Assume that on some probability space $(\Omega, \mathscr{F}, \mathbf{P}^h_{\bar{H}})$, X, a signal, and Y, an observation of X, are given as the solutions of the system (the notations will be precised below)

$$X^l_t = x^l_0 + \sum_{i=1}^{M} \int_0^t K_{H_i}(t,s) b^{l,i}(X_s)\,ds + \int_0^t K_{H_i}(t,s) a^{l,i}(X_s)\,dB^i_s,$$

$$Y^k_t = \int_0^t K_H(t,s) h^k(X_s)\,ds + \sum_{j=M+1}^{d} \int_0^t K_H(t,s)\tau^{k,j}(Y_s)\,dB^j_s,$$

we aim to compute $\pi_t(f) =_{\text{def}} \mathbf{E}_h[f(X_t) | Y_s,\ s \le t]$ for any sufficiently regular f.

The usual strategy, extensively developed for X and Y semimartingales, that is, $H_i = 1/2$, $i = 1, \dots, d$ [see, e.g., Pardoux (1989), Zakai (1969)], consists of constructing a new probability measure, later denoted by $\mathbf{P}_{\bar{H}}$, called the reference probability measure, a $\mathbf{P}_{\bar{H}}$ Brownian motion carrying the same filtration as the observation process and such that, under $\mathbf{P}_{\bar{H}}$, it is independent of the signal process. The second step is to show that the optional projection under $\mathbf{P}^h_{\bar{H}}$ can be transformed to a $\mathbf{P}_{\bar{H}}$ conditional expectation to obtain an equation for $\pi_t(f)$. The tools used are the Girsanov theorem, some results on weak solutions of stochastic differential equations, the Itô formula and two technical lemmas; see Property (P) and Lemma 3.2 below. The extension of the Itô formula to our setting is a tedious but rather straightforward extension of the work done in Decreusefond and Üstünel (1998) [hereafter designated

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DU (1998)]. As a consequence, its statement and proof are postponed to the Appendix. Actually, what is new in this part is the theorem on weak solutions of SDEs driven by a multidimensional fBm (see Theorem 3.1) and the proof that the observation process carries the whole information about the directing processes (see Theorem 3.2). Since martingale theorems cannot be used, several key computations are made within the framework of the Malliavin calculus. As a result of this preliminary work, the Zakai and Kallianpur–Striebel equations for X are established.

Unfortunately and not very surprisingly, it turns out that these equations are not closed in the sense that the filters are not given as solutions of stochastic partial differential equations as for standard diffusions [see Üstünel (1986)]. Actually, the time derivative of the filter depends here on the whole sample-path of the signal. It is thus somewhat natural to work with functionals depending on X from its very beginning. This is the objective of the second part of this paper where we consider a $\mathscr{C}([0,1];\mathbf{R}^M)$-valued, $\{\mathscr{F}_t^X,\ t \in I\}$-adapted semimartingale for the filters of which we can obtain closed equations.

The paper is organized as follows: the next section is devoted to some preliminaries on fractional Brownian motions; in Section 3, the filtering problem is posed and the Kallianpur–Striebel equation for the observation process is established. The infinite-dimensional approach is studied in Section 4. The Appendix contains the statement and proof of the Itô formula and the extension to the multidimensional setting of the results of Coutin and Decreusefond (1997) [hereafter designated CD (1997)] on existence and uniqueness of strong solutions of stochastic differential equations driven by a fractional Brownian motion.

2. Preliminaries.

DEFINITION 2.1. For any $H \in (0,1)$, the one-dimensional fractional Brownian motion (fBm) of index H (called the Hurst parameter), $\{W_t^H;\ t \in [0,T]\}$ is the unique centered Gaussian process whose covariance kernel is given by

$$R_H(s,t) = E_H\big[W_s^H W_t^H\big] \stackrel{\text{def}}{=} \frac{V_H}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big),$$

where

$$V_H \stackrel{\text{def}}{=} \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}.$$

There exist at least two approaches to define a stochastic calculus with respect to the fractional Brownian motion. The main problem is to give a sense to something like $\int_0^t u_s\, dW_s^H$. The Riemann sums approach consists of considering processes u such that

$$\sum_{t_i \in \pi} u_{t_i}\big(W_{t_{i+1}}^H - W_{t_i}^H\big) \tag{1}$$

converges in probability when the mesh of the subdivision $\pi = \{0 = t_1 < \cdots < t_n = t\}$ tends to 0. When $H > 1/2$, this can be done using different properties of the fractional Brownian motion sample-paths.

Since for $H > 1/2$ the fBm is a Dirichlet process, (1) can be given a sense using the approach developed by Föllmer (1980).

Since the fBm has $1/H$ bounded variation, one can use the work of Bertoin (1989) in which it is proved that (1) converges whenever u has $1/\beta$–bounded variation with $\beta + H > 1$ and $\beta \geq 2$. In the same vein, one can also cite the papers Dai and Heyde (1996), Lin (1996), which consider more specifically the case of the fractional Brownian motion. Young (1936) and more recently, Lyons (1994), have extended these results so that they are now applicable to the fractional Brownian motion of Hurst parameter less than 1/2.

Using the Hölder continuity of the sample-paths of W^H, when $H > 1/2$, Feyel and de la Pradelle (1996) also prove that (1) converges when u is β-Hölder continuous with $\beta + H > 1$ [see Feyel and de la Pradelle (1996)].

One could also mention the works of Ciesielski, Kerkyacharian and Roynette (1993), where a pathwise stochastic integral is defined through the notion of Besov spaces.

The approach we use here, valid for any $H \in (0, 1)$ and which has been set in DU (1998), rests on the fact that W^H is a Gaussian process so that we have at our disposal the so-called Malliavin calculus framework (or stochastic calculus of variation). For an introduction to this theory, we refer to Nualart (1995), Üstünel (1995); for details specific to fBm we refer to DU (1998).

We summarize the notations and results of DU (1998) in the multidimensional case. Let $d \in \mathbf{N}^*$ and $I = [0, T]$, $W = C_0(I, \mathbf{R}^d)$ be the real separable Banach space of continuous $\mathbf{R}^d$-valued applications , null at time 0, equipped with the supremum norm. W^* denotes the strong topological dual of W. Hereafter $\bar{H} = (H_1, \ldots, H_d)$ is fixed in $(0, 1)^d$ and $\mathbf{P}_{\bar{H}}$ is the unique probability measure on W such that the canonical process $\{W_t = (W_t^i,\ i = 1, \ldots, d),\ t \in [0, T]\}$ has independent coordinates which are one-dimensional fBm of Hurst index H_i for $i = 1, \ldots, d$. The canonical filtration is $\mathscr{F} = \{\mathscr{F}_t^{\bar{H}} = \sigma\{W_u, u \leq t\} \vee \mathscr{N}_{\bar{H}},\ t \in I\}$, where $\mathscr{N}_{\bar{H}}$ stands for the set of null sets of $(W, \mathbf{P}_{\bar{H}})$. The Cameron–Martin space (also called the reproducing kernel Hilbert space) of this process, denoted by $\mathscr{H}$, can be identified [see DU (1998) for the one-dimensional case] as the space of functions $h = (h^i)_{i=1,\ldots,d}$ of the form

$$h^i(t) \stackrel{\text{def}}{=} \int_0^t K_{H_i}(t, s)\dot{h}^i(s)\, ds \stackrel{\text{def}}{=} K_{H_i}(\dot{h}^i)(t), \tag{2}$$

where $\dot{h}^i$ belongs to $L^2(I)$ for $i = 1, \ldots, d$ and for any $H \in (0, 1)$,

$$K_H(t, r) = \frac{(t-r)^{H-1/2}}{\Gamma(H + \frac{1}{2})} F\left(\frac{1}{2} - H, H - \frac{1}{2}, H + \frac{1}{2}, 1 - \frac{t}{r}\right) 1_{[0,t)}(r). \tag{3}$$

The Gauss hypergeometric function $F(\alpha, \beta, \gamma, z)$ [see Nikiforov and Uvarov (1988)] is the analytic continuation on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}\backslash\{-1, -2, \ldots\} \times \{z \in \mathbb{C},$

$\mathrm{Arg}|1 - z| < \pi\}$ of the power series

$$\sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} z^k.$$

Here $(\alpha)_k$ denotes the Pochhammer symbol defined by

$$(a)_0 = 1 \quad \text{and} \quad (a)_k \stackrel{\text{def}}{=} \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1).$$

The norm on $\mathscr{H}$ is given by

$$\|h\|_{\mathscr{H}}^2 \stackrel{\text{def}}{=} \sum_{i=1}^{d} \|K_{H_i}^{-1}(h^i)\|_{L^2(I)}^2.$$

Hereafter, the same symbol denotes an integral operator and the kernel defining it. For instance, in the last equation, $K_{H_i}^{-1}$ is the inverse of the operator K_{H_i}. From its very definition we know that

$$K_H(t,r) = \frac{r^{1/2-H}}{\Gamma(H-1/2)} \int_r^t u^{H-1/2}(u-r)^{H-3/2}\,du\,\mathbf{1}_{[0,t]}(r) \quad \text{for } H > 1/2 \tag{4}$$

and that [cf. Samko, Kilbas and Marichev (1993), Theorem 10.4]

$$K_H f = I_{0^+}^{2H} x^{1/2-H} I_{0^+}^{1/2-H} x^{H-1/2} f \quad \text{for } H \le 1/2, \tag{5}$$

$$K_H f = I_{0^+}^{1} x^{H-1/2} I_{0^+}^{H-1/2} x^{1/2-H} f \quad \text{for } H \ge 1/2. \tag{6}$$

These formula together with some results of Samko, Kilbas and Marichev (1993) are the tools to show the properties of K_H really needed for our considerations. Actually, most of this work and related ones [CD (1997), DU (1998)] can be done for a wide class of Gaussian processes. It is sufficient that the covariance kernel R has a triangular square root K; that is,

$$R(t,s) = \int_0^{t\wedge s} K(t,r)K(s,r)\,dr, \tag{7}$$

$$K(t,s) = 0 \quad \text{if } s > t, \tag{8}$$

which satisfies properties similar to those below.

THEOREM 2.1. *For any $H \in (0,1)$, K_H satisfies the following properties*:

(i) *K_H is a Hilbert–Schmidt operator on $L^2(I;\mathbf{R})$ and maps continuously $L^2(I)$ onto a dense subset of W, namely $I_{0^+}^{H+1/2}(L^2(I;\mathbf{R}))$.*

(ii) *There exists a constant c_H such that for any t, s in I,*

$$K_H(t,s) \le c_H s^{-|H-1/2|}(t-s)^{-(1/2-H)^+}.$$

(iii) *For any* $H > 1/2$ *and any* $0 \le \gamma \le H - 1/2$,

$$I_{0^+}^{-\gamma}(K_H(\cdot, s))(t) = \frac{(t-s)^{H-1/2-\gamma}}{\Gamma(H+\frac{1}{2}-\gamma)} F\left(\frac{1}{2}-H, H, -\frac{1}{2}, H+\frac{1}{2}-\gamma, 1-\frac{t}{s}\right) 1_{[0,t)}(s)$$

and

$$\left|I_{0^+}^{-\gamma}(K_H(\cdot, s))(t)\right| \le c_{H,\gamma} s^{-|H-1/2|} \quad \text{for any } t,\ s \in I.$$

(iv) $t \mapsto K_H(t, s)$ *is* $(H - 1/2)$*-Hölder continuous for* $H > 1/2$ *and any* s.

(v) $K'_H =_{\text{def}} I_{0^+}^{-1} K_H$ *is a continuous map from* $I_{0^+}^{(1/2-H)^+}(L^2(I))$ *onto the space* $I_{0^+}^{(H-1/2)^+}(L^2(I))$.

NOTATION 1. For any $\lambda \ge 0$,

$$\text{Hol}(\lambda) = \left\{ f: I \to \mathbf{R}; \|f\|_{\text{Hol}(\lambda)} \overset{\text{def}}{=} \sup_{s \ne t} |f(t) - f(s)|(t-s)^{-\lambda} < +\infty \right\}$$

and for any f, the integrals

$$I_{a^+}^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1}\,dt, \qquad x \ge a,$$

$$I_{b^-}^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(x-t)^{\alpha-1}\,dt, \qquad x \le b,$$

where $\alpha > 0$ are, respectively, called right and left fractional integrals of order α. When $\alpha < 0$,

$$I_{a^+}^{\alpha} f \overset{\text{def}}{=} \frac{d^n}{dt^n}\left(I_{a^+}^{\alpha+n} f\right) \quad \text{where } n = -[\alpha].$$

PROOF OF THEOREM 2.1. Equation (7) is actually the definition of K_H. Equations (5), (6) and property (i) follow from Samko, Kilbas and Marichev (1993), Theorem 10.4. The upper bound (ii) has been proved in DU (1998).

For $H > 1/2$, (v) is a direct consequence of (6). For $H < 1/2$, from (5) and Lemma 10.1 of Samko, Kilbas and Marichev (1993), which stands that for any $u \in L^2(I)$ [respectively, $v \in L^2(I; s^{1-2H}\,ds)$] there exists $\hat{u} \in L^2(I)$ [respectively, $\tilde{v} \in L^2(I)$] such that

$$x^{H-1/2} I_{0^+}^{1/2-H} u = I_{0^+}^{1/2-H} x^{H-1/2} \hat{u}$$

and

$$I_{0^+}^{2H-1} x^{1/2-H} v = x^{1/2-H} I_{0^+}^{2H-1} \tilde{v},$$

respectively. It follows that

$$(K_H f)' = I_{0^+}^{2H-1} x^{1/2-H} I_{0^+}^{1-2H}(\widehat{I_{0^+}^{H-1/2} f}) = x^{1/2-H}(\widetilde{\widehat{I_{0^+}^{H-1/2} f}}),$$

again by Lemma 10.1 of Samko, Kilbas and Marichev (1993).

By (4) and Fubini's theorem,

$$
\begin{aligned}
&I_{0^+}^{-\gamma}(K_H(\cdot, s))(t)\\
&\quad= \frac{d}{dt} I_{0^+}^{1-\gamma}\biggl(\frac{s^{1/2-H}}{\Gamma(H-1/2)}\int_s^{\bullet} u^{H-1/2}(u-s)^{H-3/2}\,du\,\mathbf{1}_{[0,\cdot]}(s)\biggr)(t)\\
&\quad= \frac{s^{1/2-H}}{\Gamma(H-1/2)\Gamma(1-\gamma)}\frac{d}{dt}\biggl(\int_0^t (t-r)^{-\gamma}\int_s^r u^{H-1/2}(u-s)^{H-3/2}\,du\,dr\biggr)(t)\\
&\quad= \frac{s^{1/2-H}(1-\gamma)^{-1}}{\Gamma(H-1/2)\Gamma(1-\gamma)}\frac{d}{dt}\biggl(\int_s^t u^{H-1/2}(u-s)^{H-3/2}(t-u)^{1-\gamma}\,du\biggr)(t)\\
&\quad= \frac{s^{1/2-H}}{\Gamma(H-1/2)\Gamma(1-\gamma)}\int_s^t u^{H-1/2}(u-s)^{H-3/2}(t-u)^{-\gamma}\,du\\
&\quad= \frac{(t-s)^{H-1/2-\gamma}}{\Gamma(H+\frac{1}{2}-\gamma)}F\biggl(\frac{1}{2}-H, H-\frac{1}{2}, H+\frac{1}{2}-\gamma, 1-\frac{t}{s}\biggr)\mathbf{1}_{[0,\,t)}(s)
\end{aligned}
$$

and the first part of (iii) follows. The second part is a consequence of the regularity properties of the hypergeometric functions [see, for instance, Nikiforov and Uvarov (1988)]. As a corollary, for any $s \ge 0$ and any $0 \le \gamma \le H - 1/2$, the function $t \mapsto K_H(t, s)$ belongs to $I_{0^+}^{\gamma}(L^\infty)$ and by Besov space embeddings [see Samko, Kilbas and Marichev (1993), Feyel and de la Pradelle (1996)] it also belongs to $\mathrm{Hol}(H - 1/2)$ and

$$\bigl|K_H(t, s) - K_H(r, s)\bigr| \le c s^{-|H-1/2|}(t-s)^{H-1/2}$$

and thus property (iv) is proved. □

The definition of the so-called divergence and hence of our integral needs the definition of the Gross–Sobolev derivative.

DEFINITION 2.2. Let X be a separable Hilbert space. For an X-valued smooth cylindric functional F of the form $F(\omega) = f(l_1(\omega), \dots, l_n(\omega))x$, where f belongs to the Schwartz space on $\mathbf{R}^n$, $l_1, \dots, l_n$ to W^* and x to X, the Gross–Sobolev derivative of F, denoted by ∇F, is defined by

$$\nabla F(\omega) = \sum_{j=1}^{d}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(l_1(\omega), \dots, l_n(\omega))\,R_{H_j}(l_i^j) \otimes x\,\mathbf{e}_j,$$

where $\{\mathbf{e}_j,\ j = 1, \dots, d\}$ is the canonical basis of $\mathbf{R}^d$ and for any $l \in W^*$, l^j is the jth component of l, that is,

$$l(w) = \sum_{j=1}^{d} l^j(\langle w, \mathbf{e}_j\rangle_{\mathbf{R}^d}).$$

For any $p \ge 1$ and any $k \ge 0$, the Sobolev space $\mathbb{D}_{p,\,k,\,\bar{H}}(X)$ is the completion of the set $\mathscr{S}$ of cylindric functionals with respect to the norm,

$$\|F\|_{p,\,k,\,\bar{H}} \stackrel{\text{def}}{=} \|F\|_{L^p} + \||\nabla^{(k)}F|_{HS}\|_{L^p},$$

where $|\cdot|_{HS}$ stands for the Hilbert–Schmidt norm. For any $k<0$, $\mathbb{D}_{p,k,\bar{H}}(X)$ is the strong dual of $\mathbb{D}_{p,-k,\bar{H}}(X)$. The space $\mathbb{D}_{\infty}(X)=\bigcap_{p\geq 1,k>0}\mathbb{D}_{p,k,\bar{H}}(X)$ is the space of test functions and the space $\mathbb{D}_{-\infty}(X)=\bigcup_{p\geq 1,k<0}\mathbb{D}_{p,k,\bar{H}}(X)$ is the space of distributions. It is well known that ∇ can be extended as a continuous operator from $\mathbb{D}_{p,k,\bar{H}}$ into $\mathbb{D}_{p,k-1,\bar{H}}(\mathscr{H})$, for any $p\geq 1$ and any k.

NOTATION 2. By $\dot{\nabla}F$, we mean the inverse image of ∇F by the map $K_{\bar{H}}$ where

$$K_{\bar{H}}\colon L^2(I;\mathbf{R}^d)\to\mathscr{H}\overset{\text{not}}{=}\bigotimes_{i=1}^{n}\mathscr{H}_i$$
$$u=(u_i,\,1,\dots,d)\mapsto(K_{H_i}(u_i),\,1,\dots,d).$$

Moreover, for any one Gross–Sobolev differentiable random variable φ, for any $\psi\in L^0(\mathscr{H})$, we set

$$\nabla_{\psi}\varphi\overset{\text{def}}{=}\langle\nabla\varphi,\,\psi\rangle_{\mathscr{H}}.$$

Recall that for any u and v in $\mathbb{D}_{2,1}(\mathscr{H})$, the trace of $\nabla u\circ\nabla v$ is defined by

$$\operatorname{trace}(\nabla u\circ\nabla v)=\sum_{n,m\geq 0}\langle\nabla_{f_n}u,\,f_m\rangle_{\mathscr{H}}\langle\nabla_{f_m}v,\,f_n\rangle_{\mathscr{H}},$$

where $\{f_n=(f_n^1,\dots,f_n^d),\ n\geq 0\}$ is an orthonormal basis of $\mathscr{H}$. Expanding the scalar-product in $\mathscr{H}$ as the sum of scalar products in $\mathscr{H}_i$, we obtain

$$\operatorname{trace}(\nabla u\circ\nabla v)=\sum_{n,m\geq 0}\biggl(\sum_{i,j=1}^{d}\langle\nabla^j_{f_n^j}u^i,\,f_m^i\rangle_{\mathscr{H}_i}\biggr)\biggl(\sum_{i,j=1}^{d}\langle\nabla^j_{f_m^j}v^i,\,f_n^i\rangle_{\mathscr{H}_i}\biggr).\tag{9}$$

We say that a process u belongs to $\operatorname{Dom}_p\delta_{\bar{H}}$ when there exists c such that for any $F\in\mathbb{D}_{p^*,1,\bar{H}}$,

$$|\mathbf{E}[\langle\nabla F,u\rangle_{\mathscr{H}}]|\leq c\|F\|_{L^{p^*}(W)}$$

and we define $\delta_{\bar{H}}u$ by

$$\mathbf{E}[\langle\nabla F,u\rangle_{\mathscr{H}}]=\mathbf{E}[F\cdot\delta_{\bar{H}}u].$$

NOTATION 3. Here and hereafter, we denote by p^* the conjugate of p, that is, $1/p^*+1/p=1$.

Here $\delta_{\bar{H}}$, called the divergence operator, is a continuous operator from $\mathbb{D}_{p,k,\bar{H}}(\mathscr{H})$ into $\mathbb{D}_{p,k-1,\bar{H}}$. Since in the case of the standard Brownian motion ($d=1$, $\bar{H}=(1/2)$), the divergence coincides with the Skohorod integral, which itself is an extension of the Itô classical stochastic integral, it is somewhat natural to use the notion of divergence to define a stochastic integral with respect to the fractional Brownian motion. Henceforth, we set for any (not necessarily adapted) process $u = (u^i)_{i=1,\dots,d} \in L^2(W \times I, \mathbf{R}^d)$,

$$\int_0^1 u_s \delta_{\bar{H}} W_s \overset{\text{def}}{=} \delta_{\bar{H}}(K_{\bar{H}} u),$$

where

$$K_{\bar{H}} u \overset{\text{def}}{=} (K_{H_i} u_i,\ i = 1, \dots, d).$$

Following is a list of useful properties of $\delta_{\bar{H}}$, all excerpted from DU (1998).

1. The process

(P1) $$\left\{ \int_0^t (K_{H_i}(t,s))_{i=1,\dots,d}\ \delta_{\bar{H}} W_s = \delta_{\bar{H}}\big((R_{H_i}(t,\cdot))_{i=1,\dots,d}\big),\ t \in I \right\}$$

   is nothing but the d-dimensional fractional Brownian motion $\{(W_t^i,\ i = 1, \dots, d),\ t \in I\}$.
2. The process

(P2) $$B \overset{\text{def}}{=} \{(B_t^i)_{i=1,\dots,d},\ t \in I\} = \left\{ \int_0^t (\mathbf{1}_{[0,t]}(s))_{i=1,\dots,d}\ \delta_{\bar{H}} W_s,\ t \in I \right\}$$

   is a standard Brownian motion constructed on $(W, (\mathscr{F}_t)_{t \ge 0}, \mathbf{P}_{\bar{H}})$. Moreover, B and W have the same filtration.
3. In fact, we have for any adapted processes $u \in \operatorname{Dom} \delta_{\bar{H}}$,

(P3) $$\int_0^t u_s\, \delta_{\bar{H}} W_s = \sum_{i=1}^d \int_0^t u_s^i\, dB_s^i.$$

   It is thus justifiable to use the notation $\int u_s\, dB_s$ instead of $\delta_{\bar{H}}(K_H u)$. When u is anticipating, $\int u_s\, dB_s$ has to be understood as a Skohorod integral.
4. As a consequence, for u adapted and $u \in L^2(W \times I, \mathbf{R}^d)$, the process

$$t \mapsto \sum_{i=1}^d \int_0^t K_{H_i}(s,r) u_r^i\, dB_r^i$$

   is a martingale for any s and thus the Burkholder–Davis–Gundy inequality entails that for all $p > 1$, there exists $C_p > 0$ such that

(P4) $$\mathbf{E}\left[\left| \sum_{i=1}^d \int_0^t K_{H_i}(t,s) u_s^i\, dB_s^i \right|^p \right] \le C_p\, \mathbf{E}\left[\left| \int_0^t \sum_{i=1}^d K_{H_i}^2(t,s) (u_s^i)^2\, ds \right|^{p/2} \right].$$

COROLLARY 2.1. *Let $H \in (0,1)$ and u be adapted processes such that for some $p > 1/H$,*

$$\mathbf{E}\Big[\sup_{s \in I} |u_s|^p \Big] < \infty,$$

and $(Z_t,\ t \in I)$ be the process defined by

(10) $$Z_t = \int_0^t K_H(t,s) u_s\, dB_s.$$

The process Z admits a version with Hölder continuous sample-paths of any order less than H.

PROOF. Using Property (P4), we have

$$\begin{aligned}\mathbf{E}\big[|Z_t - Z_s|^p\big] &\le c_p \mathbf{E}\bigg[\bigg|\int_I |K_H(t,u) - K_H(s,u)|^2 |u_s|^2\,ds\bigg|^{p/2}\bigg]\\ &\le c_p V_H \mathbf{E}\Big[\sup_{s\in I}|u_s|^p\Big]|t-s|^{pH}.\end{aligned}$$

The result follows by the Kolmogorov criterion. □

LEMMA 2.1. *Let $H \in (0,1)$ and choose $p \in (1,2)$. Set*

$$\mathscr{L}_H^p \overset{\text{def}}{=} L^p(I; x^{p(H+3/2)-2}\,dx) \text{ and } \mathscr{L}_H^{p*} = L^{p^*}(I; x^{-p^* p^{-1}((H+3/2)-2)}\,dx).$$

Furthermore, let u be an $\mathbf{R}^N$-valued, adapted process such that

$$\mathbf{E}\big[\|u\|_\infty^{p^*}\big] < +\infty,$$

where p^* is the conjugate of p.

(i) *The operator $\mathscr{I} =_{\text{def}} I^1_{0^+} \circ K_H^{-1}$ is a continuous bijective map from*

$$\mathscr{L}_H^{p*} \quad \textit{into} \quad I_{0^+}^{1/2-H}(\mathscr{L}_H^{p*}).$$

(ii) *The process $\{Z_t = \int_0^t K_H(t,s)u_s\,dB_s,\ t \in I\}$ belongs to $\mathscr{L}_H^{p*}$ and $= \mathscr{I}(Z)$ has a modification equal to $\int_0^\bullet u_s\,dB_s$, $dt \otimes d\mathbf{P}_{\bar{H}}$ a.e.*

PROOF. From Samko, Kilbas and Marichev [(1993), page 188], it is known that K_H^* maps $\mathscr{L}_H^p$ into $I_{1^-}^{H+1/2}(\mathscr{L}_H^p)$ and is a bijection. It follows that $\mathscr{I} = (I_{1^-}^{-1}K_H^*)^{*-1}$ is a continuous bijective map from $(\mathscr{L}_H^{p^*})^* = \mathscr{L}_H^p$ onto $(I_{1^-}^{H+1/2}(\mathscr{L}_H^p))^* = I_{0^+}^{1/2-H}(\mathscr{L}_H^{p^*})$ and the first point follows.

Moreover, using Property (P4),

$$\begin{aligned}&\mathbf{E}\bigg[\int_I \bigg|\int_0^t K_H(t,s)u_s\,dB_s\bigg|^q t^{-qp^{-1}((H+3/2)-2)}\,dt\bigg]\\ &\quad\le \mathbf{E}\bigg[\int_I \bigg|\int_0^t K_H(t,s)^2 u_s^2\,ds\bigg|^{q/2} t^{-qp^{-1}((H+3/2)-2)}\,dt\bigg]\\ &\quad\le c\mathbf{E}\Big[\sup_{s\in I}|u_s|^q\Big]\int_I t^{-qp^{-1}((H+3/2)-2)+qH}\,dt < +\infty,\end{aligned}$$

since $p < 2$ and $H \in (0,1)$. For any $I_{1^-}^{H-1/2}(\mathscr{L}_H^p)$-valued cylindric functional G, using stochastic integration by parts and ordinary Fubini's theorem, we have

$$
\begin{aligned}
&\mathbf{E}\Big[\langle \mathscr{I}(Z),\ G\rangle_{I_{0^+}^{1/2-H}(\mathscr{L}_H^{p^*}),I_{1^-}^{H-1/2}(\mathscr{L}_H^{p})}\Big]\\
&\quad=\mathbf{E}\Big[\langle Z,\ \mathscr{I}^*(G)\rangle_{\mathscr{L}_H^{p^*},\mathscr{L}_H^{p}}\Big]\\
&\quad=\mathbf{E}\bigg[\int_I\int_0^t K_H(t,s)u_s\dot{\nabla}_s(\mathscr{I}^*(G))(t)\,ds\,dt\bigg]\\
&\quad=\mathbf{E}\bigg[\int_I u_s\int_s^1 K_H(t,s)(\mathscr{I}^*(\dot{\nabla}_sG))(t)\,dt\,ds\bigg]\\
&\quad=\mathbf{E}\bigg[\int_I u_sK_H^*\mathscr{I}^*(\dot{\nabla}_sG)(s)\,ds\bigg]=\mathbf{E}\bigg[\int_I u_sI_{1^-}^1(\dot{\nabla}_sG)(s)\,ds\bigg]\\
&\quad=\mathbf{E}\bigg[\int_I\int_0^r u_s\dot{\nabla}_sG\,ds\,dr\bigg]=\mathbf{E}\bigg[\int_I\int_0^r u_s\,dB_s\,G(r)\,dr\bigg].
\end{aligned}
$$

Hence $d\mathbf{P}_{\bar{H}}\otimes dt$ almost everywhere; we have

$$
\mathscr{I}(Z)(t)=\int_0^t u_s\,dB_s \text{ or equivalently } Z_t=(K_H\circ I_{0^+}^{-1})\bigg(\int_0^\bullet u_s\,dB_s\bigg)(t). \qquad \square
$$

3. Application to the filtering theory. As mentioned in the introduction, our purpose is to study the conditional law, $\{\pi_t,\ t\in I\}$, of a signal process $\{X_t,\ t\in I\}$ given the past of an observed process $\{Y_t,\ t\in I\}$, namely $\pi_t(f)=\mathbf{E}[f(X_t)|\sigma(Y_s,s\le t)]$ for any regular function f when X,Y are solutions of the following system of equations: for any $l\in\{1,\dots,M\}$ and any $k\in\{1,\dots,N\}$,

$$
(S_0)\qquad
\begin{aligned}
X_t^l&=x_0^l+\sum_{i=1}^M\int_0^t K_{H_i}(t,s)b^{l,i}(X_s)\,ds+\int_0^t K_{H_i}(t,s)a^{l,i}(X_s)\,dB_s^i,\\
Y_t^k&=\int_0^t K_H(t,s)h^k(X_s)\,ds+\sum_{j=M+1}^d\int_0^t K_H(t,s)\tau^{k,j}(Y_s)\,dB_s^j,
\end{aligned}
$$

where $d=N+M$, for any $i\in\{1,\dots,d\}$, $\{B_t^i,\ t\in I\}$ is the Brownian motion mentioned in properties (P_2) and (P_3) and for any $i\in\{1,\dots,M\}$, any $j\in\{M+1,\dots,d\}$, any $l\in\{1,\dots,M\}$, any $k\in\{1,\dots,N\}$, $b^{l,i}$, $a^{l,i}$, h^k and $\tau^{k,j}$ are regular deterministic functions.

HYPOTHESIS I. For technical reasons (see Theorems B.1 and 4.1), we assume hereafter that for any $i\in\{1,\dots,d\}$,

$$
H_i\ge 1/2 \quad\text{and}\quad H=H_{M+1}=\cdots=H_d.
$$

REMARK 3.1. A $K_H(t,s)$ factor in the stochastic integrals is mandatory to have meaningful equations, since $W_t=\int_0^t K_H(t,s)\,dB_s$. It is also necessary in the drift part of Y in order to apply the Girsanov theorem; see Theorem 3.1. In opposition, the $K_H(t,s)$ term in the drift part of X is just here to symmetrize the role of X and Y. Existence and uniqueness of such a system of equations are a straightforward generalization of the work done in CD (1997), hence we postpone its development to Section 4.

Let $(W, \mathscr{F}, (\mathscr{F}_t,\ t \in I), \mathbf{P}_{\bar{H}})$, the canonical probability space defined in the introduction. Consider the following d-dimensional fbm-SDE:

$$(S_1)\qquad \begin{aligned} X_t^l &= x_0^l + \sum_{i=1}^{M} \int_0^t K_{H_i}(t,s) b^{l,i}(X_s)\, ds + \int_0^t K_{H_i}(t,s) a^{l,i}(X_s)\, dB_s^i, \\ Y_t^k &= \sum_{j=M+1}^{d} \int_0^t K_H(t,s) \tau^{k,j}(Y_s)\, dB_s^j, \end{aligned}$$

where we have the following hypothesis.

HYPOTHESIS II. Here x_0^l belongs to $\mathbf{R}$, the $\mathbf{R}^M$-valued applications $b^i = (b^{l,i})_{l=1}^M$, $a^i = (a^{l,i})_{l=1}^M$, and the $\mathbf{R}^N$-valued map $\tau^j = (\tau^{k,j})_{k=1}^N$ are bounded, twice differentiable with bounded derivatives for any $i = 1, \dots, M$, and any $j = 1 + M, \dots, d$.

Theorem A.1 and Hypothesis II ensure the existence of a unique pair (X, Y) of continuous adapted processes strong solution of (S_1).

HYPOTHESIS III. Then $h\colon \mathbf{R}^N \to \mathbf{R}^N$ belongs to $C_b^1(\mathbf{R}^N, \mathbf{R}^N)$ and there exists a constant $\lambda > 0$ such that $\tau\tau^T(y) \ge \lambda I_N$, for any $y \in \mathbf{R}^N$.

DEFINITION 3.1. Let $\mathbf{P}_{\bar{H}}^h$ be the probability measure on $(W, \mathscr{F})$ defined by

$$\left.\frac{d\mathbf{P}_{\bar{H}}^h}{d\mathbf{P}_{\bar{H}}}\right|_{\mathscr{F}_t} \overset{\text{def}}{=} \mathbf{E}(L_T | \mathscr{F}_t) \overset{\text{not}}{=} L_t,$$

where

$$L_t = \exp\left[\int_0^t \sum_{j=M+1}^{d} (\tau^{-1}(Y_s) h(X_s))^j\, dB_s^j - \tfrac{1}{2} \int_0^t \|\tau^{-1}(Y_s) h(X_s)\|^2\, ds\right].$$

NOTATION 4. Consider the processes

$$\tilde{W}_t^i = W_t^i \quad \text{for } i = 1, \dots, M,$$

$$\tilde{W}_t^j = W_t^j - \sum_{k=M+1}^{d} \int_0^t K_H(t,s) (\tau^{-1}(Y_s))^{j,k} h^k(X_s)\, ds, \qquad j = M+1, \dots, d.$$

THEOREM 3.1. *Assume that hypothesis* II *and* III *hold*:

(i) *The law of the* $\mathbf{R}^d$*-valued process* $\{(\tilde{W}_t^i,\ i = 1, \dots, d),\ t \in I\}$ *under the probability measure* $\mathbf{P}_{\bar{H}}^h$ *is the same as the law of the canonical process* $\{(W_t^i,\ i = 1, \dots, d),\ t \in I\}$ *under* $\mathbf{P}_{\bar{H}}$.

(ii) *The process* $\{(X_t, Y_t),\ t \in I\}$, *under the probability measure* $\mathbf{P}^h_{\bar{H}}$, *has the same law as the process* $\{(\tilde{X}_t, \tilde{Y}_t),\ t \in I\}$ *strong solution of* (S_0).

PROOF. (i) Set $\bar{h}(x) = \sum_{k=1}^N h^k(x)\mathbf{e}_{k+M}$ and let $\{t_1, \dots, t_m\}$ be fixed in $[0,1]$. Consider the $d \cdot m$-dimensional square integrable martingale

$$Z_r^{t_1,\dots,t_m} \stackrel{\text{def}}{=} \left(\int_0^r K_{H_l}(t_j, s)\, dB_s^l;\ l = 1, \dots d,\ \ j = 1, \dots, m \right)$$

and the $\mathbf{R}^{d\cdot m}$-valued process

$$A_r^{t_1,\dots,t_m} \stackrel{\text{def}}{=} \left(\int_0^r K_{H_l}(t_j, s)\bar{h}^l(X_s)\, ds;\ l = 1, \dots d,\ \ j = 1, \dots, m \right).$$

The classical Girsanov theorem for multidimensional Brownian motion stands that $Z^{t_1,\dots,t_m} - A^{t_1,\dots,t_m}$ has under $\mathbf{P}^h_{\bar{H}}$ the same law as $Z^{t_1,\dots,t_m}$ has under $\mathbf{P}_{\bar{H}}$; in particular,

$$\mathbf{E}_h[f(Z_r^{t_1,\dots,t_m} - A_r^{t_1,\dots,t_m})] = \mathbf{E}[f(Z_r^{t_1,\dots,t_m})],$$

for any r and any bounded f from $\mathbf{R}^m \otimes \mathbf{R}^d$ into $\mathbf{R}$. Take $r = \max(t_1, \dots, t_m)$; it follows that

$$\mathbf{E}_h[f(, \dots, W_{t_j}^l - K_{H_l}(\bar{h}^l \circ X)(t_j), \dots,)] = \mathbf{E}[f(, \dots, W_{t_j}^l, \dots,)],$$

where $\mathbf{E}_h$ denotes the expectation under $\mathbf{P}^h_{\bar{H}}$ and the first point follows.

(ii) For any $t \in I$, (X_t, Y_t) is the end value of the $(\mathbf{P}_{\bar{H}}, \mathscr{F})$ semimartingale $\{(X_t(r), Y_t(r)),\ r \in I\}$ defined by

$$X_t^l(r) = x_0^l + \sum_{i=1}^M \int_0^r K_{H_i}(t,s) b^{l,i}(X_s)\, ds + \sum_{i=1}^M \int_0^r K_{H_i}(t,s) a^{l,i}(X_s)\, dB_s^i,$$

$$Y_t^k(r) = \sum_{j=M+1}^d \int_0^r K_H(t,s)\tau^{k,j}(Y_s) dB_s^j \quad \text{for any } l \text{ and } k.$$

The classical Girsanov theorem stands that $\{(X_t(r), Y_t(r),\ r \in I\}$ is a $(\mathbf{P}^h_{\bar{H}}, \mathscr{F})$ semimartingale with the following decomposition:

$$X_t^l(r) = x_0^l + \sum_{i=1}^M \int_0^r K_{H_i}(t,s) b^{l,i}(X_s)\, ds + \sum_{i=1}^M \int_0^r K_{H_i}(t,s) a^{l,i}(X_s)\, dB_s^i,$$

$$Y_t^k(r) = \int_0^r K_H(t,s) h^k(X_s) ds + \sum_{j=M+1}^d \int_0^r K_H(t,s)\tau^{k,j}(Y_s)\, d\tilde{B}_s^j,$$

where

$$\tilde{B}_t^j = B_t^j - \int_0^t (\tau^{-1}(Y_s) h(X_s))^j\, ds \quad \text{for } j = M+1, \dots, d$$

are independent standard Brownian motions. Taking $r = T$, we see that $\{(X_t, Y_t),\ t \in I\}$ is a weak solution on $(W, \mathscr{F}, \mathbf{P}^h_{\bar{H}})$ of the system (S_0) with directing processes $\{B^i, \tilde{B}^j;\ i = 1, \dots, M,\ \ j = M+1, \dots, d\}$. Since pathwise uniqueness holds for (S_0) by Theorem A.1, by a straightforward generalization of Proposition 3.20 of Revuz and Yor (1994), we obtain the uniqueness of the solution in law of the system (S_0) and the second point follows. □

We recall now the usual notations which we use hereafter.

NOTATION 5.

$$\mathscr{Y}_t = \sigma\{Y_s,\ s \le t\}, \qquad t \in I,\ \mathscr{Y} = \vee_{t \in I} \mathscr{Y}_t,$$

$$\mathscr{F}^Y_t = \sigma\{W^j_s,\ s \le t,\ j = M+1, \dots, d\}, \qquad t \in I,\ \mathscr{F}^Y = \vee_{t \in [0,T]} \mathscr{F}^Y_t,$$

$$\sigma_t(f) = \mathbf{E}_h[f(X_t) L_t | \mathscr{Y}], \qquad \pi_t(f) = \mathbf{E}[f(X_t) | \mathscr{Y}_t].$$

Here $\{\sigma_t,\ t \in I\}$ is the so-called unnormalized filter and $\{\pi_t,\ t \in I\}$ is the normalized filter; they both operate on bounded continuous real-valued functions and the well-known Kallianpur–Striebel formula holds; for all $t \in I$ $\mathbf{P}_{\bar{H}}$ and $\mathbf{P}^h_{\bar{H}}$ almost everywhere,

$$\pi_t(f) = \frac{\sigma_t(f)}{\sigma_t(1)}.$$

LEMMA 3.1. *Let U and V be two processes and $\alpha \in \mathbf{R}$; suppose that $U = I^\alpha_{0^+}(V)$ then U and V have the same filtration up to the evanescent sets of $(W, \mathbf{P}_{\bar{H}})$.*

PROOF. By symmetry it is sufficient to show that for any $t \in I$, U_t is $\sigma\{V_s,\ s \le t\}$-measurable. When $\alpha > 0$, it is clear that U_t is almost surely the limit of

$$\frac{1}{\Gamma(\alpha)} \int_0^{t-1/n} (t-s)^{\alpha-1} V_s \, ds$$

as n goes to $+\infty$. Each of these integrals (i.e., for n fixed) can be approximated by a Riemann sum involving values of V up to time $t - 1/n$ and hence is $\sigma\{V_s,\ s \le t\}$-measurable and so does U_t. When $\alpha < 0$, by definition of I^α,

$$U_t = \frac{d^n}{dt^n} I^{\alpha+n}_{0^+}(V)(t),$$

where $n = -[\alpha]$. Since any derivative can be computed as the limit of left-sided increments, it follows that U_t is $\sigma\{V_s,\ s \le t\}$-measurable for any $t \in I$. □

THEOREM 3.2. *If hypotheses* II *and* III *are fulfilled, then the σ-field* $\{\mathscr{Y}_t, t \in I\}$ *is identical to the σ-field* $\{\mathscr{F}_t^Y, t \in I\}$ *up to the evanescent sets of* $(W, \mathbf{P}_{\bar{H}})$.

PROOF. Denote by $\{\mathscr{F}_t^{\tilde{Y}}, t \geq 0\}$ the σ-field generated by the sample-paths of $\tilde{Y} =_{\text{def}} \sum_{j=M+1}^d \int_0^\bullet \tau(Y_s)^{\cdot,j}\, dB_s^j$. Since $Y = \mathscr{I}^{-1}(\tilde{Y})$, by Lemma 2.1 and a previous lemma, it follows that $\mathscr{Y} = \mathscr{F}^Y$. Moreover, by construction, we have $\mathscr{F}_t^Y = \sigma\{B_s^j, s \leq t,\ j = M+1, \dots, d\}$ hence by Kallianpur [(1980), page 219], we know that $\mathscr{F}^{\tilde{Y}} = \mathscr{F}^Y$ and the proof is thus complete. □

We now construct a regular version of the conditional probability of $\mathbf{P}_{\bar{H}}$ given the filtration $\mathscr{Y}$, which, following Pardoux (1989), enables us to treat $\{\mathbf{P}_{\bar{H}}, (\mathscr{Y}_t, t \in I)\}$ as a conditional expectation. Denote by p^X and p^Y the projections

$$p^X\colon \mathbf{R}^{M+N} \to \mathbf{R}^M, \qquad p^Y\colon \mathbf{R}^{M+N} \to \mathbf{R}^N$$

$$x = (x_1, \dots, x_M, \dots, x_d) \mapsto (x_1, \dots, x_M), \qquad x \mapsto (x_{M+1}, \dots, x_d).$$

Set $\mathbf{P}_H^{\mathscr{Y}}(\omega, \cdot)$, the probability measure on $(W, \mathscr{F})$ defined by

$$\mathbf{P}_H^{\mathscr{Y}}(\omega, \cdot) = (p^X)^*(\mathbf{P}_{\bar{H}})(\cdot) \otimes \delta_{p^Y(\omega)},$$

where $(p^X)^*(\mathbf{P}_{\bar{H}})$ is the image by the application p^X of the probability measure $\mathbf{P}_{\bar{H}}$, and δ_y is the Dirac measure at y. We denote by $\mathbf{E}[\cdot|\mathscr{Y}]$ the expectation under $\mathbf{P}_H^{\mathscr{Y}}(\omega, \cdot)$. As a consequence of the Fubini theorem, for any adapted, continuous process, dominated in L^1, $\{H_t, t \in I\}$, $\{\mathbf{E}[H_t|\mathscr{Y}], t \in I\}$ is a continuous version of the optional projection of H given the filtration $\{\mathscr{Y}_t, t \in I\}$.

LEMMA 3.2. *Let* $A = (A^1, \dots, A^M)$ *be such that* $K_{H_i}(A^i)$ *belongs to* $\mathbb{D}_{2,1}(\mathscr{H}_i)$ *for any* $i \in \{1, \dots, M\}$ *and* C *be an* $(\mathscr{F}_t, t \in I)$ *adapted, continuous process such that* $\int_I \mathbf{E}[|C_s|^2]\, ds$ *is finite. We have,* $\mathbf{P}_{\bar{H}}$ *almost everywhere, for all* $t \in I$,

$$\mathbf{E}\left[\int_0^t A_s^i\, dB_s^i|\mathscr{Y}\right] = 0, \qquad i = 1, \dots, M, \tag{11}$$

$$\mathbf{E}\left[\int_0^t C_s\, dB_s^j|\mathscr{Y}\right] = \int_0^t \mathbf{E}[C_s|\mathscr{Y}]\, dB_s^j, \qquad j = M+1, \dots, d. \tag{12}$$

PROOF. Since $\mathscr{Y} = \mathscr{F}^Y$ is the filtration of a standard Brownian motion, (12) is a consequence of Lemma 2.2.4 of Pardoux (1989). As to (11), the problem is originating from the fact that A is not supposed to be adapted. Nevertheless, fix $i \in \{1, \dots, M\}$ and remark that it is sufficient to show that for all $g \in L^2(I, \mathbf{R}^N)$,

$$\mathbf{E}\left[\int_I A_s^i\, dB_s^i \exp\left(\int_I \sum_{j=1}^N g^j(s)\, dB_s^{j+M} - \tfrac{1}{2}\|g\|^2_{L^2(I, \mathbf{R}^N)}\right)\right] = 0.$$

Using properties (P_3) and (P_4), we have

$$\begin{aligned}\Lambda_t^g &\stackrel{\text{def}}{=} \exp\biggl(\int_0^t \sum_{j=1}^N g^j(s)\,dB_s^{j+M} - \tfrac{1}{2}\|g\|^2_{L^2([0,t],\mathbf{R}^N)}\biggr)\\ &= 1 + \int_0^t \Lambda_s^g \sum_{j=1}^N g^j(s)\,dB_s^{j+M}.\end{aligned}$$

Hence, Formula (21) of Theorem 3.7 of DU (1998) yields to

$$\begin{aligned}\mathbf{E}\biggl[\int_I A_s^i\,dB_s^i\,\Lambda_T^g\biggr] &= \sum_{j=1}^N \mathbf{E}\biggl[\int_I A_s^i\,dB_s^i \cdot \int_0^t \Lambda_s^g g^j(s)\,dB_s^{j+M}\biggr]\\ &= \mathbf{E}[\langle A,\ B\rangle_{\mathscr{H}}] + \mathbf{E}[\operatorname{trace}(\nabla A \circ \nabla B)],\end{aligned}$$

where

$$A_s = K_{H_i}(A^i)(s)\mathbf{e}_i \quad \text{and} \quad B_s = \sum_{j=1}^N K_{H_j}(h^j \Lambda^{\bar{h}})(s)\mathbf{e}_{j+M}.$$

It is then clear that $\langle A,\ B\rangle_{\mathscr{H}}$ is zero and by (9), we have

$$\begin{aligned}&\operatorname{trace}(\nabla A \circ \nabla B)\\ &\quad= \sum_{k=1}^d \sum_{l,\,j>M}^d \sum_{m,\,n\ge 0} \bigl\langle (\nabla^k_{f^k_m} K_{H_i}(A^i))\mathbf{e}_i,\ f_n\bigr\rangle_{\mathscr{H}} \bigl\langle \langle \nabla^j B^l,\ f^l_m\rangle_{\mathscr{H}_l} \cdot \mathbf{e}_j,\ g_n\bigr\rangle_{\mathscr{H}}\\ &\quad= \sum_{l>M}^d \sum_{m\ge 0} \sum_{k=1}^d \bigl\langle \nabla^k_{f^k_m} K_{H_i}(A^i),\ \langle \nabla^i B^l,\ f^l_m\rangle_{\mathscr{H}_l}\bigr\rangle_{\mathscr{H}} = 0,\end{aligned}$$

since $\nabla^i B^l = 0$ for any $i \le M$. $\square$

THEOREM 3.3 (Zakai equation). *Assume that Hypotheses* II *and* III *are fulfilled. The unnormalized filter solves the following equation for all* $f \in C_b^2(\mathbf{R}^M;\mathbf{R})$, $\mathbf{P}_{\bar{H}}$ *almost everywhere for all* $t \in I$:

$$\begin{aligned}\sigma_t(f) &= f(x_0) + \sum_{i=1}^M \sum_{l=1}^M \int_0^t \mathbf{E}_h\biggl[L_s \frac{\partial f}{\partial x_l}(X_s) K'_{H_i}(b^{l,i}\circ X)(s)\,|\,\mathscr{Y}\biggr]\,ds\\ &\quad+ \sum_{i=1}^M \sum_{l=1}^M \sum_{k=1}^M \int_0^t \mathbf{E}_h\biggl[L_s \frac{\partial^2 f}{\partial x_l \partial x_k}(X_s) K'_{H_i}(a^{l,i}\circ X \cdot \dot{\nabla}^i_{\bullet} X^k_s)(s)\,|\,\mathscr{Y}\biggr]\,dr\\ &\quad+ \sum_{i=1}^M \sum_{l=1}^M \int_0^t \mathbf{E}_h\biggl[L_s \frac{\partial f}{\partial x_l}(X_s) K'_{H_i}(a^{l,i}\circ X \cdot \dot{\nabla}^i_{\bullet} \log(L_s))(s)\,|\,\mathscr{Y}\biggr]\,dr\\ &\quad+ \sum_{i=1}^M \sum_{l=1}^M \sum_{j=M+1}^d \int_0^t \mathbf{E}_h[L_s(\tau^{-1}(Y_s)h(X_s))^j f(X_s)\,|\,\mathscr{Y}]\,dB^j_s,\end{aligned} \tag{13}$$

where

$$
\dot{\nabla}_u^i \log(L_t) = \sum_{j,k=1}^{N}\sum_{l=1}^{M}\int_u^t (\tau^{-1}(Y_s))^{k,j}\frac{\partial h^k}{\partial x_l}(X_s)\dot{\nabla}_u^i X_s^l\,dB_s^{j+M}
$$

$$
\begin{aligned}
(14)\qquad & - \frac{1}{2}\sum_{j,k,m=1}^{N}\sum_{l=1}^{M}\int_u^t (\tau^{-1}(Y_s))^{j,m}(\tau^{-1}(Y_s))^{j,k}\\
&\times \frac{\partial(h^m h^k)}{\partial x_l}(X_s)\dot{\nabla}_u^i X_s^l\,ds.
\end{aligned}
$$

COROLLARY 3.1 (Kallianpur–Striebel equation). *Assume that Hypotheses* II *and* III *are fulfilled. For all* $f \in C_b^2(\mathbf{R}^M;\mathbf{R})$, *the normalized filter solves the following equation*: $\mathbf{P}_{\bar{H}}$ *almost everywhere for all* $t \in I$,

$$
\begin{aligned}
\pi_t(f) &= f(x_0) + \sum_{i=1}^{M}\sum_{l=1}^{M}\int_0^t \mathbf{E}\left[\frac{\partial f}{\partial x_l}(X_s)K'_{H_i}(b^{l,i}\circ X)(s)\,|\,\mathscr{Y}\right]ds\\
&\quad + \sum_{i=1}^{M}\sum_{l=1}^{M}\sum_{k=1}^{M}\int_0^t \mathbf{E}\left[\frac{\partial^2 f}{\partial x_l\partial x_k}(X_s)K'_{H_i}(a^{l,i}\circ X\cdot\dot{\nabla}_{\bullet}^i X_s^k)(s)\,|\,\mathscr{Y}\right]dr\\
(15)\qquad &\quad + \sum_{i=1}^{M}\sum_{l=1}^{M}\int_0^t \mathbf{E}_h\left[\frac{\partial f}{\partial x_l}(X_s)K'_{H_i}(a^{l,i}\circ X\cdot\dot{\nabla}_{\bullet}^i \log(L_s))(s)\,|\,\mathscr{Y}\right]dr\\
&\quad + \sum_{i=1}^{M}\sum_{l=1}^{M}\sum_{j=M+1}^{d}\int_0^t \mathbf{E}_h[(\tau^{-1}(Y_s)h(X_s))^j f(X_s)\,|\,\mathscr{Y}]\,d\tilde{B}_s^j,
\end{aligned}
$$

where for $j = M+1,\dots,d$,

$$
\tilde{B}_t^j = B_t^j - \sum_{k=M+1}^{d}\int_0^t \pi_s\big(\tau^{-1}(Y_s)^{k,j}h^k\big)\,ds = \int_0^t \pi_s\big(\tau^{-1}(Y_s)^{k,j}h^k\big)\,d\tilde{Y}_s
$$

are independent $(\mathbf{P}_{\bar{H}}^h, (\mathscr{Y}_t,\ t\in I))$ *standard Brownian motions and* $\tilde{Y} = \mathscr{I}(Y)$.

PROOF OF THEOREM 3.3. As a consequence of Theorems A.1 and A.2 and Lemma A.5, X satisfies the hypothesis of Theorem B.1. Following Lepingle and Mémin (1978), $\{L_t,\ t\in[0,1]\}$ is a square integrable adapted process, dominated in $L^p(W)$ for all $p>1$ and

$$
L_t = \exp\left(\sum_{i=M+1}^{d}\int_0^t (\tau^{-1}(Y_s)h(X_s))^i\,dB_s^i - \tfrac{1}{2}\int_0^t \|\tau^{-1}(Y_s)h(X_s)\|^2\,ds\right).
$$

Moreover, L_t is at least once Gross–Sobolev differentiable [see Nualart (1995)] and for any $i\in\{1,\dots,M\}$, $\dot{\nabla}_u^i \log L_t$ is given by (14) since $\nabla^j Y_s = 0$ for any $j\le M$ and any $s\in I$.

Let $(g_n,\ n\ge 0)$ be a sequence of bounded real-valued functions twice diferentiable with bounded derivatives such that $\operatorname{supp} g_n \subset [-n-1, n+1]$ and $g_n|_{[-n,n]} \equiv \mathrm{Id}_{\mathbf{R}}|_{[-n,n]}$. It follows from the boundedness of h, $\partial h/\partial x$ and τ^{-1}

that $\{g_n(L_t),\ t \in I\}$ fulfills the hypothesis of Theorem B.1. Applying the Itô formula (see Theorem B.1), we obtain

$$
\begin{aligned}
&f(X_t)g_n(L_t)\\
&\quad= f(x_0) + \sum_{i=1}^{M}\sum_{l=1}^{M}\int_0^t \frac{\partial f}{\partial x_l}(X_s)g_n(L_s)K'_{H_i}(b^{l,i}\circ X)(s)\,ds\\
&\qquad+ \sum_{i=1}^{M}\sum_{l=1}^{M}\int_0^t a^{l,i}(X_s)K'^{*}_{H_i}\Big(\frac{\partial f}{\partial x_l}\circ X\ \cdot g_n\circ L\Big)(s)\,dB_s^i\\
&\qquad+ \sum_{i=1}^{M}\sum_{l=1}^{M}\sum_{k=1}^{M}\int_0^t a^{l,i}(X_s)K'^{*}_{H_i}\Big(\frac{\partial^2 f}{\partial x_l\partial x_k}\circ X\ \cdot g_n\circ L\ \cdot \dot\nabla_s^i X_{\bullet}^k\Big)(s)\,ds\\
&\qquad+ \sum_{i=1}^{M}\sum_{l=1}^{M}\int_0^t a^{l,i}(X_s)K'^{*}_{H_i}\Big(\frac{\partial f}{\partial x_l}\circ X\ g'_n\circ L\ \cdot \dot\nabla_s^i L_{\bullet}\Big)(s)\,ds\\
&\qquad+ \sum_{j=1}^{N}\int_0^t (\tau^{-1}(Y_s)h(X_s,Y_s))^j f(X_s)g'_n(L_s)\,dB_s^{j+M}\\
&\qquad+ \frac{1}{2}\sum_{j=1}^{N}\int_0^t g''_n(L_s)|(\tau^{-1}(Y_s)h(X_s,Y_s))^j|^2\,ds\\
&\quad= f(x_0) + \sum_{i=1}^{6} A_t^{i,n}.
\end{aligned}
\tag{16}
$$

Since f is bounded and $\{L_t,\ t \in I\}$ is uniformly integrable, for any $t \in I$, $\mathbf{E}_h[f(X_t)g_n(L_t)\,|\mathscr{Y}]$ converges in $L^1(W)$ to $\mathbf{E}_h[f(X_t)L_t\,|\mathscr{Y}]$. Since f is continuous and bounded $L_{\bullet}f(X_{\bullet})$ is an $\{\mathscr{F}_t,\ t \in I\}$ adapted, continuous process, upperbounded by a random variable belonging to $L^1(\mathbf{P}_{\bar H})$, $\{\mathbf{E}_h[f(X_t)L_t\,|\,\mathscr{Y}],\ t \in I\}$ is a continuous version of $\{\sigma_t(f), t \in I\}$.

For any l, i, k; $a^{l,i}$, $b^{l,i}$ and $\partial f/\partial x_l$ are bounded; hence by Lemma 3.2, one can exchange the conditioning and the integrals in the first two summands of (16). We obtain that $\mathbf{E}_h[A_t^{2,n}\,|\,\mathscr{Y}] = 0$ and by uniform integrability of L, $A^{1,n}$ converges in $L^1(W)$ to

$$
\int_0^t \frac{\partial f}{\partial x_l}(X_s)L_sK'_{H_i}(b^{l,i}\circ X)(s)\,ds.
$$

Moreover, by Lemma A.5, for any $i \in \{1,\dots,M\}$,

$$
\dot\nabla_u^i X_s^k = \sum_{j=1}^{M} L_{V_i}^{k,j}(s,u)a^{l,i}(X_u),
$$

and according to (26), for any $q \in [1,2]$,

$$
\mathbf{E}_h\big[|\dot\nabla_u^i X_s^k|^q\big] \le c\,u^{-qH^0}.
$$

It follows from Lemma A.5 and (6) that for any $q \in [1, 1/(3/2 - \min_i H_i)) \subset [1, 2]$,

$$
\begin{aligned}
&\mathbf{E}_h\left[\left|\int_0^t a^{l,i}(X_s)K'^{*}_{H_i}\left(\frac{\partial f}{\partial x_l}\circ X\cdot(\mathrm{Id}-g_n)(L)\,\dot\nabla^i_s X^k_{\bullet}\right)(s)\,ds\right|\right]\\
&\quad=\mathbf{E}_h\left[\left|\int_0^t \frac{\partial f}{\partial x_l}(X_s)(\mathrm{Id}-g_n)(L_s)K'_{H_i}(a^{l,i}\circ X\cdot\dot\nabla^i_{\bullet}X^k_s)(s)\,ds\right|\right]\\
&\quad\le c\sup_s\big(\mathbf{E}_h[|L_s|^{q^*}\mathbf{1}_{\{|L_s|\ge n\}}]^{1/q}\big)\\
&\qquad\times\left(\int_0^t s^{q(H_i-1/2)}\int_s^t(u-s)^{q(H_i-3/2)}u^{-q(H_i-1/2)}\mathbf{E}_h[|\dot\nabla^i_u X^k_s|^q]\,du\,ds\right)^{1/q}\\
&\quad\le c\sup_s\big(\mathbf{E}_h[|L_s|^{q^*}\mathbf{1}_{\{|L_s|\ge n\}}]^{1/q}\big)\\
&\qquad\times\left(\int_0^t s^{q(H_i-1/2)}\int_s^t(u-s)^{q(H_i-3/2)}u^{-q[(H_i-1/2)+H^0]}\,du\,ds\right)^{1/q}\\
&\quad\le c\sup_s\big(\mathbf{E}_h[|L_s|^{q^*}\mathbf{1}_{\{|L_s|\ge n\}}]^{1/q}\big)\left(\int_0^t s^{-qH^0}(t-s)^{q(H_i-3/2)+1}\,ds\right)^{1/q}
\end{aligned}
$$

and the last integral is convergent because of the hypothesis on q. Thus, $A^{3,n}_t$ converges in $L^1(W)$ to

$$
\sum_{i=1}^M\sum_{l=1}^M\sum_{k=1}^M\int_0^t a^{l,i}(X_s)K'^{*}_{H_i}\left(\frac{\partial^2 f}{\partial x_l\partial x_k}\circ X\,L\dot\nabla^i_s X^k_{\bullet}\right)(s)\,ds.
$$

The term $A^{4,n}$ is handled similarly after the observation that

$$
\begin{aligned}
\dot\nabla^i_s L_t = L_t\sum_{k=1}^M\sum_{j,l=M+1}^d\int_s^t\frac{\partial h^l}{\partial x_k}(X_r)\dot\nabla^i_s X^k_r(\tau^{-1})^{j,l}(Y_s)\,dB^j_s\\
-\int_s^t\frac{h^l\partial h^l}{\partial x}(X_s)\dot\nabla^i_s X^k_r(\tau^{-1})^{j,l}(Y_s)\,ds.
\end{aligned}
$$

We have $\mathbf{P}_{\bar H}$ almost everywhere for all $t\in I$,

$$
\lim_{n\to\infty}\mathbf{E}_h[A^{4,n}_t|\mathscr{Y}]=\int_0^t\mathbf{E}_h\left[L_s\frac{\partial f}{\partial x}(X_s)K'_{H_i}(a\circ X\dot\nabla^i_{\bullet}\log(L_s))(s)\,|\,\mathscr{Y}\right]ds. \tag{17}
$$

As to $A^{5,n}$, it is a martingale directed by a standard Brownian motion and the usual proofs [see, e.g., Pardoux (1989)] can be used. It follows that $\mathbf{P}_{\bar H}$ almost everywhere, for all $t\in I$,

$$
\lim_{n\to+\infty}\mathbf{E}_h[A^{5,n}_t|\mathscr{Y}]=\sum_{j=1}^N\int_0^t\mathbf{E}_h[(\tau^{-1}(Y_s)h(X_s))^j L_s f(X_s)|\mathscr{Y}]\,dB^{j+M}_s. \tag{18}
$$

Finally, $A^{6,n}_t$ is easily seen to converge to 0 in $L^1(W)$ by uniform integrability of $\{L_s, s\in I\}$. □

Since there is no anticipative term in (13), the proof of (15) is identical to the corresponding proof in Pardoux (1989). The second part of Corollary 3.1 is a consequence of Lemma 2.1.

4. Infinite-dimensional approach. The conclusion of the previous section is that all the potential technical problems due to the lack of martingale properties can be overcome, thus the impossibility of establishing closed equations for the filters originates from the nature of the problem itself. On the other hand, there is no anticipative term in the equations we obtain; hence, it is reasonable to hope that something can be done if we filter functionals, which depends on X up to time t. Consider now the process $\mathcal{X}$ defined by

$$\mathcal{X}_r(t) = x_0 + \sum_{i=1}^{M} \int_0^r K_{H_i}(t,s) b^i(X_s)\,ds + \sum_{i=1}^{M} \int_0^r K_{H_i}(t,s) a^i(X_s)\,dB_s^i. \tag{19}$$

Denote by $\mathbb{W}$ the Banach space of continuous $\mathbf{R}^M$-valued functions equipped with the sup norm. For any $\alpha > 0$, let $\mathbb{V}_\alpha$ be the Hilbert space $I_{0^+}^{\alpha}(L^2(I;\mathbf{R}^M))$ whose norm is given by

$$\|f\|_{\mathbb{V}_\alpha} \stackrel{\text{def}}{=} \|I_{0^+}^{-\alpha} f\|_{L^2(I;\mathbf{R}^M)}.$$

LEMMA 4.1. *The process $\mathcal{X}$ is a $\{\sigma(B_s^i,\ i=1,\dots,M,\ s\le t),\ t\in I\}$-adapted $\mathbb{W}$-valued process.*

PROOF. Now $\mathcal{X}$ is clearly $\{\mathcal{F}_t^X,\ t\in I\}$-adapted. Moreover, for any $r\in I$, by the techniques used above, for any $p\ge 1$,

$$\mathbf{E}_h[|\mathcal{X}_r(t) - \mathcal{X}_r(s)|^p] \le c(t-s)^{p\min_i H_i}$$

and the Kolmogorov criterion ensures that $\mathcal{X}_r$ belongs to $\mathbb{W}$. □

THEOREM 4.1. *Let $F\colon \mathbb{W}\to\mathbf{R}$ be twice $\mathbb{V}_\alpha$-differentiable for some $0<\alpha<\min_i(H_i-1/2)$ such that $D_\alpha F$ and $D_\alpha^2 F$ are continuous from $\mathbb{W}$ into $\mathbb{V}_\alpha$ and $\mathbb{V}_\alpha\otimes\mathbb{V}_\alpha$, respectively. If $\mathcal{X}$ is given by* (19) *then*

$$\begin{aligned} F(\mathcal{X}_t) &= F(x_0) + \sum_{i=1}^{M}\int_0^t \langle D_\alpha F(\mathcal{X}_s),\ K_{H_i}(\cdot,s) b^i(X_s)\rangle_{\mathbb{V}_\alpha}\,ds \\ &\quad + \sum_{i=1}^{d}\int_0^t \langle D_\alpha F(\mathcal{X}_s),\ K_{H_i}(\cdot,s) a^i(X_s)\rangle_{\mathbb{V}_\alpha}\,dB_s^i \\ &\quad + \tfrac{1}{2}\sum_{i=1}^{M}\int_0^t (K_{H_i}(\cdot,s)a^i(X_s))^*\, D_\alpha^2 F(\mathcal{X}_s)(K_{H_i}(\cdot,s)a^i(X_s))\,ds \end{aligned}$$

Here x_0 is identified with the constant element of $\mathbb{W}$ equal to x_0 everywhere. Note that the scalar products in $\mathbb{V}_\alpha$ can be explicitly computed using Theorem 2.1(iii).

PROOF. Let $\{v_i,\ 1 \le i \le M\}$ be an orthonormal family of $\mathbb{V}_\alpha$ and $\mathscr{B}$ be a $\mathbb{W}$-valued standard Brownian motion. One can always consider that

$$\begin{aligned}\chi_r(t) = x_0 &+ \sum_{i=1}^{M} \int_0^r K_{H_i}(t,s) b^i(X_s)\, ds \\ &+ \sum_{i=1}^{M} \int_0^r K_{H_i}(t,s) a^i(X_s) \otimes v_i \, d\mathscr{B}_s.\end{aligned} \tag{20}$$

By Theorem 2.1(v), for any s and almost any ω,

$$t \mapsto K_{H_i}(t,s) b^i(X_s) \quad \text{and} \quad t \mapsto K_{H_i}(t,s) a^i(X_s) \otimes v_i$$

belong to $\mathbb{V}_\alpha$ and $\mathbb{V}_\alpha \otimes \mathbb{V}_\alpha$, respectively. Moreover, by the very definition of the $\mathbb{V}_\alpha$ norm,

$$\begin{aligned}\mathbf{E}_h\left[\int_I |K_{H_i}(\cdot,s) b^i(X_s)|^2_{\mathbb{V}_\alpha}\, ds\right] &= \mathbf{E}_h\left[\int\int_{I\times I} |I_{0^+}^{-\alpha}(K_{H_i}(\cdot,s))(t) b^i(X_s)|^2\, dt\, ds\right] \\ &\le c \int_I s^{-2|H_i - 1/2|}\, ds < +\infty,\end{aligned}$$

by Theorem 2.1(vi). Consequently, we also have

$$\mathbf{E}_h\left[\int_I |K_{H_i}(\cdot,s) b^i(X_s) \otimes v_i|^2_{\mathbb{V}_\alpha\otimes\mathbb{V}_\alpha}\, ds\right] = \mathbf{E}_h\left[\int_I |K_{H_i}(\cdot,s) b^i(X_s)|^2_{\mathbb{V}_\alpha}\, ds\right] < +\infty.$$

Hence the two maps,

$$s \mapsto K_{H_i}(\cdot,s) b^i(X_s) \quad \text{and} \quad s \mapsto K_{H_i}(\cdot,s) a^i(X_s) \otimes v_i,$$

belong to $L^2(\mathbb{V}_\alpha)$ and $L^2(\mathbb{V}_\alpha \otimes \mathbb{V}_\alpha)$, respectively. The space $\mathbb{V}_\alpha$ is densely embedded in $\mathbb{W}$; hence $(\iota, \mathbb{V}_\alpha, \mathbb{W})$ (where ι denotes the embedding from $\mathbb{V}_\alpha$ into $\mathbb{W}$) is a Wiener space and we are in position to apply the Itô formula for Banach-valued processes given in Kuo (1975). We get

$$\begin{aligned}F(\chi_t) = F(x_0) &+ \sum_{i=1}^{M} \int_0^t \langle D_\alpha F(\chi_s),\ K_{H_i}(\cdot,s) b^i(X_s)\rangle_{\mathbb{V}_\alpha}\, ds \\ &+ \int_0^t \left\langle \left(\sum_{i=1}^{M} K_{H_i}(\cdot,s) a^i(X_s)\otimes v_i\right)^* D_\alpha F(\chi_s),\ d\mathscr{B}_s\right\rangle \\ &+ \frac{1}{2}\int_0^t \operatorname{trace}\Bigg(\left(\sum_{i=1}^{M} K_{H_i}(\cdot,s) a^i(X_s)\otimes v_i\right)^* D^2_\alpha F(\chi_s) \\ &\qquad\qquad \times \left(\sum_{j=1}^{M} K_{H_j}(\cdot,s) a^j(X_s)\otimes v_j\right)\Bigg)\, ds.\end{aligned}$$

Since v_i is orthogonal to v_j and $\mathscr{B}(v_i) = B^i$, the result follows. □

Define the infinite-dimensional filters by

$$\tilde{\sigma}_t(F) \stackrel{\text{def}}{=} \mathbf{E}_h[F(\mathcal{X}_t)L_t \,|\, \mathscr{Y}_t] \quad \text{and} \quad \tilde{\pi}_t(F) \stackrel{\text{def}}{=} \mathbf{E}[F(\mathcal{X}_t) \,|\, \mathscr{Y}_t] = \tilde{\sigma}_t(F)/\tilde{\sigma}_t(1).$$

THEOREM 4.2 (Infinite-dimensional Zakai equation). *Let $F\colon \mathbb{W} \to \mathbf{R}$ be twice $\mathbb{V}_\alpha$-differentiable for some $0 < \alpha < \min_i(H_i - 1/2)$ such that $D_\alpha F$ and $D^2_\alpha F$ are continuous and bounded from $\mathbb{W}$ into $\mathbb{V}_\alpha$ and $\mathbb{V}_\alpha \otimes \mathbb{V}_\alpha$, respectively. Assume that Hypothesis* II, III *hold*; *we have*

$$\begin{aligned}\tilde{\sigma}_t(F) = F(x_0) &+ \sum_{j=M+1}^{d} \int_0^t \tilde{\sigma}_s\big(F \times (\tau^{-1}(Y_s)h \circ p_s)^j\big)\, dB^j_s \\ &+ \sum_{i=1}^{M} \int_0^t \tilde{\sigma}_s\big(\langle D_\alpha F,\ K_{H_i}(\cdot, s) b^i \circ p_s\rangle_{\mathbb{V}_\alpha}\big)\, ds \\ &+ \tfrac{1}{2} \sum_{i=1}^{M} \int_0^t \tilde{\sigma}_s\big((K_{H_i}(\cdot, s)a^i \circ p_s)^*\, D^2_\alpha F\, (K_{H_i}(\cdot, s)a^i \circ p_s)\big)\, ds,\end{aligned} \tag{21}$$

where $p_s(x) = x_s$ for any $x \in \mathbb{W}$.

PROOF. Applying the standard integration by parts formula for semimartingales, we have

$$F(\mathcal{X}_t)L_t = \int_0^t F(\mathcal{X}_s)\, dL_s + \int_0^t L_s\, dF(\mathcal{X}_s)$$

because the independence of the two sigma fields $\sigma\{B^i,\ 1 \le i \le M\}$ and $\sigma\{B^j,\ M+1 \le d \le d\}$ implies that $\langle F(\mathcal{X}),\ L\rangle_t = 0$. We thus obtain

$$\begin{aligned}L_t F(\mathcal{X}_t) = F(x_0) &+ \sum_{i=1}^{M} \int_0^t L_s \big\langle D_\alpha F(\mathcal{X}_s),\ K_{H_i}(\cdot, s) b^i(X_s)\big\rangle_{\mathbb{V}_\alpha}\, ds \\ &+ \sum_{i=1}^{M} \int_0^t L_s \big\langle D_\alpha F(\mathcal{X}_s),\ K_{H_i}(\cdot, s) a^i(X_s)\big\rangle_{\mathbb{V}_\alpha}\, dB^i_s \\ &+ \tfrac{1}{2} \sum_{i=1}^{M} \int_0^t L_s (K_{H_i}(\cdot, s)a^i(X_s))^*\, D^2_\alpha F(\mathcal{X}_s)(K_{H_i}(\cdot, s)a^i(X_s))\, ds \\ &+ \sum_{j=M+1}^{d} \int_0^t F(\mathcal{X}_s) L_s (\tau^{-1}(Y_s)h(X_s))^j\, dB^j_s.\end{aligned}$$

It remains once again to commute the conditional expectation with the integrals. Since $D_\alpha F$ and a^i are bounded,

$$\begin{aligned}&\big|\big\langle D_\alpha F(\mathcal{X}_s),\ K_{H_i}(\cdot, s)a^i(X_s)\big\rangle_{\mathbb{V}_\alpha}\big| \\ &\quad \le \sup_{x \in B} \|D_\alpha F(x)\|_{\mathbb{V}_\alpha} \|a^i\|_\infty \bigg(\int_0^1 |I^{-\gamma}_{0^+} K_{H_i}(\cdot, s)(t)|^2\, dt\bigg)^{1/2} \\ &\quad \le \sup_{x \in B} \|D_\alpha F(x)\|_{\mathbb{V}_\alpha} \|a^i\|_\infty\ s^{1/2-H}\end{aligned}$$

by Theorem 2.1(iii). Thus,

$$\int_I \mathbf{E}_h[|\langle D_\alpha F(\chi_s),\ K_{H_i}(\cdot, s)a^i(X_s)\rangle_{\mathbb{V}_\alpha}|^2]\, ds < +\infty$$

so that we can apply Lemma 3.2. The other terms are handled similarly and the proof is complete. □

It is then routine to prove the following.

COROLLARY 4.1 (Infinite-dimensional Kallianpur–Striebel equation). *With the hypothesis of the previous theorem, we have*

$$\begin{aligned}\tilde{\pi}_t(F) = F(x_0) &+ \sum_{i=1}^{M} \int_0^t \tilde{\pi}_s\big(\langle D_\alpha F,\ K_{H_i}(\cdot, s)b^i \circ p_s\rangle_{\mathbb{V}_\alpha}\big)\, ds \\ &+ \tfrac{1}{2}\sum_{i=1}^{M} \int_0^t \tilde{\pi}_s\big((K_{H_i}(\cdot, s)a^i \circ p_s)^*\, D_\alpha^2 F(\cdot)(K_{H_i}(\cdot, s)a^i \circ p_s)\big)\, ds \qquad (22)\\ &+ \int_0^t \big(\tilde{\pi}_s(F\cdot \tau^{-1}(Y_s)h\circ p_s) - \tilde{\pi}_s(F)\cdot\tilde{\pi}_s(\tau^{-1}(Y_s)h\circ p_s)\big)\, d\tilde{Y}_s.\end{aligned}$$

We have obtained a closed equation for $\tilde{\pi}(F)$; it thus remains to solve it, and the original problem is solved observing that

$$\pi_t(f) = \tilde{\pi}_t(f\circ p_t),\ \text{that is, } \pi_t = p_t^* \tilde{\pi}_t.$$

Numerical procedures to derive an approximation of $\tilde{\pi}_t(F)$ are the subject of our current investigations.

APPENDIX A

Multidimensional stochastic differential equations. In this section we extend the results of CD (1997) in the multidimensional case. By an fBm-SDE, we mean an equation of the form,

$$\text{(E)}\qquad X_t^l = x^l + \sum_{i=1}^{d}\left\{\int_0^t K_{H_i}(t, s)b(s, X_s)^{l,i}\, ds + \int_0^t K_{H_i}(t, s)\sigma(s, X_s)^{l,i}\, dB_s^i\right\},$$

where $b^{l,i}$ and $\sigma^{l,i}$ are deterministic functions for $i = 1, \dots, d$ and $l = 1, \dots, m$, $x^l \in \mathbf{R}$, and $\{B_t^i,\ t \in I,\ i = 1, \dots, d\}$ is the Brownian motion mentioned in properties (P2) and (P3). Set

$$H^0 = \sup_{i=1,\dots,d} |H_i - \tfrac{1}{2}|,\ \Delta_H = \{p \geq 1,\ pH^0 < 1\} \text{ and } \alpha_p = (1 - pH^0)^{-1}$$

By a solution of the differential equation (E), we mean a $\mathbf{R}^m$-valued adapted stochastic process $X = \{X_t,\ t \in I\}$ such that

$$\big(t \mapsto E[\| X_t \|^2]\big) \in \bigcup_{\alpha > \alpha_2} L^\alpha(I) \overset{\text{not}}{=} L_+^{\alpha_2}. \qquad (23)$$

THEOREM A.1. *Let b and σ be L-Lipschitz continuous with respect to their second variable, uniformly with respect to their first variable*: *for all t in* $[0, T]$; *for all x, y in $\mathbf{R^m}$, and any $i = 1, \dots, d$, $l = 1, \dots, m$,*

$$|b^{l,i}(t,x) - b^{l,i}(t,y)| + |\sigma^{l,i}(t,x) - \sigma^{l,i}(t,y)| \le L \parallel x - y \parallel .$$

Assume also that there exist x_0 and y_0 in $\mathbf{R}^m$ and $\alpha_b > \alpha_1$, $\alpha_\sigma > 2\alpha_2$ such that for any l and i,

$$\text{(H1)} \qquad b^{l,i}(\cdot, x_0) \in L^{\alpha_b}(I, \mathbf{R}^m) \quad \textit{and} \quad \sigma^{l,i}(\cdot, y_0) \in L^{\alpha_\sigma}(I, \mathbf{R}^m).$$

The differential equation (E) *has a unique solution. Moreover, for this solution and for any $p \in \Delta_H \cap [2, +\infty)$,*

$$\big(t \mapsto \mathbf{E}[\parallel X_t \parallel^p]\big) \quad \textit{is bounded on } I.$$

Whenever b and σ satisfy the hypothesis of Theorem A.1 *with $\alpha_b \ge 2$ and the boundedness of σ as supplementary conditions, the solution of* (E) *has almost surely continuous sample-paths.*

The proof is based on a standard Picard approximation scheme [as in CD (1997)] where the Gronwall lemma is replaced by the following lemma.

LEMMA A.1. *For any $p \ge 1$, consider*

$$K_1^p(t,s) \stackrel{\text{def}}{=} \sup_{i=1,\dots,d} |K_{H_i}(t,s)|^p,$$

$$K_{n+1}^p(t,s) \stackrel{\text{def}}{=} \int_s^t K_n^p(t,u) K_1^p(u,s)\, du.$$

Set

$$H^1 = \max_j (1/2 - H_j)^+ \quad \textit{and} \quad H^- = \min_j (H_j - 1/2).$$

R1. *For $p \in \Delta_H$,*

$$\sup_{t \in I} \sup_i \int_0^t |K_{H_i}(t,s)|^p\, ds < +\infty.$$

R2. *For $p \in \Delta_H$, the resolvent series,*

$$\sum_{n=1}^{+\infty} z^n (K_n^p \mathbf{1})(t) = \sum_{n=1}^{+\infty} z^n \int_0^t K_1^p(t, s_1)\, ds_1 \times \int_0^{s_1} K_1^p(s_1, s_2)\, ds_2 \cdots \int_0^{s_{n-1}} K_1^p(s_{n-1}, s_n)\, ds_n$$

converges for all $z \in \mathbb{C}$. More precisely, if $T \ge 1$,

$$(K_n^p \mathbf{1})(t) \le c_H^{np} \prod_{j=1}^n B(j(1+pH^-) + 1 - pH^0,\ 1 - pH^1) T^{j(1+pH^-)}.$$

R3. *For any* $\phi \in L_+^{\alpha_p} =_{\text{not}} \bigcup_{\alpha > \alpha_p} L^\alpha(I)$,

$$t \mapsto \int_0^t K_1^p(t,s)\phi(s)\,ds \text{ is bounded with respect to } t \text{ on } I.$$

R4. *For any* $\alpha > 0$,

$$\int_s^t K_2^2(t,u)\,(u-s)_+^{-\alpha}\,du \le B(1-\alpha,\ 1-2H^1)\,s^{-2H_0}(t-s)^{-\alpha+1-2H^1},$$

when $\alpha = 0$, *we have*

$$\int_s^t K_2^2(t,u)\,du \le B(1-2H^0,\ 1-2H^1)\,(t-s)^{1-2(H^0+H^1)}.$$

PROOF. By homogeneity and according to Theorem 2.1(ii), for any β and any i,

$$\int_0^t |K_{H_i}(t,s)|^p s^\beta\,ds \le c_H B(1-p(1/2-H_i)^+,\ \beta - p|H_i - 1/2| + 1) \\ \times \left(\frac{t}{T}\right)^{\beta+1+p(H_i-1/2)} T^{\beta - p(|H_i-1/2|+(1/2-H_i)^+)+1}.$$

Since $|t/T| \le 1$ and $T \ge 1$, using

$$\begin{aligned}\int_0^t &|K_{H_i}(t,s)|^p s^\beta\,ds \\ &\le c_H^p B(\beta+1-pH^0,\ 1-pH^1) \\ &\quad \times \exp\Bigl(\bigl(\beta+1+p\min_i(H_i-1/2)\bigr)\ln t \\ &\qquad\qquad - p\min_i\bigl((H_i-1/2)+|H_i-1/2|+(1/2-H_i)^+\bigr)\ln T\Bigr).\end{aligned}$$

Notice that $p \in \Delta_H$ entails that $1 + p\min_i(H_i - 1/2) > 0$; thus (R1) follows by taking $\beta = 0$. By induction on n, we get

$$K_n^p(\mathbf{1})(t) \le c_H^{np} \prod_{j=1}^n B(j(1+pH^-)+1-pH^0,\ 1-pH^1)T^{j(1+pH^-)},$$

for any $t \in I$. By the usual criterions of convergence of series, one sees that $\sum_{n=1}^{+\infty} z^n(K_n^p\mathbf{1})(t)$ converges for all z provided that

$$1 - pH^1 > 0 \quad \text{and} \quad p(H^0 + H^1) < 2.$$

By the way, since $1/2 - H_i \le |H_i - 1/2|$, both conditions are satisfied when p belongs to Δ_H. Point (R3) follows from Hölder inequality and (2). As to point (4), according to Theorem 2.1(ii),

$$\begin{aligned}\int_s^t K_2^2(t,u)\,(u-s)_+^{-\alpha}\,du &\le \int_s^t (t-u)^{-2H^1} u^{-2H^0}(u-s)_+^{-\alpha}\,du \\ &\le s^{-2H^0}\int_s^t (t-u)^{-2H^1}(u-s)_+^{-\alpha}\,du\end{aligned}$$

and the result follows by a change of variable. When $\alpha = 0$, one can obtain a finer upper bound, bounding u^{-2H^0} by $(u-s)^{-2H^0}$. $\square$

THEOREM A.2. *Let b and σ be once continuously differentiable with respect to their space variable, with bounded derivative, and satisfy* (H1); *assume furthermore that σ is bounded. For any $t \in I$, the value at t of the solution of* (E) *belongs to $\mathbb{D}_{2,1}$. Moreover, for any $\xi \in \mathscr{H}$,*

$$\begin{aligned}\langle \nabla X_t^l, \xi\rangle_{\mathscr{H}} &= \sum_{i=1}^d \langle K_{H_i}(K_{H_i}(t,\cdot)\sigma^{l,i}\circ X, \xi^i\rangle_{\mathscr{H}_i} \\ &\quad + \sum_{i,j=1}^d \sum_{k=1}^m \int_0^t K_{H_j}(t,u)\frac{\partial b^{l,j}}{\partial x^k}(u, X_u)\langle \nabla^i X_u^k, \xi^i\rangle_{\mathscr{H}_i}\,du \\ &\quad + \sum_{i,j=1}^d \sum_{k=1}^m \int_0^t K_{H_j}(t,u)\frac{\partial \sigma^{j,l}}{\partial x^k}(u, X_u)\langle \nabla^i X_u^k, \xi^i\rangle_{\mathscr{H}_i}\,dB_u^j.\end{aligned}$$

PROOF. It is a straightforward generalization of the proof of Theorem 4.1 of CD (1997) The method consists of showing that the terms of the Picard sequence constructed in the definition of the solution of (S_0) form a bounded sequence of $\mathbb{D}_{2,1}$. According to Üstünel [(1995), Proposition 3, page 37], X thus belongs to $\mathbb{D}_{2,1}$ and the expression of ∇X is easy to derive. In order to control the $\mathbb{D}_{2,1}$ norms, the Gronwall lemma is replaced by (R2) and (R3).

THEOREM A.3. *The equation*

$$\begin{aligned}\xi_t^l &= \sum_{i=1}^d K_{H_i}(K_{H_i}(t,\cdot)\sigma^{l,i}\circ X)\mathbf{e}_i \\ &\quad + \sum_{i,j=1}^d \sum_{k=1}^m \int_0^t K_{H_j}(t,u)\frac{\partial b^{l,j}}{\partial x^k}(u, X_u)\xi_u^k \mathbf{e}_i\,du \\ &\quad + \sum_{i,j=1}^d \sum_{k=1}^m \int_0^t K_{H_j}(t,u)\frac{\partial \sigma^{j,l}}{\partial x^k}(u, X_u)\xi_u^k \mathbf{e}_i\,dB_u^j,\end{aligned} \tag{24}$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ is the canonical basis of $\mathbf{R}^d$, has at most one solution in the set of $\mathscr{H}$-valued processes which satisfy

$$t \mapsto \mathbf{E}[\|\xi_t\|_H^2] \in L_+^{\alpha_2}.$$

THEOREM A.4. *For any $l = 1, \dots, m$ and $i = 1, \dots, d$, set $V_0^{l,i}(t,s) = K_{H_i}(t,s)$ and*

$$\begin{aligned}V_{n+1}^{l,i}(t,s) &= \sum_{j=1}^d \sum_{k=1}^m \int_s^t K_{H_j}(t,u)\frac{\partial b^{l,j}}{\partial x_k}(u, X_u)V_n^{k,i}(u,s)\,du \\ &\quad + \sum_{j=1}^d \sum_{k=1}^m \int_s^t K_{H_j}(t,u)\frac{\partial \sigma^{l,j}}{\partial x_k}(u, X_u)V_n^{k,i}(u,s)\,dB_u^j.\end{aligned}$$

Consider the $\mathbf{R}^{m\times m}$*-valued process*

$$L_V(s,t) \stackrel{\text{def}}{=} \sum_{n=0}^{+\infty} V_n(t,s).$$

L_V *is the unique solution of the system of equations,*

$$\begin{aligned} L^{l,i}(t,s) &= K_{H_i}(t,s) + \sum_{j=1}^{d}\sum_{k=1}^{m}\int_s^t K_{H_j}(t,u)\frac{\partial b^{l,j}}{\partial x_k}(u,X_u)L^{k,i}(u,s)\,du \\ &\quad + \sum_{j=1}^{d}\sum_{k=1}^{m}\int_s^t K_{H_j}(t,u)\frac{\partial \sigma^{l,j}}{\partial x_k}(u,X_u)L^{k,i}(u,s)\,dB_u^j \end{aligned} \tag{25}$$

such that $(t\mapsto \mathbf{E}[|L(t,s)|^2])$ *belongs to* $L_+^{\alpha_2}$. *For any* $t,\ s\in I$, *we have*

$$\mathbf{E}[|L_V(t,s)|^2] \le c(t-s)^{-2H^1}s^{-2\nu_H}, \tag{26}$$

where

$$\nu_H = \left(\left\lceil \frac{2H^1}{1-2H^1}\right\rceil + 1\right)H^0. \tag{27}$$

THEOREM A.5 (Variation of the parameter formula). *Assume that the hypotheses of Theorem* A.2 *hold and that* $H_i \ge 1/2$ *for all* $i \in \{1,\dots,d\}$. *The following representation formula holds*:

$$\langle \nabla^i X_t^l,\ \xi^i\rangle_{\mathscr{H}_i} = \sum_{k=1}^{m}\int_0^t L_{V_i}^{l,k}(t,s)\sigma^{k,i}(X_s)K_{H_i}^{-1}\xi(s)\,ds.$$

PROOF. Since $H_i \ge 1/2$, $H^1 = 0$ and $\nu_H = H^0$, we have

$$\begin{aligned} \mathbf{E}\left[\left|\sum_{k=1}^{m}\int_0^t L_{V_i}^{l,k}(t,s)\sigma^{k,i}(X_s)K_{H_i}^{-1}\xi(s)\,ds\right|^2\right] &\le c\sum_{k=1}^{m}\mathbf{E}\left[\int_0^t L_{V_i}^{l,k}(t,s)^2\,ds\right]\|\xi\|_{\mathscr{H}}^2 \\ &\le c\int_0^t s^{-2H^0}\,ds\|\xi\|_{\mathscr{H}}^2. \end{aligned}$$

Hence, $t\mapsto \sum_{k=1}^m \int_0^t L_{V_i}^{l,k}(t,s)\sigma^{k,i}(X_s)K_{H_i}^{-1}\xi(s)\,ds$ belongs to $L_+^{\alpha_2}$ and is a true solution of (24). According to Theorem A.2, it is equal to $\nabla^i X_t^l$. □

APPENDIX B

Itô formula. We prove a multidimensional version of the Itô formula which is slightly different from DU (1998) in the one-dimensional case. Actually, it turns out that X does not fulfill the integrability hypothesis of Theorem 5.1 of DU (1998) but satisfies some strong regularity properties which are sufficient to prove the Itô formula. The proof here follows the same lines as the proof of the anticipative Itô formula in Nualart (1995). We say that a process X fulfills hypothesis IV if the following holds.

HYPOTHESIS IV. X is a continuous process, belonging to $\mathbb{D}_{2,1}(\mathscr{H})$ such that $(s^{H^0}\dot{\nabla}_s X_t,\ t \in I)$ is a uniformly continuous process in $L^2(W)$, uniformly with respect to s.

Note that according to Theorem A.5, when Hypothesis I holds, the solution of (E) satisfies IV.

THEOREM B.1. *Assume that Hypothesis I holds. Let* $X = (X^l)_{l=1,\dots,M}$ *be given by*

$$X^l = x_0 + \sum_{i=1}^d \left\{ \int_0^t K_{H_i}(t,s) b_s^{l,i}\,ds + \int_0^t K_{H_i}(t,s) a_s^{l,i}\,dB_s^i + \int_0^t c_s^{l,i}\,dB_s^i \right\},$$

where B^i *are the Brownian motions mentioned in properties* (P2) *and* (P3), $a^{l,i}$, $b^{l,i}$ *and* $c^{l,i}$ *are adapted processes,* $a^{l,i}$, $b^{l,i}$ *are bounded and* $c^{l,i}$ *belongs to* $L^2(W \times I)$, *for any* i *and any* l. *Assume furthermore that* X *fulfills Hypothesis* IV *and that for any* i, l,

$$\mathbf{E}\left[\int_0^1 \int_0^s |\dot{\nabla}_r^i a_s^{l,i}|^2 (sr)^{1-2H_i}\,dr\,ds\right] < +\infty. \tag{28}$$

For all $f \in C^2(\mathbf{R}^M, \mathbf{R})$ *with bounded derivatives, we have for all* $t \in I$ $\mathbf{P}_{\bar{H}}$ *almost everywhere,*

$$\begin{aligned} f(X_t) = f(X_0) &+ \sum_{i=1}^d \sum_{l=1}^M \int_0^t \frac{\partial f}{\partial x_l}(X_s) K'_{H_i}(b^{l,i})(s)\,ds \\ &+ \sum_{i=1}^d \sum_{l=1}^M \int_0^t a_s^{l,i} K'^{*}_{H_i}\left(\frac{\partial f}{\partial x_l} \circ X\right)(s)\,dB_s^i \\ &+ \sum_{i=1}^d \sum_{l,k=1}^M \int_0^t \frac{\partial^2 f}{\partial x_l \partial x_k}(X_s) K'_{H_i}(a^{l,i} \dot{\nabla}_{\bullet}^i X_s^k)(s)\,ds \\ &+ \sum_{i=1}^d \sum_{l=1}^M \int_0^t \frac{\partial f}{\partial x_l}(X_s) c^{l,i}(X_s)\,dB_s^i \\ &+ \frac{1}{2}\sum_{i=1}^d \sum_{l,k=1}^M \int_0^t \frac{\partial f}{\partial x_l \partial x_k}(X_s) c_s^{l,i} c_s^{i,k}\,ds. \end{aligned}$$

REMARK B.1. One should remark that the null set out of which the previous Itô formula holds depends on t; hence when a formula for processes is wanted, one should verify that both sides of the equality are continuous with respect to t. Before we start the proof it is useful to recall two expressions which follow immediately from the definitions of K_{H_i} and $I_{0^+}^{H_i-1/2}$,

$$
\begin{aligned}
&K_{H_i}(t+\alpha, s) - K_{H_i}(t, s) \\
&\quad = s^{1/2-H_i} \int_t^{t+\alpha} u^{H_i-1/2}(u-s)^{H_i-3/2}\,du\ \mathbf{1}_{[0,t]}(s) \\
&\qquad + s^{1/2-H_i} \int_s^{t+\alpha} u^{H_i-1/2}(u-s)^{H_i-3/2}\,du\ \mathbf{1}_{[t,t+\alpha]}(s),
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&\Gamma(H_i - 1/2) I_{t^-}^{H_i-1/2}(f(u))(s) \\
&\quad = \int_s^t f(u)(u-s)^{H_i-3/2}\,du \\
&\quad = \sum_{n=0}^{\Delta-1} \mathbf{1}_{[0,t_n]}(s) \int_{t_n}^{t_{n+1}} f(u)(u-s)^{H_i-3/2}\,du \\
&\qquad + \mathbf{1}_{[t_n,t_{n+1}]}(s) \int_s^{t_n} f(u)(s-u)^{H_i-3/2}\,du.
\end{aligned}
\tag{30}
$$

for any subdivision of $[0, t]$.

PROOF. Let $\Delta \in \mathbf{N}^*$, and $\Pi_\Delta =_{\text{def}} \{t_n,\ n = 0, \ldots, \Delta\}$ a subdivision of $[0, t]$ with $t_0 = 0$ and $t_\Delta = t$, the mesh of which is $|\Pi_\Delta| =_{\text{def}} \sup_{i=0,\ldots,\Delta-1}(t_{i+1} - t_i)$. We have

$$
f(X_t) - f(X_0) = \sum_{n=0}^{\Delta-1} f(X_{t_{n+1}}) - f(X_{t_n}).
$$

According to the Taylor expansion, we can write

$$
\begin{aligned}
f(X_t) - f(X_0) &= \sum_{n=0}^{\Delta-1}\sum_{l=1}^{M} \frac{\partial f}{\partial x_l}(X_{t_n})(X_{t_{n+1}}^l - X_{t_n}^l) \\
&\quad + \sum_{n=0}^{\Delta-1}\sum_{l,k=1}^{M} \int_0^1 \frac{\partial^2 f}{\partial x_l \partial x_k}(uX_{t_n} + (1-u)X_{t_{n+1}})\,du \\
&\qquad \times (X_{t_{n+1}}^l - X_{t_n}^l)(X_{t_{n+1}}^k - X_{t_n}^k) \\
&= \sum_{n=0}^{\Delta-1}\left(\sum_{l=1}^{M} S_1^{n,l} + \sum_{k,l=1}^{M} S_2^{n,l,k}\right).
\end{aligned}
$$

We first deal with the first-order terms $S_1^{n,l}$,

$$
\begin{aligned}
S_1^{n,l} &= \sum_{i=1}^{d} \frac{\partial f}{\partial x_l}(X_{t_n}) \int_I (K_{H_i}(t_{n+1}, s) - K_{H_i}(t_n, s)) b_s^{l,i}\,ds \\
&\quad + \sum_{i=1}^{d} \frac{\partial f}{\partial x_l}(X_{t_n}) \int_I (K_{H_i}(t_{n+1}, s) - K_{H_i}(t_n, s)) a_s^{l,i}\,dB_s^i \\
&\quad + \sum_{i=1}^{d} \frac{\partial f}{\partial x_l}(X_{t_n}) \int_{t_n}^{t_{n+1}} c_s^{l,i}\,dB_s^i = \sum_{j=1}^{3}\sum_{i=1}^{d} B_j^{l,i,n}.
\end{aligned}
$$

STEP 1. We show that if we set

$$\mathscr{E}_1^{l,i,\Delta} = \sum_{n=0}^{\Delta-1} B_1^{l,i,n} - \int_0^t \frac{\partial f}{\partial x_l}(X_s) s^{H_i-1/2} I_{0^+}^{H_i-1/2}(u^{1/2-H_i} b_u^{l,i})(s)\,ds, \tag{31}$$

then $\mathbf{P}_{\bar{H}}$ a.e., $\mathscr{E}_1^{l,i,\Delta}$ goes to 0 when the mesh of the subdivision $|\Pi_\Delta|$ goes to 0, because, observe that

$$\begin{aligned}\int_0^t \frac{\partial f}{\partial x_l}(X_s) s^{H_i-1/2} I_{0^+}^{H_i-1/2}(u^{1/2-H_i} b_u^{l,i})(s)\,ds \\ = \int_0^t u^{1/2-H_i} b_u^{l,i} I_{t-}^{H_i-1/2}\left(\frac{\partial f}{\partial x_l}(X_s) s^{H_i-1/2}\right)(u)\,du.\end{aligned}$$

According to (29),

$$\begin{aligned}&\Gamma(H_i - 1/2) B_1^{l,i,\Delta} \\ &\quad = \sum_{n=0}^{\Delta-1} \int_0^{t_n} \frac{\partial f}{\partial x_l}(X_{t_n}) s^{1/2-H_i} b_s^{l,i} \int_{t_n}^{t_{n+1}} u^{H_i-1/2}(u-s)^{H_i-3/2}\,du\,ds \\ &\qquad + \int_{t_n}^{t_{n+1}} \frac{\partial f}{\partial x_l}(X_{t_n}) s^{1/2-H_i} b_s^{l,i} \int_s^{t_{n+1}} u^{H_i-1/2}(u-s)^{H_i-3/2}\,du\,ds.\end{aligned}$$

Applying (30) to the function

$$f(u) = \sum_{n=0}^{\Delta-1} \frac{\partial f}{\partial x_l}(X_{t_n}) u^{H_i-1/2}\, \mathbf{1}_{[t_n,\, t_{n+1})}(u),$$

we obtain

$$B_1^{l,i,\Delta} = \int_0^t s^{1/2-H_i} b_s^{l,i} I_{t^-}^{H_i-1/2}\left(u^{H_i-1/2} \sum_{n=0}^{\Delta-1} \frac{\partial f}{\partial x_l}(X_{t_n}) \mathbf{1}_{[t_n,\, t_{n+1}]}(u)\right)(s)\,ds.$$

Since $(X_s,\ s \in I)$ is a continuous process, b and $\partial f/\partial x_l$ are bounded, for all $s \in [0, 1]$,

$$I_{t^-}^{H_i-1/2}\left(u^{H_i-1/2} \sum_{n=0}^{\Delta-1} \frac{\partial f}{\partial x_l}(X_{t_n}) \mathbf{1}_{[t_n,\, t_{n+1}]}(u)\right)(s)$$

converges in $L^1(W)$ and almost everywhere to

$$I_{t^-}^{H_i-1/2}\left(u^{H_i-1/2} \frac{\partial f}{\partial x_l}(X_u)\right)(s)$$

when the mesh of the subdivision Π^Δ goes to zero. Since b is bounded, by dominated convergence, the first point is established.

STEP 2. We prove that $\sum_{n=0}^{\Delta-1} B_2^{l,i,n}$ goes to

$$\int_0^t s^{1/2-H_i} a_s^{l,i} I_{t^-}^{H_i-1/2}\left(\frac{\partial f}{\partial x_l}(X_u)u^{H_i-1/2}\right)(s)\,dB_s^i \\ -\sum_{k=1}^{M}\int_0^t s^{1/2-H_i} a_s^{l,i} I_{t^-}^{H_i-1/2}\left(\frac{\partial^2 f}{\partial x_l \partial x_k}(X_u)u^{H_i-1/2}\dot{\nabla}_s^i X_u^k\right)(s)d_s$$

in L^2 when the mesh of the subdivision Π^Δ goes to zero. Since a and $\partial f/\partial x_l$ are bounded, the properties of the Skohorod integral [see, for instance, Nualart (1995)], allow writing

$$\begin{aligned}\frac{\partial f}{\partial x_l}(X_{t_n})&\int_I [K_{H_i}(t_{n+1},s)-K_{H_i}(t_n,s)]a^{l,i}(X_s)\,dB_s^i \\ &=\int_I \frac{\partial f}{\partial x_l}(X_{t_n})[K_{H_i}(t_{n+1},s)-K_{H_i}(t_n,s)]a^{l,i}(X_s)\,dB_s^i \\ &\quad+\sum_{k=1}^{M}\int_I \frac{\partial^2 f}{\partial x_l\partial x_k}(X_{t_n})\dot{\nabla}_s^i X_{t_n}^k[K_{H_i}(t_{n+1},s)-K_{H_i}(t_n,s)]a^{l,i}(X_s)\,ds.\end{aligned}$$

Proceeding as above, we obtain

$$\begin{aligned}\sum_{n=0}^{\Delta-1} B_2^{l,i,n} &= \int_I s^{1/2-H_i} a_s^{l,i} I_{t^-}^{H_i-1/2}\left(\sum_{n=0}^{\Delta-1}\frac{\partial f}{\partial x_l}(X_{t_n})\mathbf{1}_{[t_n,t_{n+1}]}(u)u^{H_i-1/2}\right)(s)\,dB_s^i \\ &\quad+\int_I s^{1/2-H_i} a_s^{l,i} I_{t^-}^{H_i-1/2}\left(\dot{\nabla}_s^i\left[\sum_{n=0}^{\Delta-1}\frac{\partial f}{\partial x_l}(X_{t_n})\right]\mathbf{1}_{[t_n,t_{n+1}]}(u)u^{H_i-1/2}\right)(s)\,ds.\end{aligned}$$

Set

$$\begin{aligned}\mathscr{A}_s^{l,i,\Delta} &= s^{1/2-H_i} a_s^{l,i} I_{t^-}^{H_i-1/2}\left(\sum_{n=0}^{\Delta-1}\frac{\partial f}{\partial x_l}(X_{t_n})\mathbf{1}_{[t_n,t_{n+1}]}(u)u^{H_i-1/2}\right)(s), \\ \mathscr{A}_s^{l,i} &= s^{1/2-H_i} a_s^{l,i} I_{t^-}^{H_i-1/2}\left(\frac{\partial f}{\partial x_l}(X_u)u^{H_i-1/2}\right)(s).\end{aligned}$$

For the stochastic integral, recall that

$$\begin{aligned}&\mathbf{E}\left[\left|\int_0^t(\mathscr{A}_s^{l,i,\Delta}-\mathscr{A}_s^{l,i})\,dB_s^i\right|^2\right] \\ &\qquad=\mathbf{E}\left[\int_0^t|\mathscr{A}_s^{l,i,\Delta}-\mathscr{A}_s^{l,i}|^2\,ds\right] \\ &\qquad\quad+\iint_{[0,t]^2}\dot{\nabla}_r^i(\mathscr{A}_s^{l,i,\Delta}-\mathscr{A}_s^{l,i})\dot{\nabla}_s^i(\mathscr{A}_r^{l,i,\Delta}-\mathscr{A}_r^{l,i})\,dr\,ds.\end{aligned}\tag{32}$$

Since $a^{l,i}$ is bounded, we have

$$\mathbf{E}\left[\int_I |\mathscr{A}_s^{l,i,\Delta} - \mathscr{A}_s^{l,i}|^2 ds\right] \le \|a^{l,i}\|_\infty \int_0^t \int_0^t s^{1-2H_i}\, ds$$

$$\mathbf{E}\left[\left| I_{t^-}^{H_i-1/2}\left(u^{H_i-1/2}\left[\sum_{n=1}^{\Delta-1}\left\{\frac{\partial f}{\partial x_l}(X_{t_n}) - \frac{\partial f}{\partial x_l}(X_u)\right\}\mathbf{1}_{[t_n,\,t_{n+1}]}(u)\right]\right)(s)\right|^2\right] ds.$$

Recall that for $g \in L^\infty(I)$ and $H_i > 1/2$,

$$\left|I_{t^-}^{H_i-1/2}(u^{H_i-1/2}g)(s)\right| \le c\|g\|_\infty (t-s)^{2H_i-1} \le c\|g\|_\infty, \tag{33}$$

since $H_i - 1/2 \ge 0$, for any i. Considering that $\partial F/\partial x_l$ is bounded, it follows by dominated convergence that

$$\mathbf{E}\left[\int_I |\mathscr{A}_s^{l,i,\Delta} - \mathscr{A}_s^{l,i}|^2\, ds\right] \longrightarrow\ 0 \text{ as } |\Pi^\Delta| \to 0.$$

Since a is adapted, the trace term in (32) (i.e., the rightmost summand) can be decomposed in the sum,

$$\begin{aligned}
&\iint_{[0,\,t]^2} (sr)^{1/2-H_i} a_r^{l,i} \dot\nabla_r^i a_s^{l,i} I_{t^-}^{H_i-1/2}\big(\zeta_u^\Delta u^{H_i-1/2}\big)(s)\\
&\qquad \times I_{t^-}^{H_i-1/2}\big(\dot\nabla_s^i \zeta_u^\Delta u^{H_i-1/2}\big)(r)\, dr\, ds\\
&\qquad + \int_0^t \int_0^t (sr)^{1/2-H_i} a_s^{l,i} a_r^{l,i} I_{t^-}^{H_i-1/2}\big(\dot\nabla_r^i \zeta_u^\Delta u^{H_i-1/2}\big)(s)\\
&\qquad\qquad \times I_{t^-}^{H_i-1/2}\big(\dot\nabla_s^i \zeta_u^\Delta u^{H_i-1/2}\big)(r)\, dr\, ds,
\end{aligned}$$

where

$$\zeta_u^\Delta = \sum_{n=0}^{\Delta-1}\left\{\frac{\partial f}{\partial x_l}(X_{t_n}) - \frac{\partial f}{\partial x_l}(X_u)\right\}\mathbf{1}_{[t_n,\,t_{n+1}]}(u).$$

Denote by C_1 and C_2 the two summands of the last sum. By the Cauchy–Schwarz inequality, (33) and (28) and Fubini's theorem,

$$\begin{aligned}
\mathbf{E}[|C_1|] &\le c\|a\|_\infty \|\zeta^\Delta\|_\infty \mathbf{E}\left[\iint_{[0,\,t]^2} \left|\dot\nabla_r^i a_s (sr)^{1/2-H_i}\right|^2 dr\, ds\right]^{1/2}\\
&\qquad \times \mathbf{E}\left[\int_0^t \int_r^t \left|I_{t^-}^{H_i-1/2}\big(\dot\nabla_s^i \zeta_u^\Delta u^{H_i-1/2}\big)(r)\right|^2 dr\, ds\right]^{1/2}\\
&\le c\,\mathbf{E}\left[\int_0^t \int_0^s \left(\int_r^t (u-r)^{H_i-3/2} \dot\nabla_s^i \zeta_u^\Delta\, u^{H_i-1/2}\, du\right)^2 dr\, ds\right]^{1/2}\\
&\le c\,\mathbf{E}\left[\int_0^t \int_r^t (t-r)^{H_i-1/2} \int_r^t |\dot\nabla_r^i \zeta_u^\Delta|^2 u^{2H_i-1}(u-r)^{H_i-3/2}\, du\, dr\, ds\right]^{1/2}.
\end{aligned}$$

Moreover,

$$\mathbf{E}[|\dot\nabla_r^i \zeta_u^\Delta|^2] \le c \sum_{n=1}^{\Delta-1} \mathbf{E}[\|\dot\nabla_r^i X_{t_n} - \dot\nabla_r^i X_u\|^2]\mathbf{1}_{[t_n,\,t_{n+1}]}(u),$$

hence according to Hypothesis IV, (33) and the dominated convergence theorem, one concludes that C_1 tends to 0 in $L^1(W)$ with the mesh of the subdivision. On the other hand, by the Cauchy–Schwarz inequality again,

$$\mathbf{E}[|C_2|] \le c\mathbf{E}\left[\iint_{[0,\,t]^2} \left|I_{t^-}^{H_i-1/2}(\dot{\nabla}_r^i \zeta_u^\Delta u^{H_i-1/2})(s)\right|^2 (sr)^{1-2H_i}\,ds\,dr\right]^{1/2};$$

thus the same majorizations can be used again and it follows that C_2 converges to 0 in $L^1(W)$. Following the same reasoning, one shows that

$$\int_0^t s^{1/2-H_i} a_s^{l,\,i} I_{t^-}^{H_i-1/2}(\zeta_u^\Delta u^{H_i-1/2})(s)\,ds$$

goes to 0 in $L^1(W)$ and thus completes the proof of the second step.

STEP 3. One can prove by the standard semimartingale techniques that

$$\sum_{n=0}^{\Delta-1} C^{l,\,i,\,n} - \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_l}(X_s) c_s^{l,\,i}\,dB_s^i, \tag{34}$$

when the mesh of the subdivision Π^Δ goes to 0 in $L^2(W)$.

STEP 4. We deal now with the second-order terms. Since $a^{l,\,i}$ and $b^{l,\,i}$ are bounded, following CD (1997), the processes $\{\int_0^t K_{H_i}(t,s) b_s^{l,\,i}\,ds,\ t \in I\}$ and $\{\int_0^t K_{H_i}(t,s) a_s^{l,\,i}\,dB_s^i,\ t \in I\}$ are Hölder continuous in L^2, and they have null quadratic variation. The only terms which do not go to 0 when the mesh of the subdivision Π^Δ goes to 0 in $L^2(W, \mathbf{P}_{\bar{H}})$ are of the form

$$\sum_{n=0}^{\Delta-1} \frac{1}{2} \int_{t_n}^{t_{n+1}} c_s^{l,\,i}\,dB_s^i \int_{t_n}^{t_{n+1}} c_s^{i,\,k}\,dB_s^i \int_0^1 \frac{\partial^2 f}{\partial x_k \partial x_l}(uX_{t_n} + (1-u)X_{t_{n+1}}).$$

As in the classical Itô formula, they converge to

$$\frac{1}{2} \sum_{l,\,k=1}^{M} \int_0^t \frac{\partial f}{\partial x_l \partial x_k}(X_s) c_s^{l,\,i} c_s^{i,\,k}\,ds$$

when the mesh of the subdivision Π^Δ goes to 0, in $L^1(W)$. □

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